Uncertainty Quantification for multiscale kinetic equations with random inputs

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Where do kinetic equations sit in physics

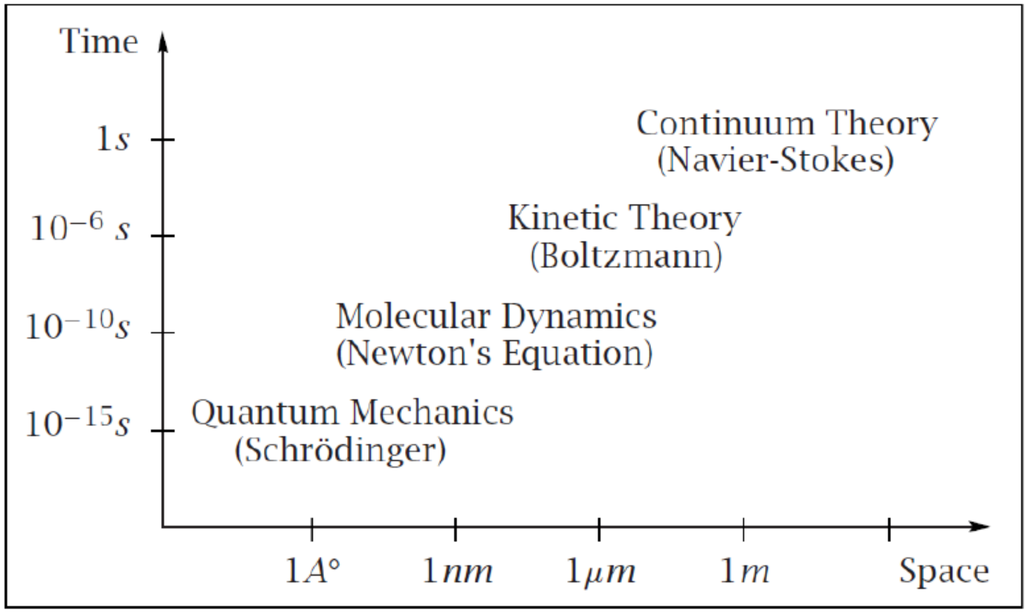


Figure 1. Different laws of physics are required to describe properties and processes of fluids at different scales.

- E & Engquist, AMS Notice (2003)

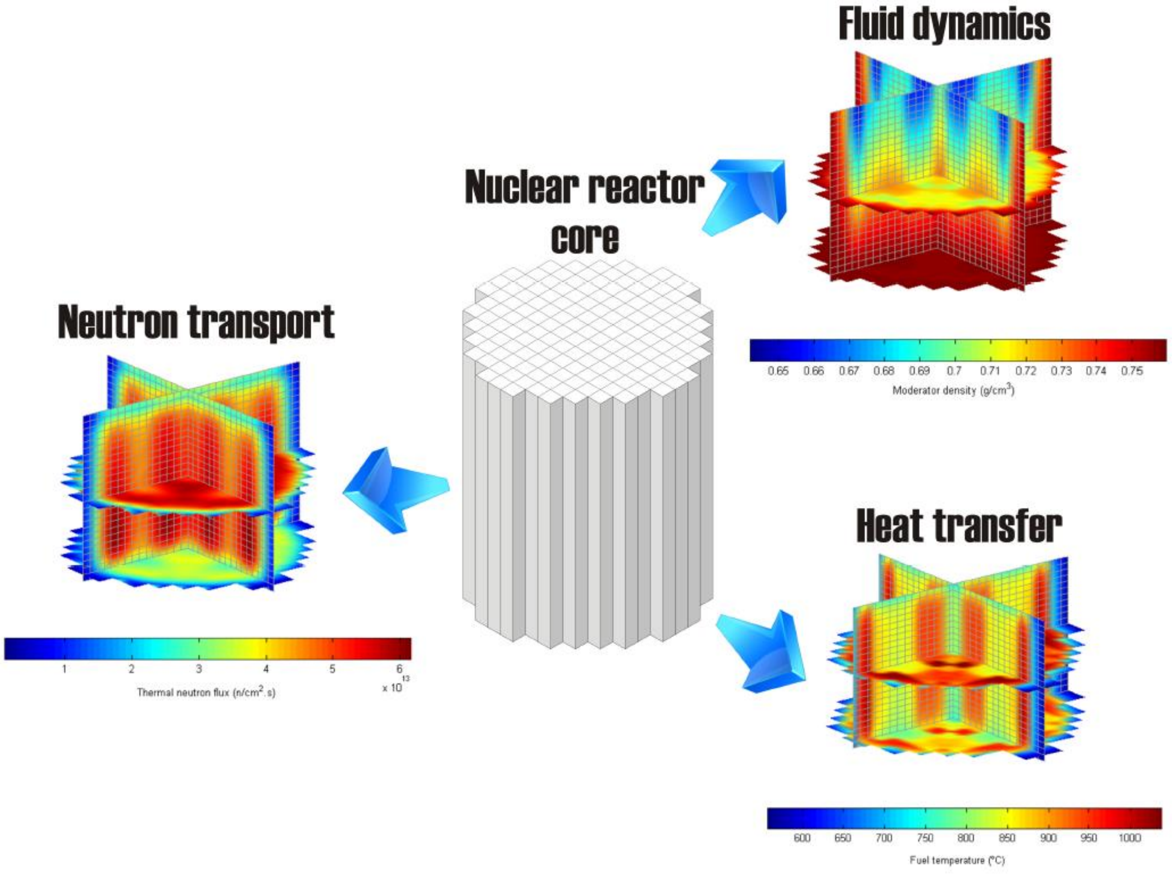
Kinetic equations with applications

- Rarefied gas—astronautics (Boltzmann equation)
- Plasma (Vlasov-Poisson, Landau, Fokker-Planck,…)
- Semiconductor device modeling
- Microfluidics
- Nuclear reactor (nentron transport)
- Astrophysics, medical imaging (radiative transfer)
- Multiphase flows
- Environmental science, energy, social science, neuronal networks, biology, …

Challenges in kinetic computation

- High dimension (phase space, 6d for Boltzmann)
- Multiple scales
- uncertainty

Multiscale phenomena



Uncertainty in kinetic equations

- Kinetic equations are usually derived from N-body Newton's second law, by mean-field limit, BBGKY hierachy, Grad-Boltzmann limit, etc.
- Collision kernels are often empirical
- Initial and boundary data contain uncertainties due to measurement errors or modelling errors; geometry, forcing
- While UQ has been popular in solid mechanics, CFD, elliptic equations, etc. (Abgrall, Babuska, Ghanem, Gunzburger, Hesthaven, Karniadakis, Knio, Mishra, Neim, Nobile, Tempone, Xiu, Webster, Schwab, etc.) there has been almost no effort for kinetic equations

Data for scattering cross-section

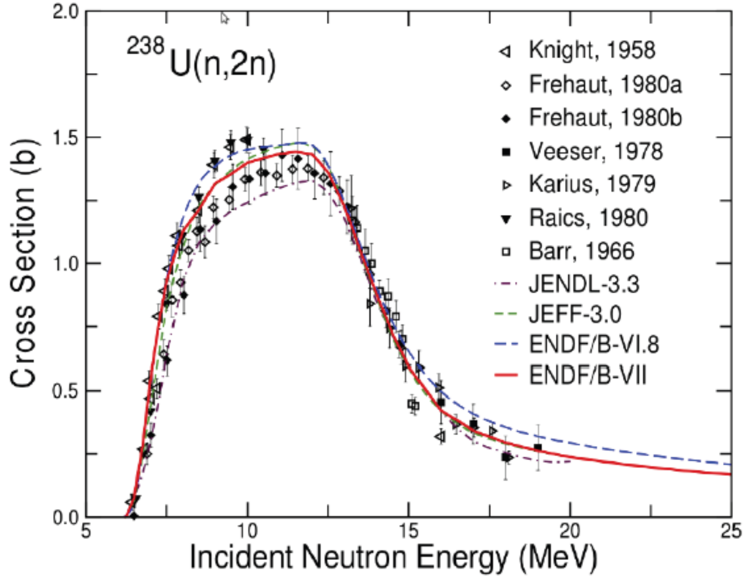


Figure 2: Example of uncertainty associated with a nuclear cross-section (from (Chadwick et al., 2006)). Figure contains values corresponding to several data libraries and measurements.

UQ for kinetic models

For kinetic models, the only thing certain is their uncertainty

- Quantify the propagation of the uncertainty
- efficient numerical methods to study the uncertainty
- understand its statistical moments
- sensitivity analysis, long-time behavior of the uncertainty
- Control of the uncertainty
- dimensional reduction of high dimensional uncertainty
- …

Example: random linear neutron transport equation (Jin-Xiu-Zhu JCP'14)

$$\epsilon\partial_t f(v) + v\partial_x f(v) = \frac{\sigma(x,z)}{\epsilon}\left[\frac{1}{2}\int_{-1}^{1} f(v')\,dv' - f(v)\right],$$

$\sigma(x,z)$ the scattering cross-section, is random

Diffusion limit: Larsen-Keller, Bardos-Santos-Sentis, Bensoussan-Lions-Papanicolaou (for each z)

as $\epsilon \to 0^+$ f→ $\rho(t,x) = \frac{1}{2}\int_{-1}^{1} f(v')\,dv'$

$$\rho_t = \partial_x\left[\frac{1}{3\sigma(x,z)}\partial_x\rho\right]$$

Polynomial Chaos (PC) approximation

- The PC or generalized PC (gPC) approach first introduced by Wiener, followed by Cameron-Martin, and generalized by Ghanem and Spanos, Xiu and Karniadakis etc. has been shown to be very efficient in many UQ applications when the solution has enough regularity in the random variable
- Let z be a random variable with pdf $\rho(z) > 0$
- Let $\Phi_m(z)$ be the orthonormal polynomials of degree m corresponding to the weight $\rho(z) > 0$

$$\int \Phi_i(z)\Phi_j(z)\rho(z)\,dz = \delta_{ij}$$

The Wiener-Askey polynomial chaos for random variables (table from Xiu-Karniadakis SISC 2002)

	Random variables ζ	Wiener-Askey chaos $\{\Phi(\zeta)\}$	Support
Continuous	Gaussian	Hermite-Chaos	$(-\infty, \infty)$
	Gamma	Laguerre-Chaos	$[0, \infty)$
	Beta	Jacobi-Chaos	$[a, b]$
	Uniform	Legendre-Chaos	$[a, b]$
Discrete	Poisson	Charlier-Chaos	$\{0, 1, 2, \dots\}$
	Binomial	Krawtchouk-Chaos	$\{0, 1, \dots, N\}$
	Negative Binomial	Meixner-Chaos	$\{0, 1, 2, \dots\}$
	Hypergeometric	Hahn-Chaos	$\{0, 1, \dots, N\}$

TABLE 4.1

The correspondence of the type of Wiener-Askey polynomial chaos and their underlying random variables ($N \geq 0$ is a finite integer).

Generalized polynomial chaos stochastic Galerkin (gPC-sG) methods

- Take an orthonormal polynomial basis $\{\Phi_j(z)\}$ in the random space

- Expand functions into Fourier series and truncate:

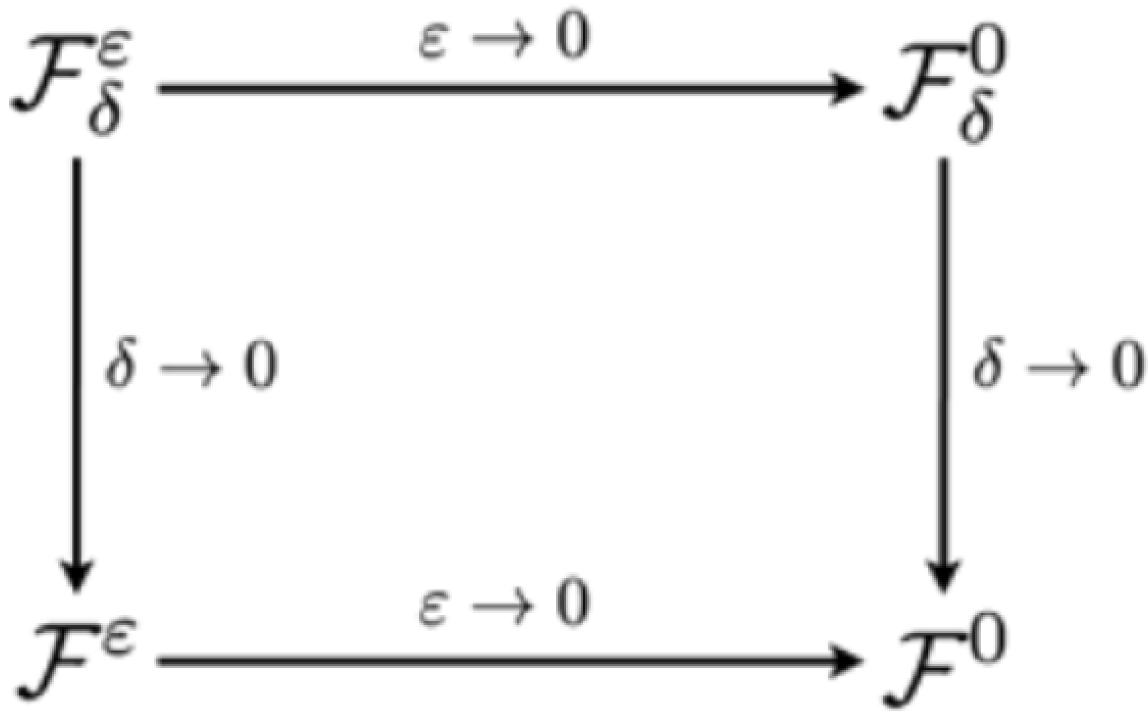
$$f(z) = \sum_{j=0}^{\infty} f_j \phi_j(z) \approx \sum_{j=0}^{K} f_j \phi_j(z) := f^K(z).$$

- Substitute into system, Galerkin projection. Then one gets a deterministic system of the gPC coefficients $(f_0, \ldots, f_K)$

Accuracy and efficiency

- We will consider the gPC-stochastic Galerkin (gPC-SG) method
- Under suitable regularity assumptions this method has a spectral accuracy
- Much more efficient than Monte-Carlo samplings (halfth-order)
- Our regularity analysis is also important for stochastic collocation method
- How to deal with multiple scales?

A multiscale paradigm:
Asymptotic-preserving (AP) methods- Jin ‘99



Stochastic AP schemes (s-AP)

2.1. Stochastic asymptotic preserving scheme. We now consider the same problem subject to random inputs.

$$\partial_t u^\epsilon = \mathcal{L}^\epsilon(t, x, z, u^\epsilon; \epsilon), \tag{2.3}$$

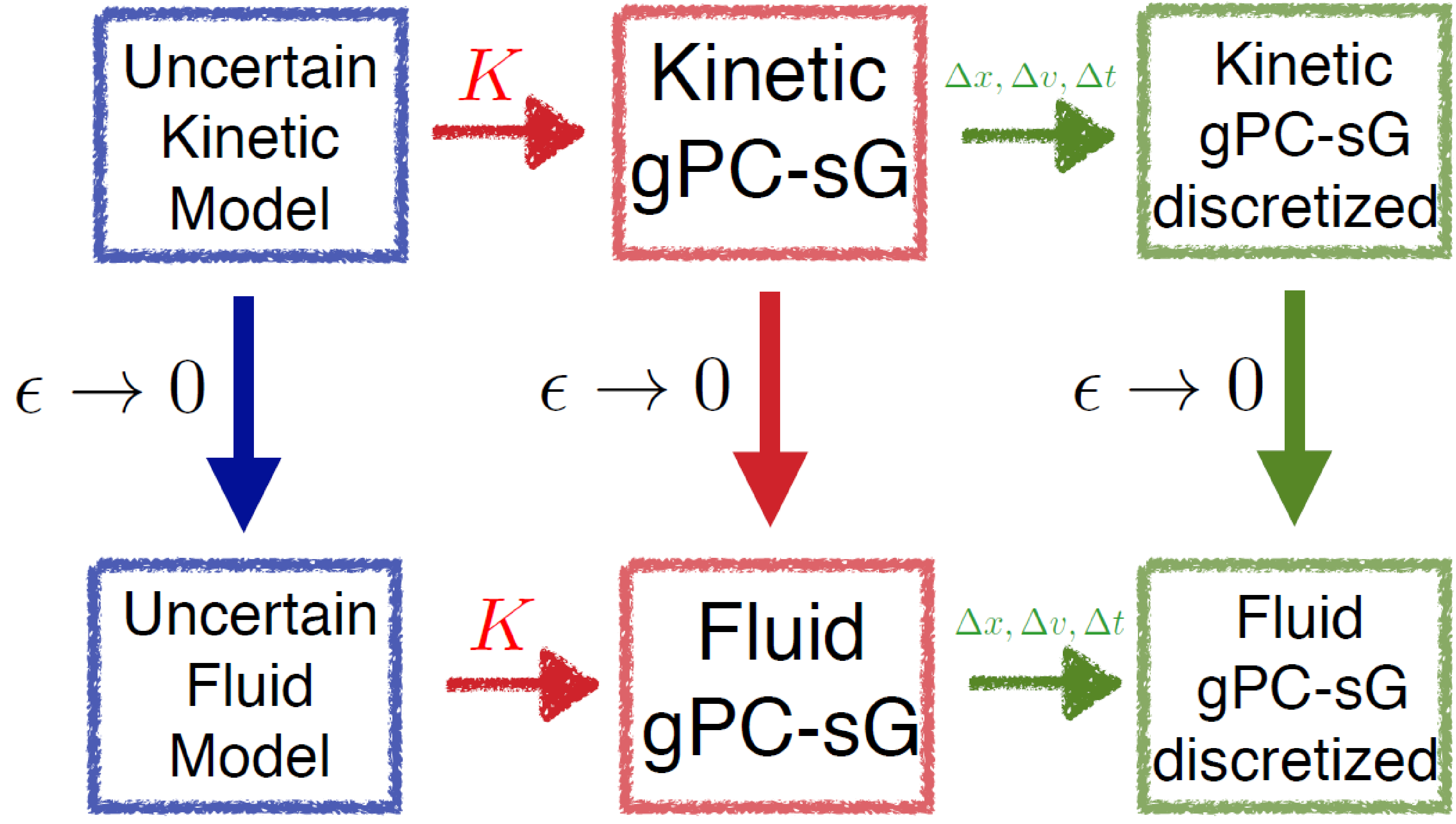
where $z \in I_z \subseteq \mathbb{R}^d$, $d \geq 1$, are a set of random variables equipped with probability density function ρ. These random variables characterize the random inputs into the system. As $\epsilon \to 0$, the diffusive limit becomes

$$\partial_t u = \mathcal{L}(t, x, z, u). \tag{2.4}$$

We now extend the concept of deterministic AP to the stochastic case. To avoid the cluttering of notations, let us now focus on the discretization in the random space I_z.

DEFINITION 2.1 (Stochastic AP). *Let $\mathcal{S}$ be a numerical scheme for* (2.3), *which results in a solution $v^\epsilon(z) \in V_z$ in a finite dimensional linear function space V_z. Let $v(z) = \lim_{\epsilon\to 0} v^\epsilon(z)$ be its asymptotic limit. We say that the scheme $\mathcal{S}$ is* strongly asymptotic perserving *if the limiting solution $v(z)$ satisfies the limiting equation* (2.4) *for almost every $z \in I_z$; and it is* weakly asymptotic perserving *if the limiting solution $v(z)$ satisfies the limiting equation* (2.4) *in a weak form.*

Stochastic AP



Linear transport equation with random coeffcients

$$\epsilon\partial_t f + v\partial_x f = \frac{\sigma(x,z)}{\epsilon}\left[\frac{1}{2}\int_{-1}^{1} f(v')\,dv' - f\right],$$

To understand its diffusion limit, we first split this equation into two equations for $v > 0$:

$$\begin{aligned}
\epsilon\partial_t f(v) + v\partial_x f(v) &= \frac{\sigma(x,z)}{\epsilon}\left[\frac{1}{2}\int_{-1}^{1} f(v')\,dv' - f(v)\right],\\
\epsilon\partial_t f(-v) - v\partial_x f(-v) &= \frac{\sigma(x,z)}{\epsilon}\left[\frac{1}{2}\int_{-1}^{1} f(v')\,dv - f(-v)\right],
\end{aligned} \tag{3.6}$$

and then consider its even and odd parities

$$\begin{aligned}
r(t,x,v) &= \frac{1}{2}[f(t,x,v) + f(t,x,-v)],\\
j(t,x,v) &= \frac{1}{2\epsilon}[f(t,x,v) - f(t,x,-v)].
\end{aligned} \tag{3.7}$$

Diffusion limit

The system (3.6) can then be rewritten as follows:

$$\begin{cases} \partial_t r + v\partial_x j = \dfrac{\sigma(x,z)}{\epsilon^2}(\overline{r} - r), \\ \partial_t j + \dfrac{v}{\epsilon^2}\partial_x r = -\dfrac{\sigma(x,z)}{\epsilon^2} j. \end{cases} \tag{3.8}$$

where

$$\overline{r}(t,x) = \int_0^1 r dv.$$

As $\epsilon \to 0^+$, (3.8) yields

$$r = \overline{r}, \qquad j = -\frac{v}{\sigma(x,z)}\partial_x \overline{r}.$$

Substituting this into system (3.8) and integrating over v, one gets the limiting diffusion equation ([23, 1]):

$$\partial_t \overline{r} = \partial_x \left[\frac{1}{3\sigma(x,z)} \partial_x \overline{r} \right]. \tag{3.9}$$

gPC approximations

$$r_N(x,z,t) = \sum_{m=1}^{M} \hat{r}_m(t,x)\Phi_m(z), \quad j_N(x,z,t) = \sum_{m=0}^{M} \hat{j}_m(t,x)\Phi_m(z) \tag{6.1}$$

be the Nth-order gPC expansion for the solutions and

$$\hat{\mathbf{r}} = (\hat{r}_1, \cdots, \hat{r}_M)^T, \quad \hat{\mathbf{j}} = (\hat{j}_1, \cdots, \hat{j}_M)^T,$$

$$\begin{cases} \partial_t \hat{\mathbf{r}} + v\partial_x \hat{\mathbf{j}} = \dfrac{1}{\epsilon^2}\mathbf{S}(x)(\overline{\mathbf{r}} - \hat{\mathbf{r}}), \\ \partial_t \hat{\mathbf{j}} + \dfrac{v}{\epsilon^2}\partial_x \hat{\mathbf{r}} = -\dfrac{1}{\epsilon^2}\mathbf{S}(x)\hat{\mathbf{j}}, \end{cases} \tag{6.2}$$

where

$$\overline{\mathbf{r}}(x,t) = \int_0^1 \hat{\mathbf{r}}\, dv,$$

and $\mathbf{S}(x) = (s_{ij}(x))_{1\le i,j\le M}$ is a $M \times M$ matrix with entries

$$s_{ij}(x) = \int \sigma(x,z)\Phi_i(z)\Phi_j(z)\rho(z)dz. \tag{6.3}$$

Vectorized version of the deterministic problem (we can do APUQ!)

- One can now use deterministic AP schemes to solve this system

- Why s-AP?

- When $\epsilon \to 0$ the gPC-SG for transport equation becomes the gPC-SG for the limiting diffusion equation

gPC-SG for limiting diffusion equations

- For diffusion equation:

$$u_t = \partial_x [a(x,z)\partial_x u]$$

- Galerkin approximation:

$$u(x,z,t) = \sum_{m=0}^{M} \hat{u}_m(t,x)\Phi_m(z)$$

- moments: $\mathbb{E}[u] = \hat{u}_0 \,,$ $\mathrm{Var}[u] = \sum_{m=0}^{M} \hat{u}_m^2$
- Let $\hat{\mathbf{u}} = (\hat{u}_1, \cdots, \hat{u}_M)^T$ then

$$\partial_t \hat{\mathbf{u}} = \partial_x \left(\mathbf{A}\partial_x \hat{\mathbf{u}}\right) \qquad \mathbf{A} = (a_{ij})_{M\times M}$$ symm. pos. def

$$a_{ij}(x) = \int a(x,z)\Phi_i(z)\Phi_j(z)\rho(z)dz.$$

Uniform spectral accuracy (via coercivity: uniform expotential decay to the local equilibrium: *Jin-J.-G. Liu-Ma* (RMS '17)

- Define the following norms

$$\langle f, g\rangle_\omega = \int_{\mathbb{R}^d} f(z) g(z)\, \omega(z)\, \mathrm{d}z, \quad \|f\|_\omega^2 = \langle f, f\rangle_\omega$$

$$\|f(t,x,v,\cdot)\|_{H^k}^2 := \sum_{\alpha\le k} \|D^\alpha f(t,x,v,\cdot)\|_\omega^2$$

$$\|f(t,\cdot,\cdot,\cdot)\|_{\Gamma(t)}^2 := \int_Q \|f(t,x,v,\cdot)\|_\omega^2 \,\mathrm{d}x\,\mathrm{d}v.$$

Uniform regularity

- The regularity in the random space is preserved in time, uniformly in ε

$$D^k f(t,x,v,z) := \partial_z^k f(t,x,v,z)$$

Theorem 4.1 (Uniform regularity). *Assume*

$$\sigma(z) \geq \sigma_{\min} > 0\,.$$

If for some integer $m \geq 0$,

$$\|D^k \sigma(z)\|_{L^\infty} \leq C_\sigma, \qquad \|D^k f_0\|_{\Gamma(0)} \leq C_0, \qquad k = 0, \ldots, m,$$

then

$$\|D^k f\|_{\Gamma(t)} \leq C, \qquad k = 0, \cdots, m, \qquad \forall t > 0,$$

where C_σ, C_0 *and* C *are constants independent of* ε.

- A good problem to use the gPC-SG for UQ

Key estimates

Energy estimate: We will establish the following energy estimate by using Mathematical Induction with respect to k: for any $k \geq 0$, there exist k constants $c_{kj} > 0$, $j = 0, \ldots, k-1$ such that

$$\varepsilon^2 \partial_t \Big(\|D^k f\|^2_{\Gamma(t)} + \sum_{j=0}^{k-1} c_{kj} \|D^j f\|^2_{\Gamma(t)} \Big) \leq \begin{cases} -2\sigma_{\min} \big\| [f] - f \big\|^2_{\Gamma(t)}, & k = 0, \\ -\sigma_{\min} \big\| D^k([f] - f) \big\|^2_{\Gamma(t)}, & k \geq 1. \end{cases}$$

Theorem 4.2 **(Estimate on $[\boldsymbol{f}] - \boldsymbol{f}$).** *With all the assumptions in Theorem 4.1 and Lemma 4.2, for a given time $T > 0$, the following regularity result of $[f] - f$ holds:*

$$\begin{aligned} &\|D^k([f] - f)\|^2_{\Gamma(t)} \\ &\leq \mathrm{e}^{-\sigma_{\min} t / 2\varepsilon^2} \|D^k([f_0] - f_0)\|^2_{\Gamma(0)} + C'(T)\varepsilon^2 \\ &\leq C(T)\varepsilon^2, \end{aligned} \tag{54}$$

for any $t \in (0, T]$ and $0 \leq k \leq m,$, where $C'(T)$ and $C(T)$ are constants depending on T.

uniform spectral convergence (sAP)

Theorem 4.3 (Uniformly convergence in ε). *Assume*

$$\sigma(z) \geq \sigma_{\min} > 0 \,.$$

If for some integer $m \geq 0$,

$$\|\sigma(z)\|_{H^k} \leq C_\sigma, \quad \|D^k f_0\|_{\Gamma(0)} \leq C_0, \quad \|D^k(\partial_x f_0)\|_\omega \leq C_x, \quad k = 0, \ldots, m, \tag{82}$$

Then the error of the whole gPC-SG method is

$$\|f - f_N\|_{\Gamma(t)} \leq \frac{C(T)}{N^k}, \tag{83}$$

where $C(T)$ is a constant independent of ε.

Uniform stability

- For a fully discrete scheme based on the deterministic micro-macro decomposition (f=M + g) based approach (Klar-Schmeiser, Lemou-Mieusseun) approach, we can also prove the following uniform stability:

$$\Delta t \leq \frac{\sigma_{\min}}{3}\Delta x^2 + \frac{2\varepsilon}{3}\Delta x,$$

Numerical tests

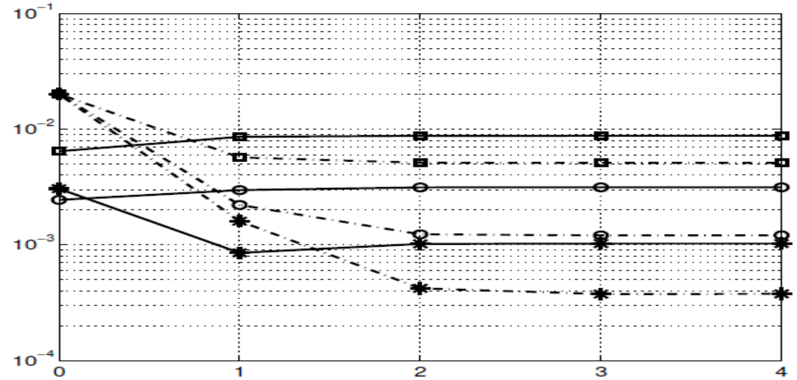


FIG. 8.13. *The linear transport equation: Errors of the mean (solid line) and standard deviation (dash line) of $\overline{r}$ (circle) with respect to the gPC order at $\epsilon = 10^{-8}$: $\Delta x = 0.04$ (squares), $\Delta x = 0.02$ (circles), $\Delta = 0.01$ (stars).*

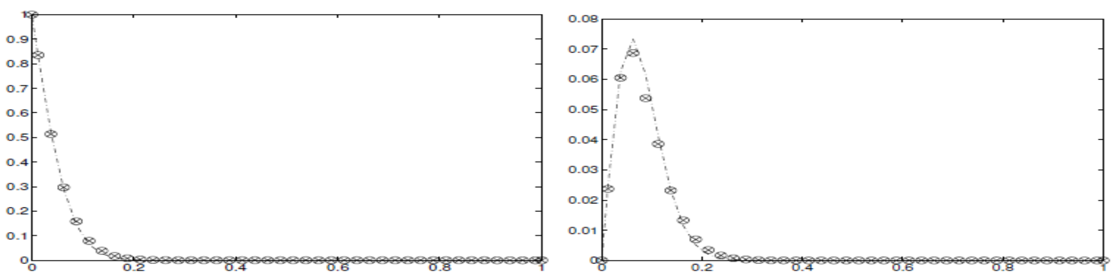


FIG. 8.14. *The linear transport equation: The mean (left) and standard deviation (right) of $\overline{r}$ at $\epsilon = 10^{-8}$, obtained by the gPC Galerkin at order $N = 4$ (circles), the stochastic collocation method (crosses), and the limiting analytical solution (8.6).*

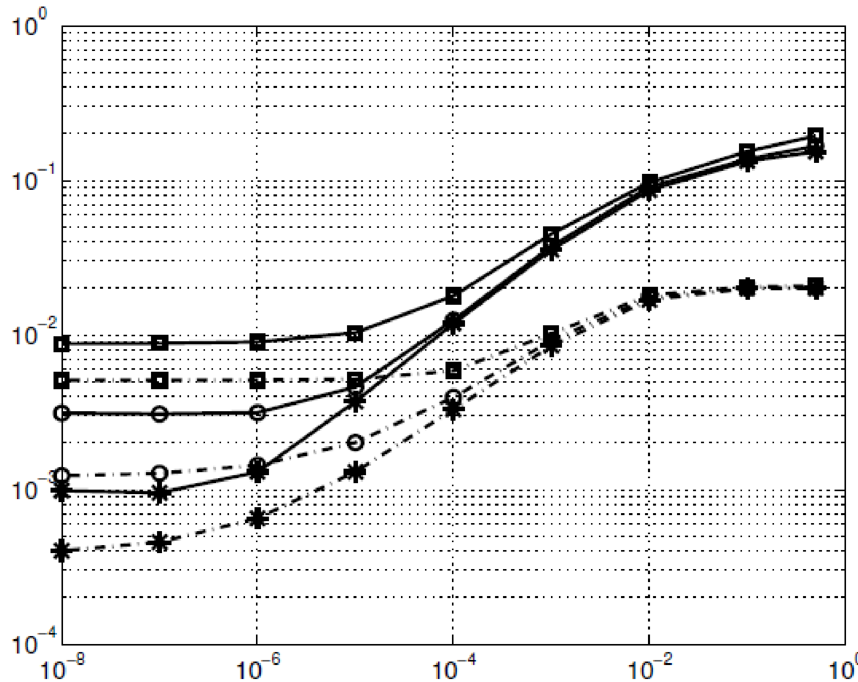


Fig. 8.15. *The linear transport equation: Differences in the mean (solid line) and standard deviation (dash line) of* $\overline{r}$ *with respect to* ϵ^2*, between the limiting analytical solution* (8.6) *and the 4th-order gPC solution with* $\Delta x = 0.04$ *(squares),* $\Delta x = 0.02$ *(circles) and* $\Delta x = 0.01$ *(stars).*

Nonlinear collisional kinetic equations (*Liu Liu-J*, SIAM MMS '18)

One can extend hypocoercivity theory developed by Villani, Guo, Mouhut, Briant, etc. in velocity space for deterministic problems to study the following properties in random space:

regularity, sensitivity in random parameter, long-time behavior (exponential decay to global equilibrium, spectral convergence and exponential decay of numerical error for gPC-SG

linear kinetic equation: Qin Li & Li Wang (SIAM/ASA J. UQ '18)

Hypocoercivity-a toy model (the linear Fokker-Planck equation)

$$\partial_t f + \varepsilon^a v \partial_x f = \varepsilon^b \partial_{vv} f$$

$(x, v) \in \mathbb{T}^2$, $\mathbb{T}^2$ is a two-dimensional periodic box

suppose $\int_{\mathbb{T}} f \, dv \, dx = 0$.

- Basic energy estimates:

$$\frac{1}{2}\partial_t \|f\|^2 + \varepsilon^b \|\partial_v f\|^2 = 0 \,,$$
$$\frac{1}{2}\partial_t \|\partial_v f\|^2 + \varepsilon^b \|\partial_{vv} f\|^2 = -\varepsilon^a \langle \partial_x f, \partial_v f \rangle \,,$$
$$\frac{1}{2}\partial_t \|\partial_x f\|^2 + \varepsilon^b \|\partial_{xv} f\|^2 = 0 \,,$$

- No dissipation in x
- Key idea: consider the mixed derivative terms:

$$\partial_t \langle \partial_x f, \partial_v f \rangle + \varepsilon^a \|\partial_x f\|^2 = -2\varepsilon^b \langle \partial_{xv} f, \partial_{vv} f \rangle$$

Herou-Nier (04), Villani (09)

- Introduce a Lyapunov functional

$$|||f|||^2 = \|f\|^2 + \gamma_1 \varepsilon_1^{\beta} \min\left(1, \frac{t}{\varepsilon^b}\right) \|\partial_v f\|^2 + \gamma_2 \varepsilon_2^{\beta} \min\left(1, \frac{t}{\varepsilon^b}\right)^3 \|\partial_x f\|^2$$
$$+ 2\gamma_3 \varepsilon_3^{\beta} \min\left(1, \frac{t}{\varepsilon^b}\right)^2 \langle \partial_x f, \partial_v f \rangle .$$

With appropriate choices of constants $\gamma_1, \gamma_2, \gamma_3$ and powers $\beta_1, \beta_2, \beta_3, a, b$, we get

$$|||f(t)|||^2 + \theta \int_s^t D_u(f(u))\, du \le |||f(s)|||^2 ,$$

for $\theta > 0$, and D_u are positive dissipation terms. By using Poincare inequality

$$|||f(t)||| \le \|f(0)\| e^{-\kappa(\varepsilon)t} \qquad \text{for some } \kappa(\varepsilon) > 0.$$

Nonlinear collisional kinetic euqations

$$\begin{cases} \partial_t f + \frac{1}{\epsilon^\alpha} v \cdot \nabla_x f = \frac{1}{\epsilon^{1+\alpha}} \mathcal{Q}(f), \\ f(0,x,v,z) = f_{in}(x,v,z), \qquad x \in \Omega \subset \mathbb{T}^d, \ v \in \mathbb{R}^d, z \in I_z \subset \mathbb{R}. \end{cases}$$

perturbative setting $\quad f = \mathcal{M} + \epsilon M h$

(avoid compressible Euler limit, thus shocks):

Global Maxwellian $\quad \mathcal{M} = \frac{1}{(2\pi)^{\frac{d}{2}}} e^{-\frac{|v|^2}{2}} \qquad M = \sqrt{\mathcal{M}}$

Euler (acoustic scaling)

$$\partial_t h + v \cdot \nabla_x h = \frac{1}{\epsilon} \mathcal{L}(h) + \mathcal{F}(h,h)$$

(incompressible) Navier-Stokes scaling

$$\partial_t h + \frac{1}{\epsilon} v \cdot \nabla_x h = \frac{1}{\epsilon^2} \mathcal{L}(h) + \frac{1}{\epsilon} \mathcal{F}(h,h)$$

Why it works: hypocoercivity decay of the linaar part dominates the bounded (weaker) nonlinear part

hypocoercivity

$$\langle h, \mathcal{L}(h)\rangle_{L^2_v} \leq -\lambda\, ||h^\perp||_{\Lambda^2_v}$$

$$h^\perp = h - \Pi_{\mathcal{L}}(h)$$

$\Pi_{\mathcal{L}}(h)$ is the orthogonal projection in L^2_v on $N(\mathcal{L})$

$$||h||_{\Lambda_v} = ||h(1+|v|)^{\gamma/2}||_{L^2}$$

$$||\cdot||_{\Lambda} := ||\, ||\cdot||_{\Lambda_v}\, ||_{L^2_x}.$$

Boundedness of the nonlinear term

$$\left|\langle \partial^m \partial_l^j \mathcal{F}(h,h),\ f\rangle_{L^2_{x,v}}\right| \leq \begin{cases} \mathcal{G}^{s,m}_{x,v,z}(h,h)\,||f||_\Lambda\,, & \text{if } j \neq 0, \\ \mathcal{G}^{s,m}_{x,z}(h,h)\,||f||_\Lambda\,, & \text{if } j = 0. \end{cases}$$

there exists a z-independent $C_{\mathcal{F}} > 0$ such that

$$\sum_{|m|\leq r} (\mathcal{G}^{s,m}_{x,v,z}(h,h))^2 \leq C_{\mathcal{F}}\,||h||^2_{H^{s,r}_{x,v}}\,||h||^2_{H^{s,r}_\Lambda},$$

$$\sum_{|m|\leq r} (\mathcal{G}^{s,m}_{x,z}(h,h))^2 \leq C_{\mathcal{F}}\,||h||^2_{H^{s,r}_x L^2_v}\,||h||^2_{H^{s,r}_\Lambda}\,.$$

$$||h||^2_{H^{s,r}_{x,v}} = \sum_{|m|\leq r} ||\partial^m h||^2_{H^s_{x,v}}\,, \qquad ||h||^2_{H^{s,r}_\Lambda} = \sum_{|m|\leq r} ||\partial^m h||^2_{H^s_\Lambda}$$

$$||h||^2_{H^{s,r}_x L^2_v} = \sum_{|m|\leq r} ||\partial^m h||^2_{H^s_x L^2_v} \qquad ||h||^2_{H^{s,r}_\Lambda} = \sum_{|m|\leq r} ||\partial^m h||^2_{H^s_\Lambda}$$

$$||h||^2_{H^s_\Lambda} = \sum_{|j|+|l|\leq s} ||\partial_l^j h||^2_\Lambda$$

Convergence to global equilibrium (random initial data)

$$Assume\ ||h_{in}||_{H^s_{x,v}L^\infty_z} \leq C_I,\ then$$

- For incompressible N-S scaling:

$$||h_\epsilon||_{H^{s,r}_{x,v}L^\infty_z} \leq C_I\, e^{-\tau_s t}, \qquad ||h_\epsilon||_{H^s_{x,v}H^r_z} \leq C_I\, e^{-\tau_s t}.$$

- For Euler (acoustic) scaling:

$$||h_\epsilon||_{\mathcal{H}^{s,r}_\epsilon L^\infty_z} \leq \delta_s\, e^{-\epsilon\,\tau_s t}, \qquad ||h_\epsilon||_{\mathcal{H}^s_\epsilon H^r_z} \leq \delta_s\, e^{-\epsilon\,\tau_s t}$$

Random collision kernel

$$B(|v-v_*|, \cos\theta, z) = \phi(|v-v_*|)\, b(\cos\theta, z), \qquad \phi(\xi) = C_\phi\, \xi^\gamma, \text{ with } \gamma \in [0,1],$$

$$\forall \eta \in [-1,1], \qquad |b(\eta,z)| \le C_b,\ |\partial_\eta b(\eta,z)| \le C_b, \text{ and } |\partial_z^k b(\eta,z)| \le C_b^*, \forall\, 0 \le k \le r\,.$$

- Need to use a weighted Sobolev norm in random space as in Jin-Ma-J.G. Liu

$$||g||_{L^{2,r*}_{x,v}} := \sum_{m=0}^{r} \widetilde{C}_{m,r+1}\, ||\partial^m g||_{L^2_{x,v}},$$

- Similar decay rates can be obtained

gPC-SG approximation

$$f(t,x,v,z) \approx \sum_{|\mathbf{k}|=1}^{K} f_{\mathbf{k}}(t,x,v)\psi_{\mathbf{k}}(z) := f^K(t,x,v,z),$$

$$h(t,x,v,z) \approx \sum_{|\mathbf{k}|=1}^{K} h_{\mathbf{k}}(t,x,v)\psi_{\mathbf{k}}(z) := h^K(t,x,v,z).$$

- Perturbative setting

$$f_{\mathbf{k}} = \mathcal{M} + \epsilon M h_{\mathbf{k}}$$

$$\begin{cases} \partial_t h_{\mathbf{k}} + \dfrac{1}{\epsilon} v \cdot \nabla_x h_{\mathbf{k}} = \dfrac{1}{\epsilon^2}\mathcal{L}_{\mathbf{k}}(h^K) + \dfrac{1}{\epsilon}\mathcal{F}_{\mathbf{k}}(h^K, h^K), \\ h_{\mathbf{k}}(0,x,v) = h^0_{\mathbf{k}}(x,v), \qquad x \in \Omega \subset \mathbb{T}^d,\ v \in \mathbb{R}^d, \end{cases}$$

- Assumptions: z bounded

$$|\partial_z b| \leq O(\epsilon)$$

(following R. Shu-Jin) $||\psi_k||_{L^\infty} \leq C k^p, \qquad \forall\, k,$

Let $q > p + 2$, *define the energy* E^K *by*

$$E^K(t) = E^K_{s,q}(t) = \sum_{k=1}^{K} ||k^q h_k||^2_{H^s_{x,v}},$$

Regularity and exponential decay

(i) Under the incompressible Navier-Stokes scaling,

$$E^K(t) \le \eta\, e^{-\tau t} \qquad ||h^K||_{H^s_{x,v}L^\infty_z} \le \eta\, e^{-\tau t}$$

(ii) Under the acoustic scaling,

$$E^K(t) \le \eta\, e^{-\epsilon\tau t}, \qquad ||h^K||_{H^s_{x,v}L^\infty_z} \le \eta\, e^{-\epsilon\,\tau t}$$

gPC-SG error

Theorem 5.3. *Suppose the assumptions on the collision kernel and basis functions in Theorem 5.1 are satisfied, then*

(i) Under the incompressible Navier-Stokes scaling,

$$||h^e||_{H_z^s} \le C_e \frac{e^{-\lambda t}}{K^r}, \tag{5.22}$$

(ii) Under the acoustic scaling,

$$||h^e||_{H_z^s} \le C_e \frac{e^{-\epsilon\lambda t}}{K^r}, \tag{5.23}$$

with the constants C_e, $\lambda > 0$ *independent of* K *and* ϵ.

$$||h(x,v,\cdot)||^2_{H_z^s} = \int_{I_z} ||h||^2_{H^s_{x,v}} \, \pi(z)\, dz$$

A general framework

- This framework works for general linear and nonlinear collisional kinetic equations

- Linear and nonlinear Boltzmann, Landau, relaxation-type quantum Boltzmann, etc.

gPC-SG for many different kinetic equations

- Boltzmann: a fast algorithm for collision operator (J. Hu-Jin, JCP '16), sparse grid for high dimensional random space (J. Hu-Jin-R. Shu '16): initial regularity in the random space is preserved in time; but not clear whether it is unifornly stable in the compressible Euler limit (s-AP?): gPC-SG for nonlinear hyperbolic system is not globally hyperbolic! (APUQ is open)
- Landau equation (J. Hu-Jin-R. Shu, '16)
- Radiative heat transfer (APUQ OK: Jin-H. Lu JCP'17): proof of regularity in random space for linearized problem (nonlinear? Open)
- Kinetic-incompresssible fluid couple models for disperse two phase flow: (efficient algorithm in multi-D: Jin-R. Shu. Theory open)
- Vlasov-Poisson system (Landau Damping) : R. Shu-J

conclusion

- gPC-SG allows us to treat kinetic equations with random inputs in the deterministic AP framework
- Many different kinetic equations can be solved this way;
- Hypocoercivity based regularity and sensitivity analysis can be done for general linear and nonlinear collision kinetic equations and Vlasov-Poisson-Fokker-Planck system, which imply (uniform) spectral convergence of gPC methods
- Kinetic equations have the good regularity in the random space, even for the nonlinear kinetic equation: good problem for UQ!
- Many kinetic ideas useful for UQ problems: mean-field approximations; moment closure; etc. （APUQ is one example）
- Many open questions, very few existing works
- Kinetic equations are good problems for UQ;

** UQ + Multiscale **