Runge-Kutta Methods

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Intended Learning Outcomes

Our classroom norms

- Get ready to engage in learning actively.
- Raise your hand to ask a question at any point during this session.

Objectives

- Use a variety of RK methods to solve practical problems
- Test the rate of convergence of various RK methods
- Construct multistage RK methods (time permitting)

Recall: First-order Initial Value Problem (IVP)

First-order IVP

$$\begin{cases} \dfrac{dy}{dt} = f(t, y), & \leftarrow \text{ordinary differential equation (ODE)} \\ y(t_0) = y_0. & \leftarrow \text{initial condition (IC)} \end{cases} \tag{1}$$

Example

An object falling is subject to two external forces: gravitational and drag. Then, by Newton's second law of motion, we have an equation of the form

$$\begin{cases} m\dfrac{dv}{dt} = mg - bv^2, \\ v(0) = 0, \end{cases} \quad \rightarrow \begin{cases} \dfrac{dv}{dt} = f(t, v) = g - \beta v^2, \\ v(0) = 0. \end{cases}$$

Recall: Euler's Method

Taylor series methods

$$Y_{n+1} = Y_n + h\varphi(t_n, Y_n, h)$$

where

$$\varphi(t, y, h) = f(t, y) + \frac{h}{2!}\frac{\partial f}{\partial t}(t, y) + \frac{h^2}{3!}\frac{\partial^2 f}{\partial t^2}(t, y) + \dots$$

Euler's Method (RK1)

$$Y_{n+1} = Y_n + hf(t_n, Y_n), \quad t_{n+1} = t_n + h$$

This method is not as accurate as other RK methods, as we shall see.

Note

Y_n is the approximate value of $y(t_n)$, and is sometimes denoted as $Y_n(h)$ to emphasize that the stepsize h was used.

Test of Convergence

Order of a method

If a numerical method is convergent, the global truncation error $E_h = \max_n |Y_n(h) - y(t_n)| \to 0$ as $h \to 0$. If $E_h = \mathcal{O}(h^p) = Ch^p$, we say that the numerical method is of order p.

Finding the rate of convergence p

- If the analytic solution is available, estimate the slope of $E_h(h)$ in log-log plot of E_h vs h. This gives the value of p.
- Otherwise, compute the ratios of the difference in approximations with different step sizes:

$$p = \log_2 \left(\frac{\max_n |Y_n(h) - Y_n(h/2)|}{\max_n |Y_n(h/2) - Y_n(h/4)|} \right)$$

In this case, $E_h \approx 2^p E_{h/2}$.

Example

Use Euler's method with $h = 2^{-k}$, $k = 2, \ldots, 9$ to approximate the solution to

$$\begin{cases} \dfrac{dy}{dt} = 32 - y^2, \\ y(0) = 0 \end{cases}$$

on the interval $[0, 1]$. Use the analytical solution

$$y(t) = \sqrt{32}\frac{e^{2\sqrt{32}t} - 1}{e^{2\sqrt{32}t} + 1}$$

to verify that the order of Euler's method is 1.

Midpoint method (RK2)

Midpoint method (RK2)

$$\begin{aligned} K_1 &= hf(t_n, Y_n) \\ K_2 &= hf\left(t_n + \frac{h}{2}, Y_n + \frac{1}{2}K_1\right) \\ Y_{n+1} &= Y_n + K_2. \end{aligned}$$

Idea

Take "trial step" to the middle of the interval using Euler's method. Then use the values of t and Y at the midpoint to take the full step.

Example

Use RK2 method with $h = 2^{-k}$, $k = 2, \ldots, 9$ to approximate the solution to

$$\begin{cases} \dfrac{dy}{dt} = 32 - y^2 \\ y(0) = 0 \end{cases}$$

on the interval $[0, 1]$. Verify that the order of the method is 2 without using the analytical solution.

RK4 method

Classic fourth-order Runge-Kutta method (RK4)

$$
\begin{aligned}
K_1 &= hf(t_n, Y_n), \\
K_2 &= hf\left(t_n + \frac{h}{2}, Y_n + \frac{1}{2}K_1\right), \\
K_3 &= hf\left(t_n + \frac{h}{2}, Y_n + \frac{1}{2}K_2\right), \\
K_4 &= hf\left(t_n + h, Y_n + K_3\right), \\
Y_{n+1} &= Y_n + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4).
\end{aligned}
$$

Activity

Use RK4 method with $h = 2^{-k}$, $k = 2, \ldots, 9$ to approximate the solution to

$$\begin{cases} \dfrac{dy}{dt} = 32 - y^2 \\ y(0) = 0 \end{cases}$$

on the interval $[0, 1]$. Verify that the order of the method is 4 with and without using the analytical solution.

Analytical solution:

$$y(t) = \sqrt{32}\,\frac{e^{2\sqrt{32}t} - 1}{e^{2\sqrt{32}t} + 1}$$

RK methods

Recall: Taylor series methods

$$Y_{n+1} = Y_n + h\varphi(t_n, Y_n, h)$$

where $$\varphi(t,y,h) = f(t,y) + \frac{h}{2!}\frac{\partial f}{\partial t}(t,y) + \frac{h^2}{3!}\frac{\partial^2 f}{\partial t^2}(t,y) + \dots$$

m-stage explicit RK method

$$\varphi(t,y,h) = \sum_{i=1}^{m} c_i K_i, \quad K_i = f\left(t + a_i h, y + h\sum_{j=1}^{i-1} b_{ij} K_j\right)$$

The coefficients a_i, b_{ij}, c_i fully characterize RK method.

Idea

The coefficients are chosen to cancel out error terms in a Taylor series.
Optimal choice of these coefficients requires a deep mathematical analysis.

Motivation: RK2

From $y' = f(t, y)$ we can use the Fundamental Theorem of Calculus to obtain that

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))dt. \tag{2}$$

Let $h = t_{n+1} - t_n$. Using the midpoint formula for an integral we obtain that

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt = hf\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right) + \mathcal{O}(h^3), \tag{3}$$

so that

$$y(t_{n+1}) = y(t_n) + hf\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right) + \mathcal{O}(h^3). \tag{4}$$

However, we do not know $y\left(t_n + \frac{h}{2}\right)$! We can use Euler's method to approximate $y\left(t_n + \frac{h}{2}\right)$ and obtain

$$y\left(t_n + \frac{h}{2}\right) = y(t_n) + \frac{h}{2} f(t_n, y(t_n)) + \mathcal{O}(h^2). \tag{5}$$

As we insert (5) into (4), we obtain the algorithm for the second order RK method (RK2).

Motivation: RK4

From $y' = f(t, y)$ we can use the Fundamental Theorem of Calculus to obtain that

$$y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(t, y(t))dt. \tag{6}$$

Let $h = t_{n+1} - t_n$. Using the Simpson rule for an integral we obtain that

$$\int_{t_n}^{t_{n+1}} f(t, y(t))dt = \frac{h}{6}\left[f(t_n, y(t_n)) + 4f\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right) + f(t_{n+1}, y(t_{n+1}))\right] + \mathcal{O}(h^5), \tag{7}$$

so that

$$y(t_{n+1}) = y(t_n) + \frac{h}{6}\left[f(t_n, y(t_n)) + 4f\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right) + f(t_{n+1}, y(t_{n+1}))\right] + \mathcal{O}(h^5). \tag{8}$$

However, we do not know $y\left(t_n + \frac{h}{2}\right)$ and $y(t_{n+1})$! The fourth-order RK4 splits midpoint evaluations in two steps, that is we have

$$y(t_{n+1}) = y(t_n) + \frac{h}{6}\left[f(t_n, y(t_n)) + 2f\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right) + 2f\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right)\right. \tag{9}$$

$$\left. + f(t_{n+1}, y(t_{n+1}))\right] + \mathcal{O}(h^5). \tag{10}$$

Continuation

We have:

$$y(t_{n+1}) = y(t_n) + \frac{h}{6}\left[f(t_n, y(t_n)) + 2f\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right) + 2f\left(t_n + \frac{h}{2}, y\left(t_n + \frac{h}{2}\right)\right)\right. \tag{11}$$

$$\left. + f(t_{n+1}, y(t_{n+1}))\right] + \mathcal{O}(h^5). \tag{12}$$

The first two function evaluations are as for RK2; that is, we make

$$K_1 = hf(t_n, Y_n) \tag{13}$$

which is the slope at t_n, and then we compute the slope at the midpoint using Euler's method to predict $y\left(t_n + \frac{h}{2}\right)$:

$$K_2 = hf\left(t_n + \frac{h}{2}, Y_n + \frac{1}{2}K_1\right). \tag{14}$$

Then the improved slope at the midpoint is used to further improve the slope used for $y\left(t_n + \frac{h}{2}\right)$:

$$K_3 = hf\left(t_n + \frac{h}{2}, Y_n + \frac{1}{2}K_2\right). \tag{15}$$

Finally, with the latter slope we can in turn predict:

$$y(t_{n+1}) = y(t_n) + K_3 + \mathcal{O}(h^5). \tag{16}$$

As we put everything together, we obtain the algorithm for the fourth-order RK method (RK4).

Final remarks

Note

Higher-order IVPs can be reduced to a system of first-order ones. E.g.,

$$\text{(second-order)}\ \frac{d^2y}{dt^2} = f\left(t, y, \frac{dy}{dt}\right) \Leftrightarrow \begin{cases} \frac{dy}{dt} = v, \\ \frac{dv}{dt} = f(t, y, v). \end{cases}$$

Other numerical methods for solving IVPs

- Richardson extrapolation, Bulirsch-Stoer method: extrapolate a computed result to a smaller stepsize
- Predictor-corrector methods: store solution, extrapolate it, then correct using derivative information

Adaptive stepsize

Changes in stepsize during the computation are often used to achieve greater accuracy with minimal effort.

Summary

- Runge-Kutta methods can be used to solve ODEs coming from applications like physics and engineering
- Convergence and order of numerical methods can be tested by computations with different step sizes
- Runge-Kutta methods of various orders can be built using FTC or Taylor series