Data science with multilayer networks: Mathematical foundations and applications

CDSE Days
University at Buffalo, State University of New York
Monday April 9, 2018

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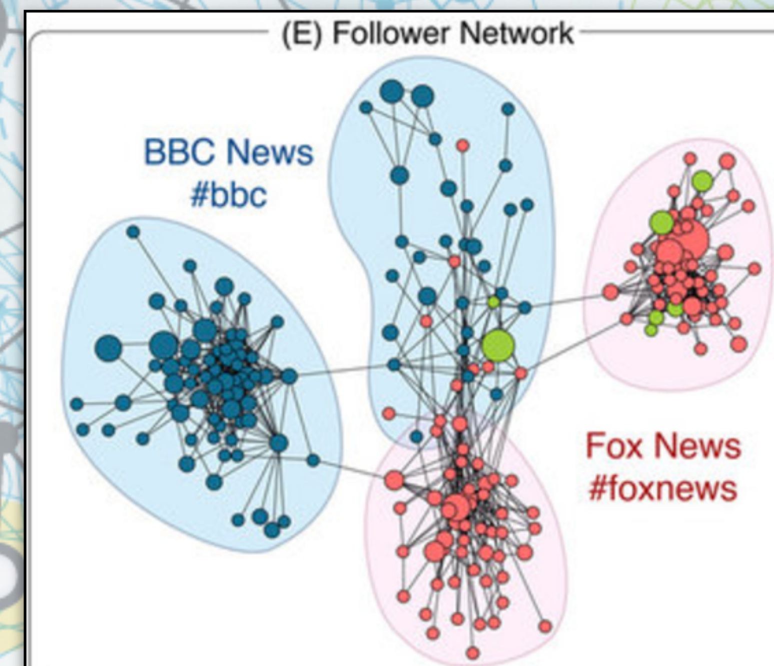
Large-scale networks arise in numerous research fields to describe social networks, biological systems, technologies, economics, …

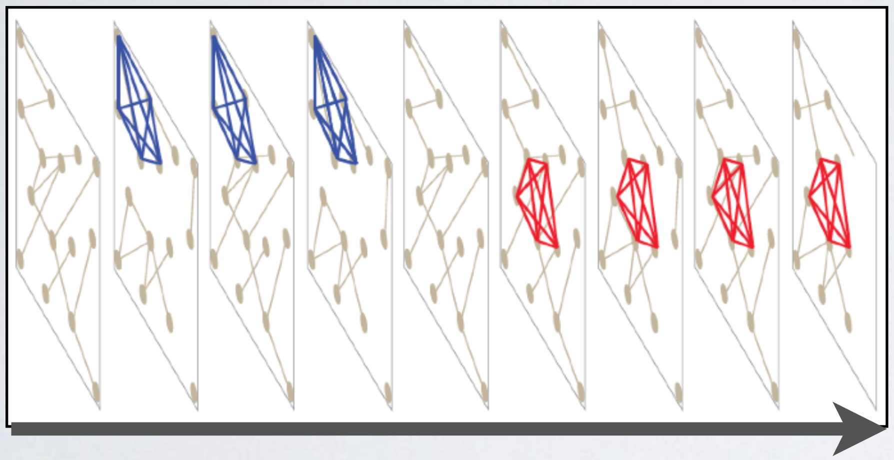


There are myriad techniques for extracting insights from structural patterns

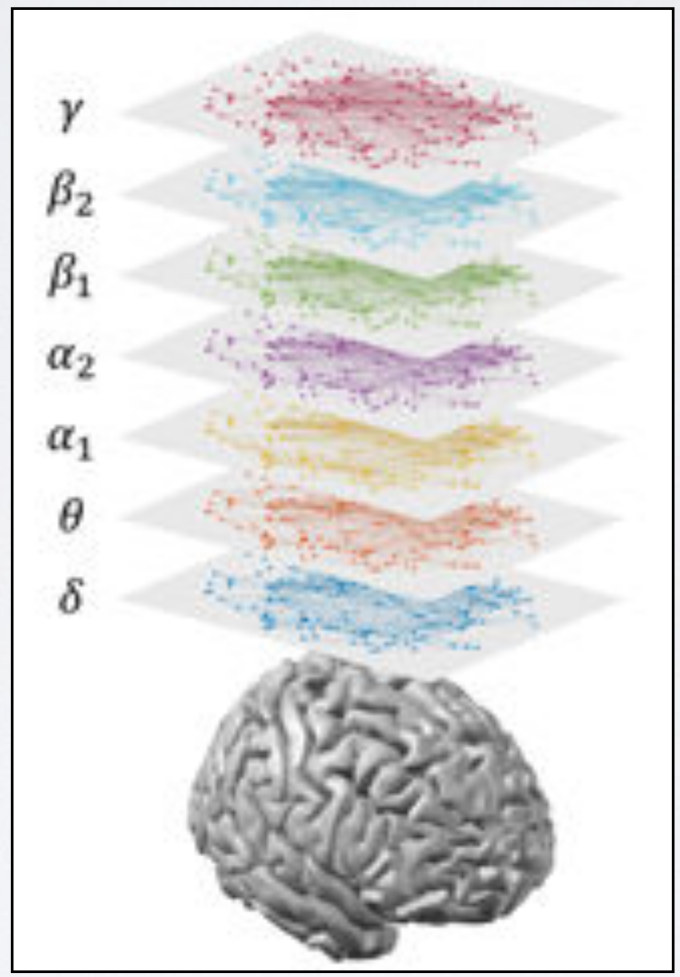
MULTILAYER NETWORKS

- A more comprehensive modeling framework

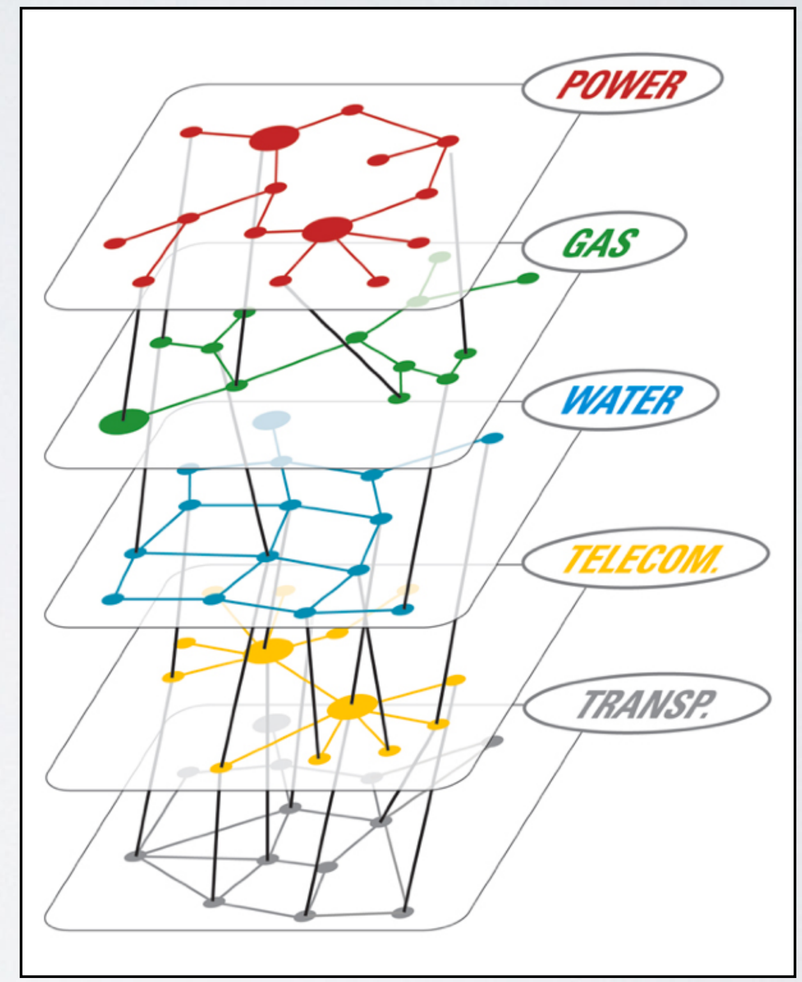
time-varying networks



complementary data



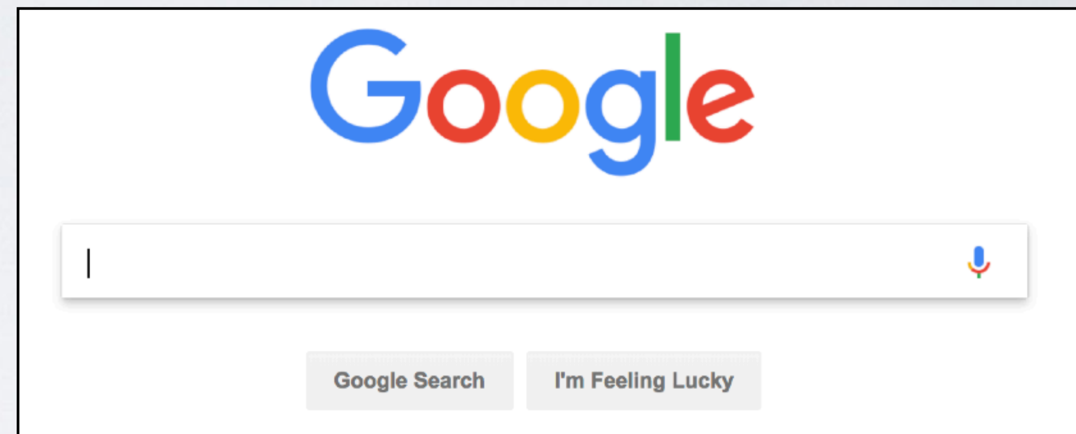
interdependent systems

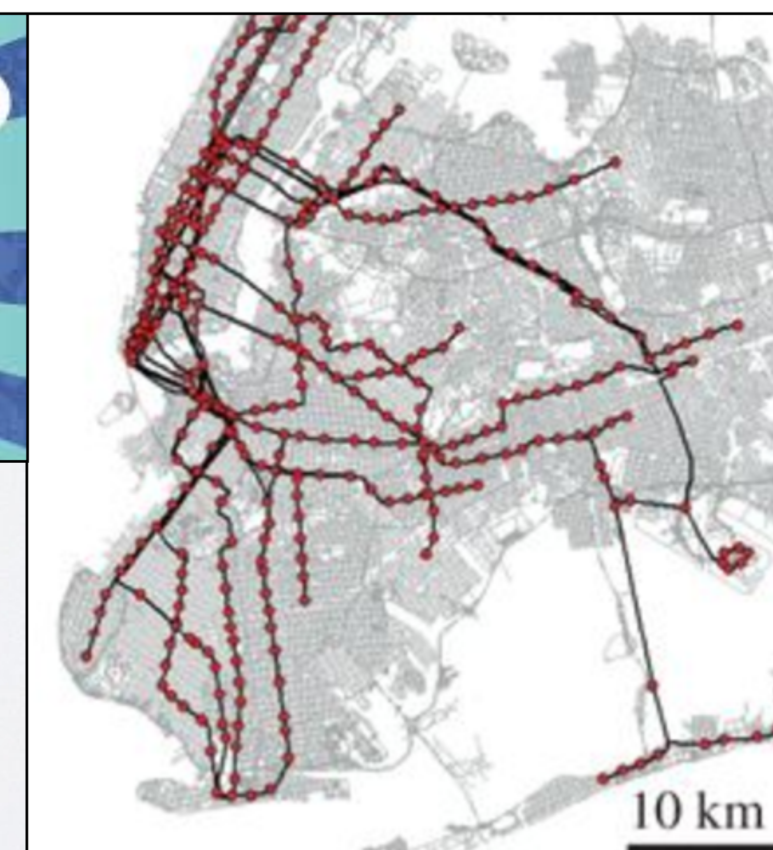


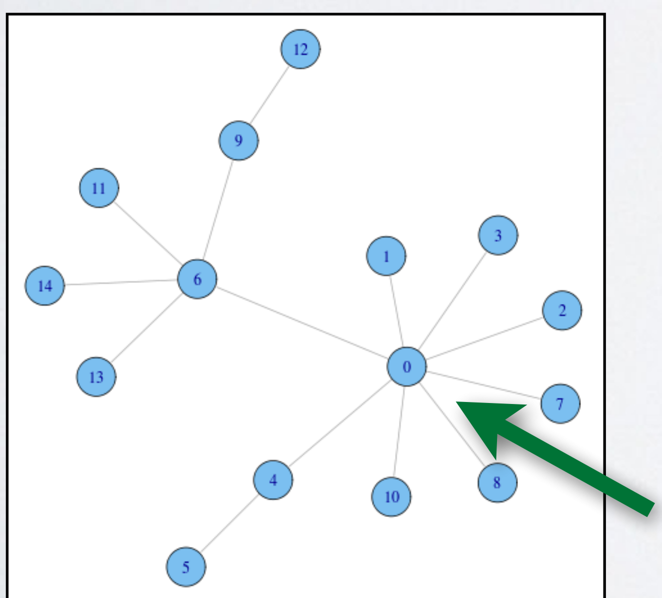
- See reviews
  - Kivelä et al. (2014) Multilayer networks. J. of Complex Networks 2(3), 203-271.
  - Boccaletti et al. (2014) The structure and dynamics of multilayer networks. Physics Reports 544(1), 1-122.

CENTRALITY AND RANKING

- **Centrality Analysis -** ranking nodes according to their importances
  - Google PageRank for web search
  - Identifying influential persons
  - Points of fragility in complex systems
  - Ranking universities, academics, etc.
  - Drug targets in biological networks







most important

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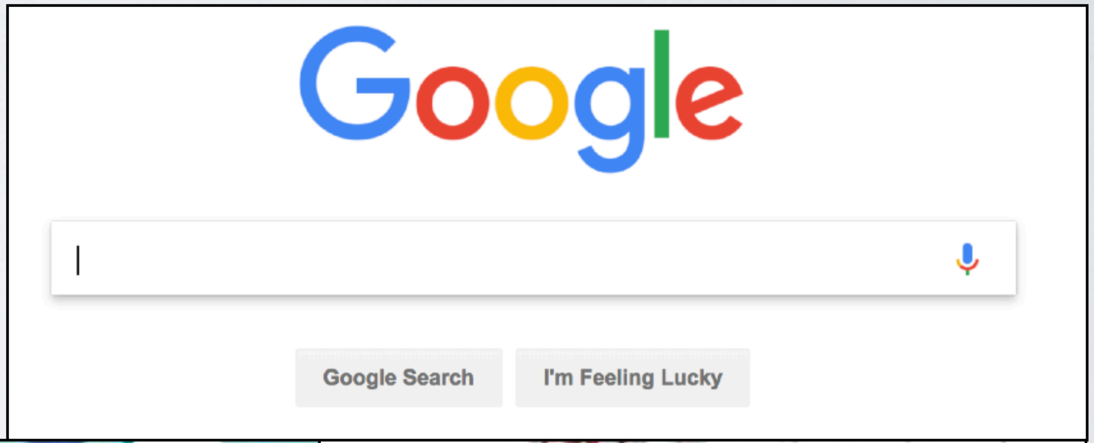

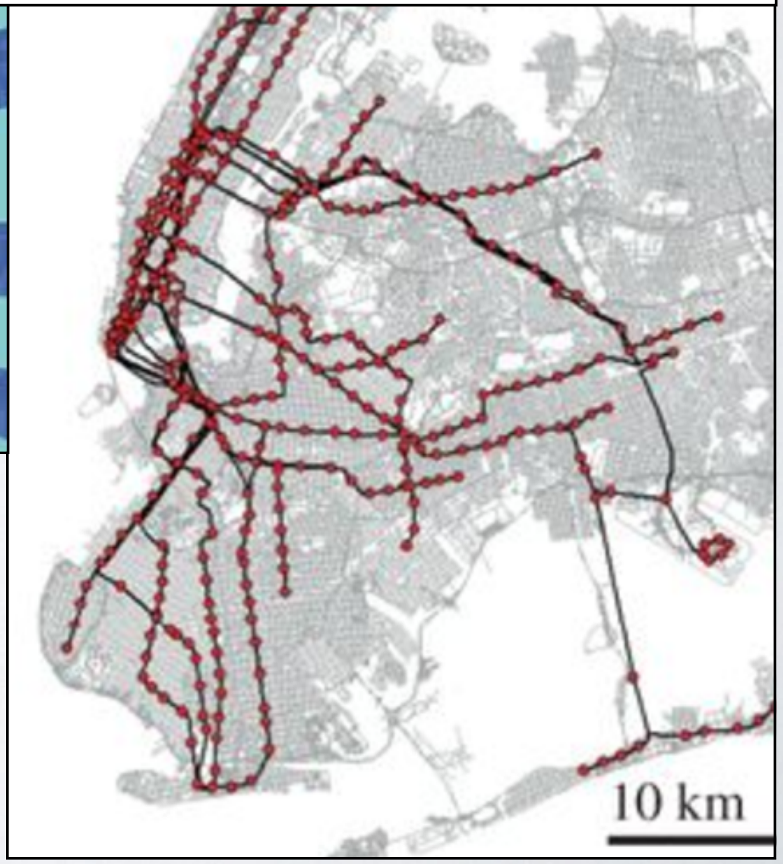

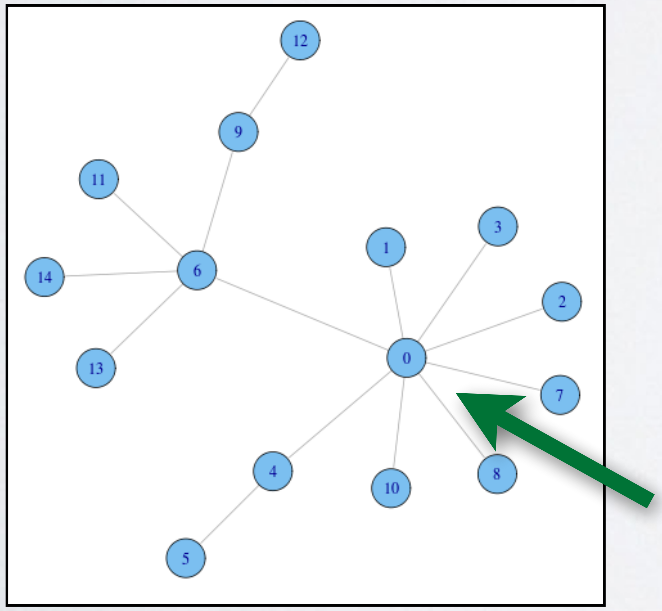

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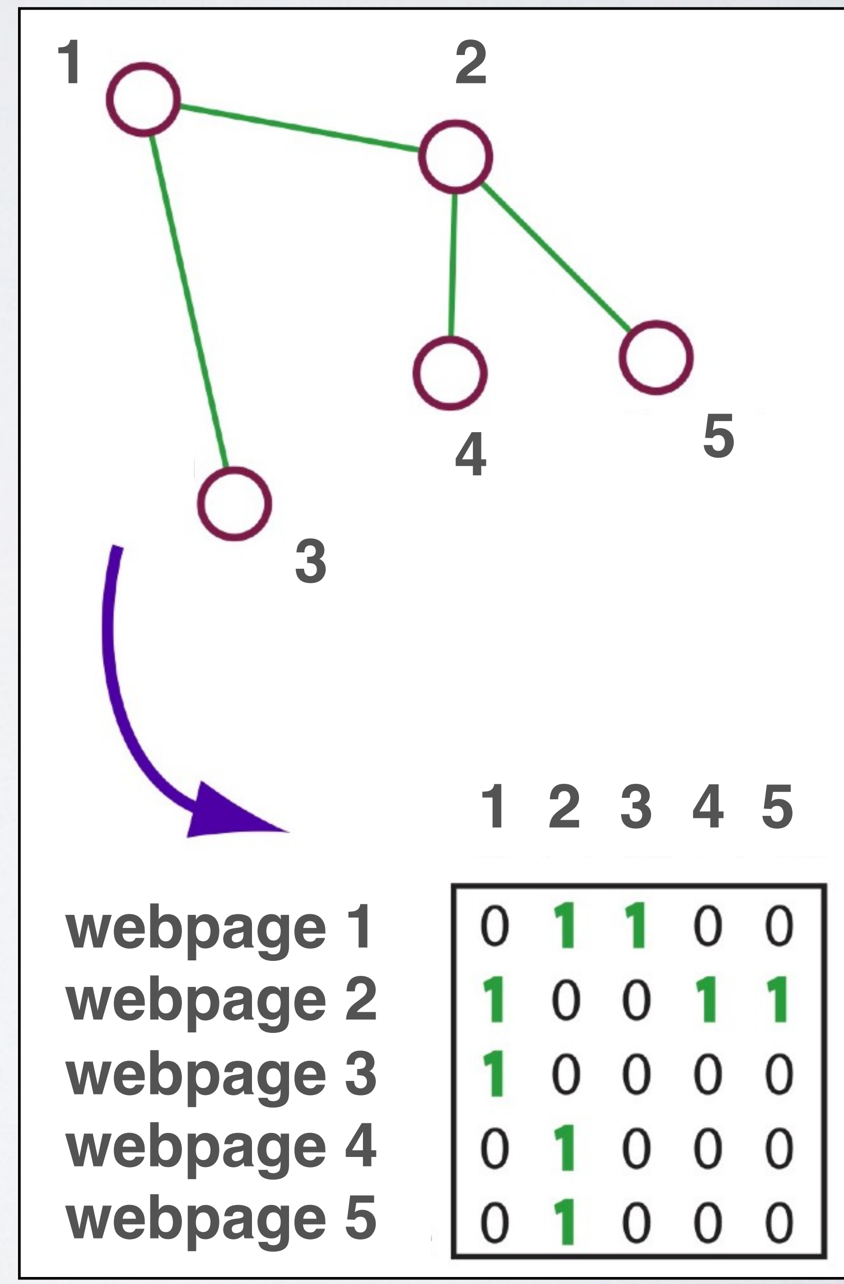
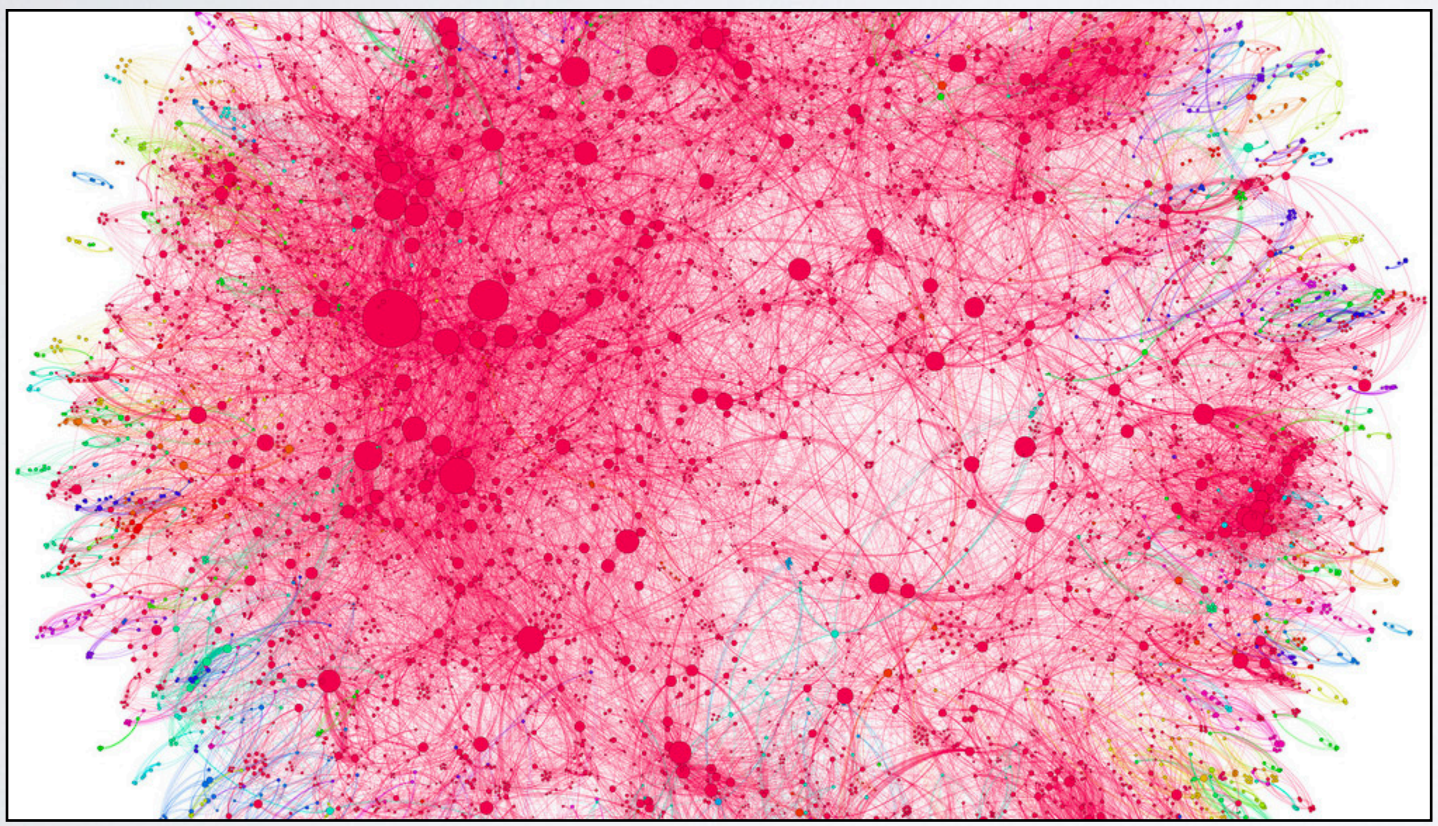
GOOGLE'S PAGERANK ALGORITHM

- Adjacency matrix A_{ij} - encodes network

example web graph



massive web graph



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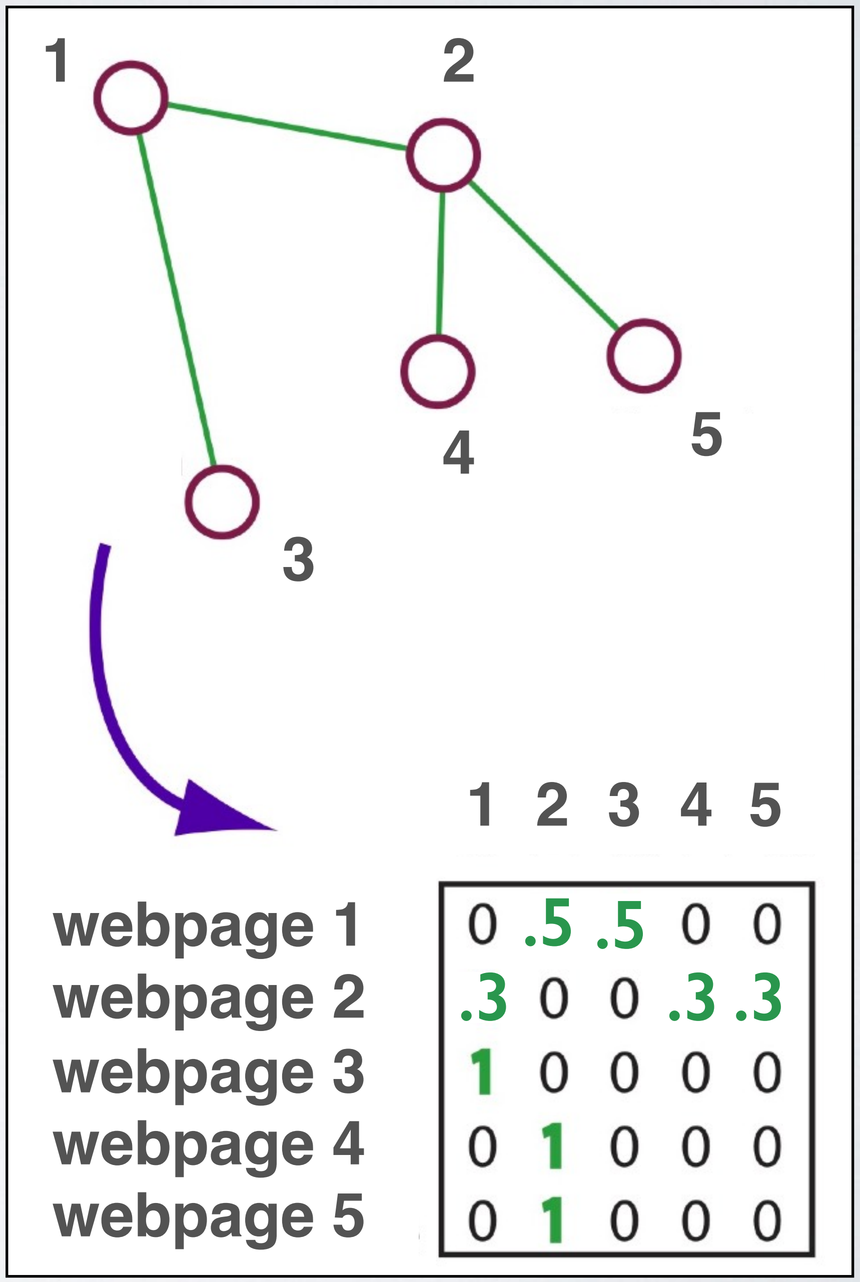

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$$C_{ij}^{(PR)} = (1-\alpha)P_{ij} + \alpha$$

$$\alpha = 0.15$$

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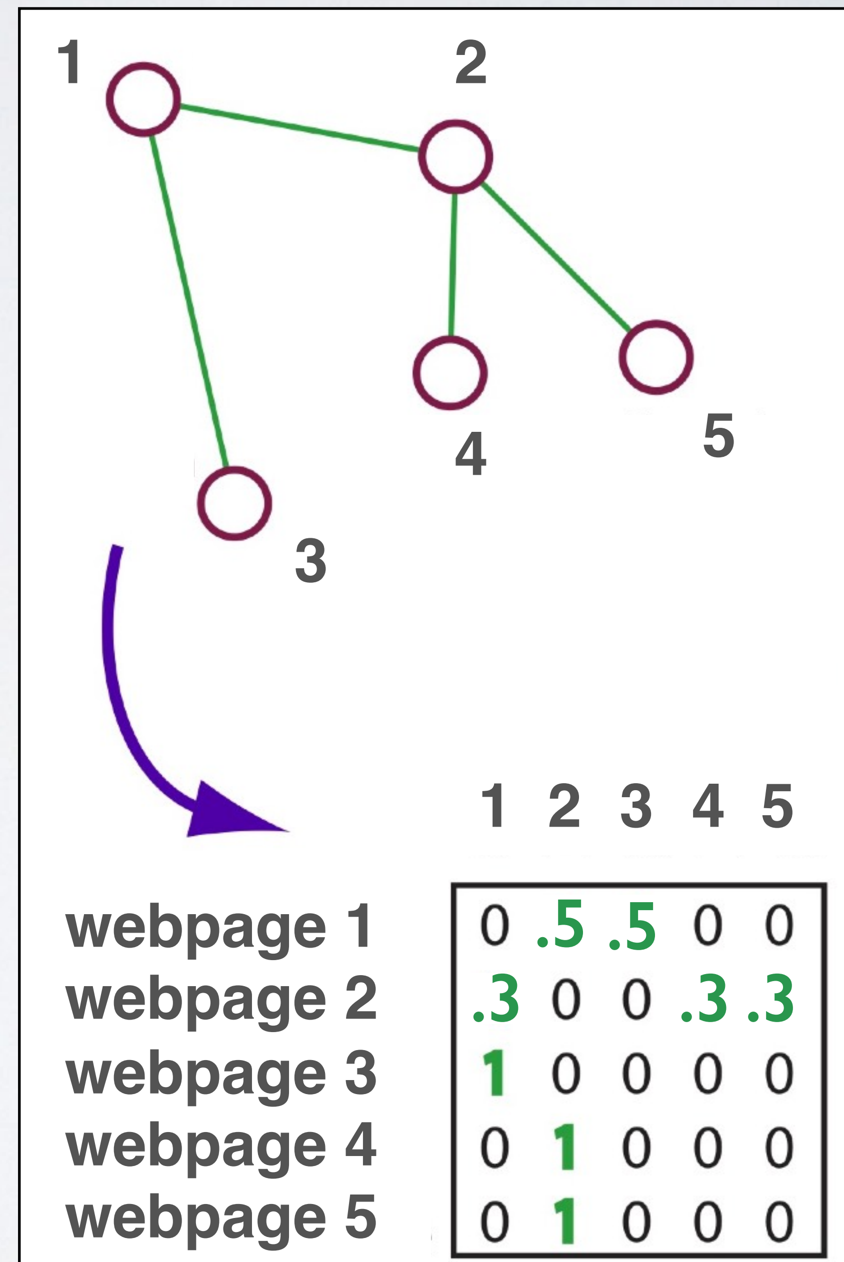

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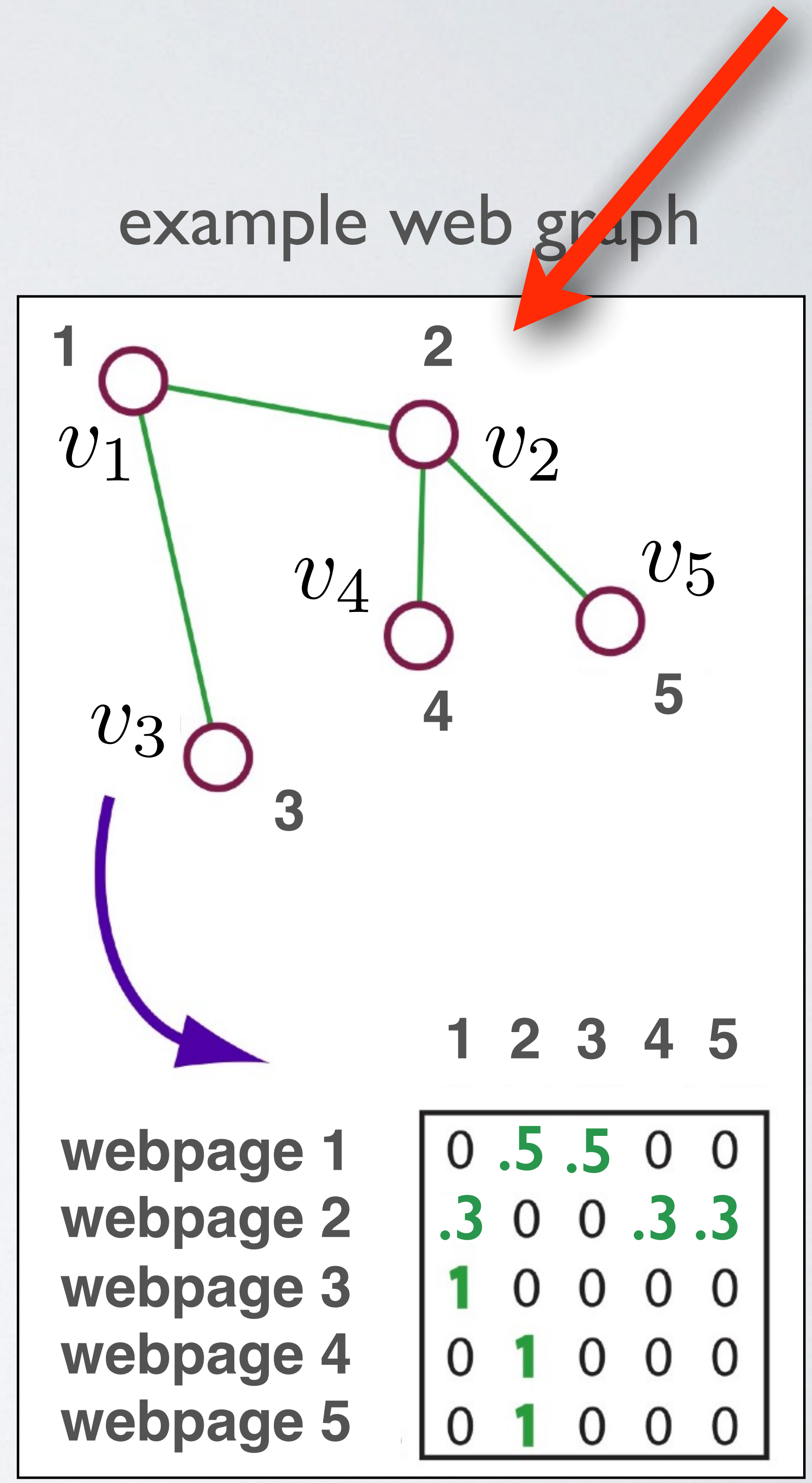
- PageRank vector $\mathbf{v}$ solves

$$C^{(PR)}\mathbf{v} = \lambda_{max}\mathbf{v}$$

- $\{v_i\}$ gives stationary distribution of random web surfers

example web graph



EIGENVECTOR-BASED CENTRALITIES

- Node rankings are indicated by a centrality score, which is computed as the dominant eigenvector of a "centrality matrix" C

$$C\mathbf{v} = \lambda_{max}\mathbf{v}$$

- Examples
  - Google's PageRank centrality -Brin & Page, 1998

$$C^{(PR)} = (1-\alpha)P + \alpha$$

  - Hub and Authority scores for directed networks -Kleinberg, 1999

$$C^{(hub)} = A^T A$$

$$C^{(auth)} = AA^T$$

  - Eigenvector centrality for undirected networks

$$C^{(evec)} = A$$

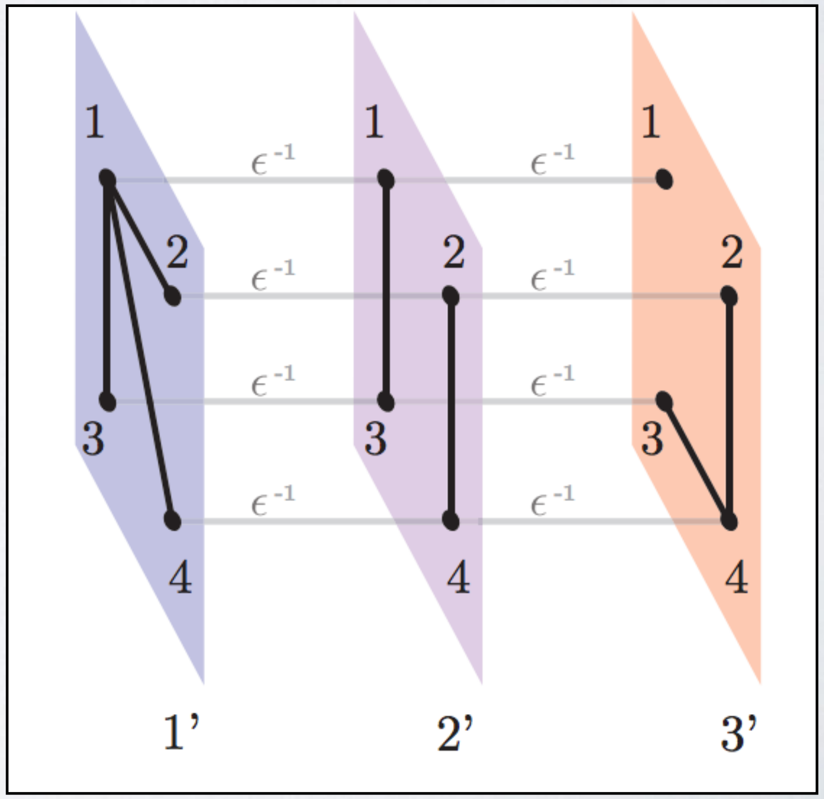
SUPRA-CENTRALITY MATRIX

- We introduce "supra-centrality" matrices as a temporal generalization of centrality
  - Consider T centrality matrices $\mathbf{C}^{(t)}$ for T time layers
  - We place them as diagonal blocks in a matrix and couple them together with inter-layer "identity" edges of weight ϵ^{-1}

$$\mathbb{C}(\epsilon) = \begin{bmatrix} \mathbf{C}^{(1)} & \epsilon^{-1}\mathbf{I} & 0 & \cdots \\ \epsilon^{-1}\mathbf{I} & \mathbf{C}^{(2)} & \epsilon^{-1}\mathbf{I} & \ddots \\ 0 & \epsilon^{-1}\mathbf{I} & \mathbf{C}^{(3)} & \ddots \\ \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

$$\mathbb{C}(\epsilon)\mathbf{v}(\epsilon) = \lambda_{\max}\mathbf{v}(\epsilon)$$

coupled time layers



JOINT, *MARGINAL* AND *CONDITIONAL* CENTRALITY

- We introduce a vocabulary for centrality motivated by statistics

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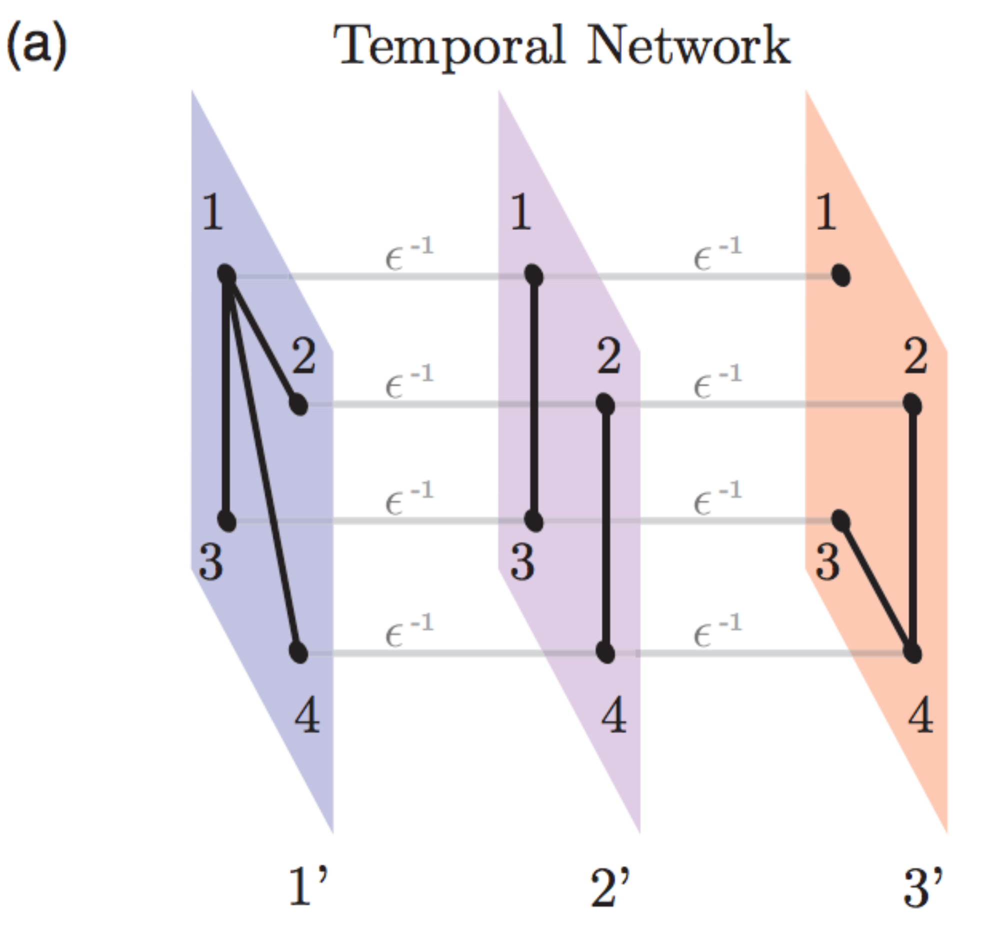

(b) Centralities for $\epsilon = 0.5$

Time Layer

Node Index	1'	2'	3'	MNC
1	0.3570	0.3392	0.1679	0.8641
2	0.2516	0.3298	0.2357	0.8171
3	0.2471	0.3207	0.2312	0.7990
4	0.2655	0.3579	0.2927	0.9162
MLC	1.1213	1.3476	0.9275	

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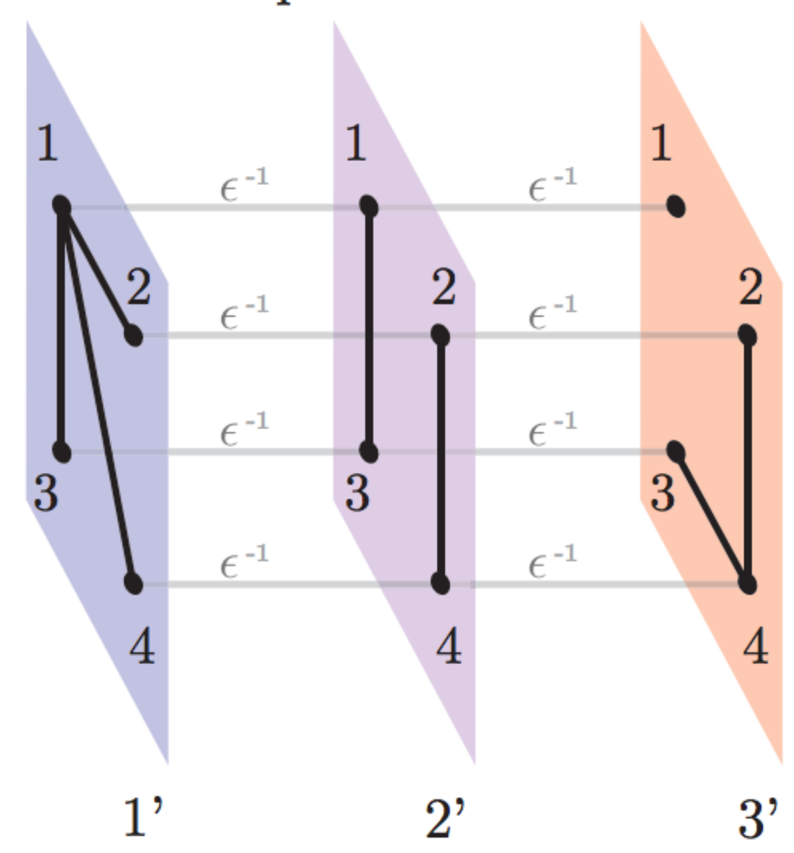
(a) Temporal Network



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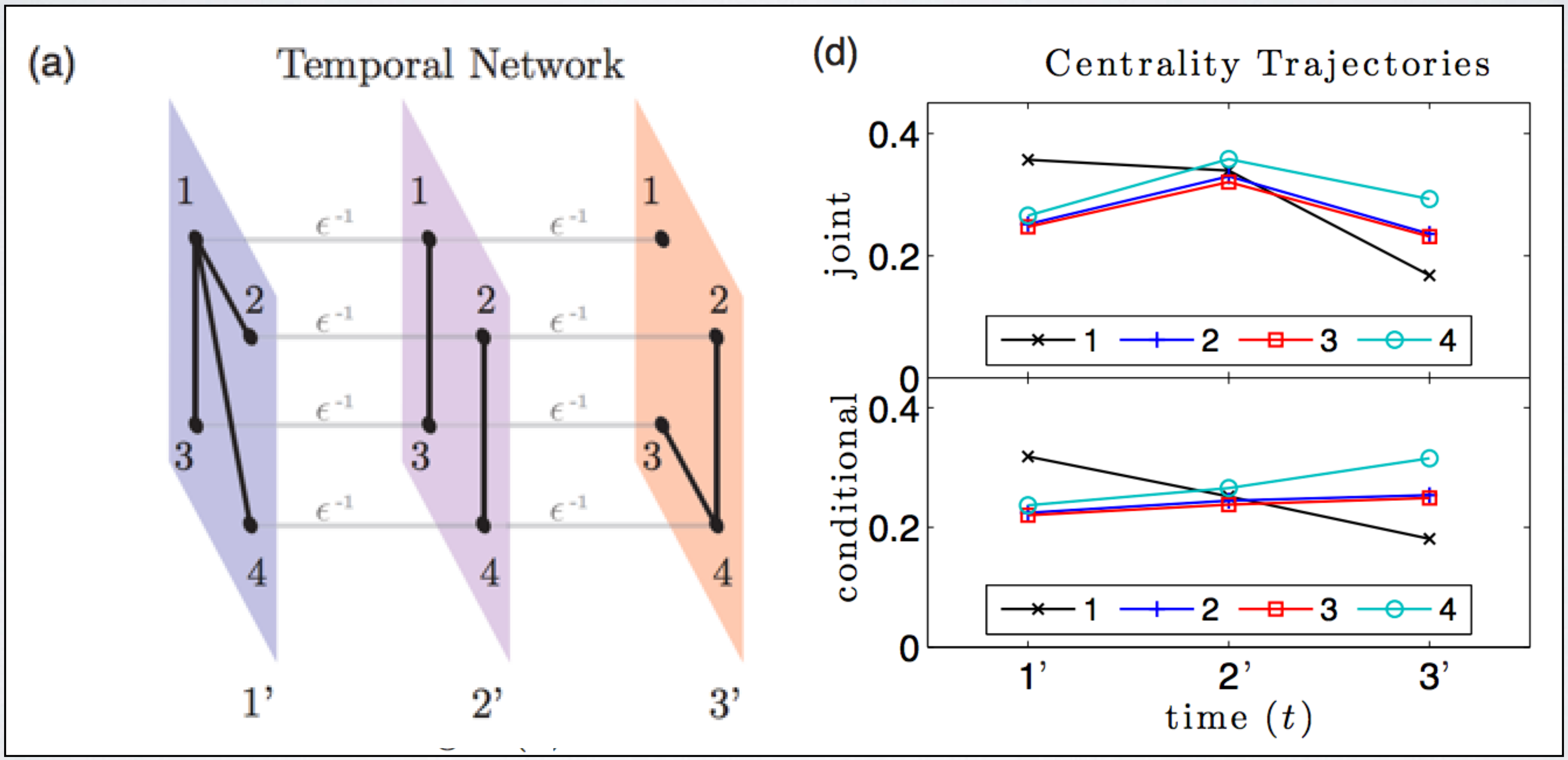
$$\frac{0.2516}{1.1213} = 0.2244$$

CENTRALITY TRAJECTORIES

- Conditional centrality reveals how the importances of nodes change with time



STRONG-COUPLING LIMIT: TIME-AVERAGED CENTRALITY

- We analyze the strong coupling limit $\epsilon \to 0^+$ which yields layer aggregation
- We conduct a singular perturbation analysis for $\mathbb{C}(\epsilon)\mathbf{v}(\epsilon) = \lambda_{\max}\mathbf{v}(\epsilon)$

$$\mathbf{v}(\epsilon) = \mathbf{v}_0 + \epsilon\mathbf{v}_1 + \epsilon^2\mathbf{v}_2 + \dots$$
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- This yields two layer-aggregated centralities:
  - Time-averaged centrality $\{\alpha_i\}$, which describe the conditional centralities when they become constant across the layers
  - First-order mover scores $\{m_i\}$, which measure the extent of centrality change across the layers

CASE STUDY 1:

- We analyzed data from the Internet Movie Database (IMDb) to study the *golden age of hollywood,* 1920–1960,
- We consider 55 actors (26 male and 29 female) during the time period 1909–2009
- An edge from i to j in layer t indicates the number of movies in which i and j co-star and actor j is listed first in the movies *billing order*

CASE STUDY I:

Top Time-Averaged Centralities

Rank	Actor	α_i	m_i
1	Gable, Clark	0.3683	136.32
2	Marx, Groucho	0.3627	163.34
3	Marx, Harpo	0.2844	112.28
4	Garland, Judy	0.2820	100.28
5	Tracy, Spencer	0.2681	98.20
6	Stewart, James	0.2371	78.78
7	Crawford, Joan	0.2369	90.58
8	Astaire, Fred	0.2103	73.29
9	Marx, Chico	0.2055	86.39
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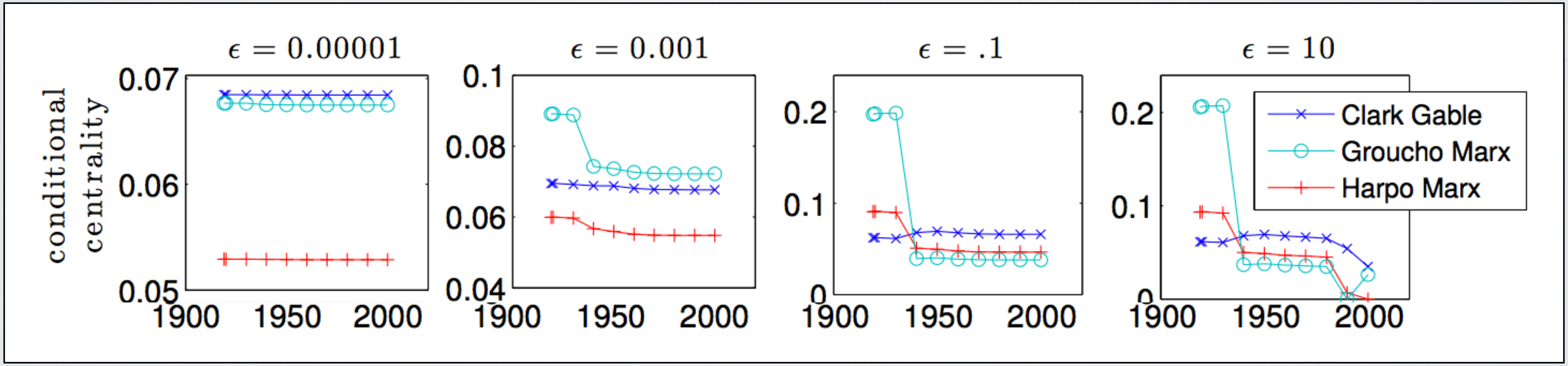
CASE STUDY 1:

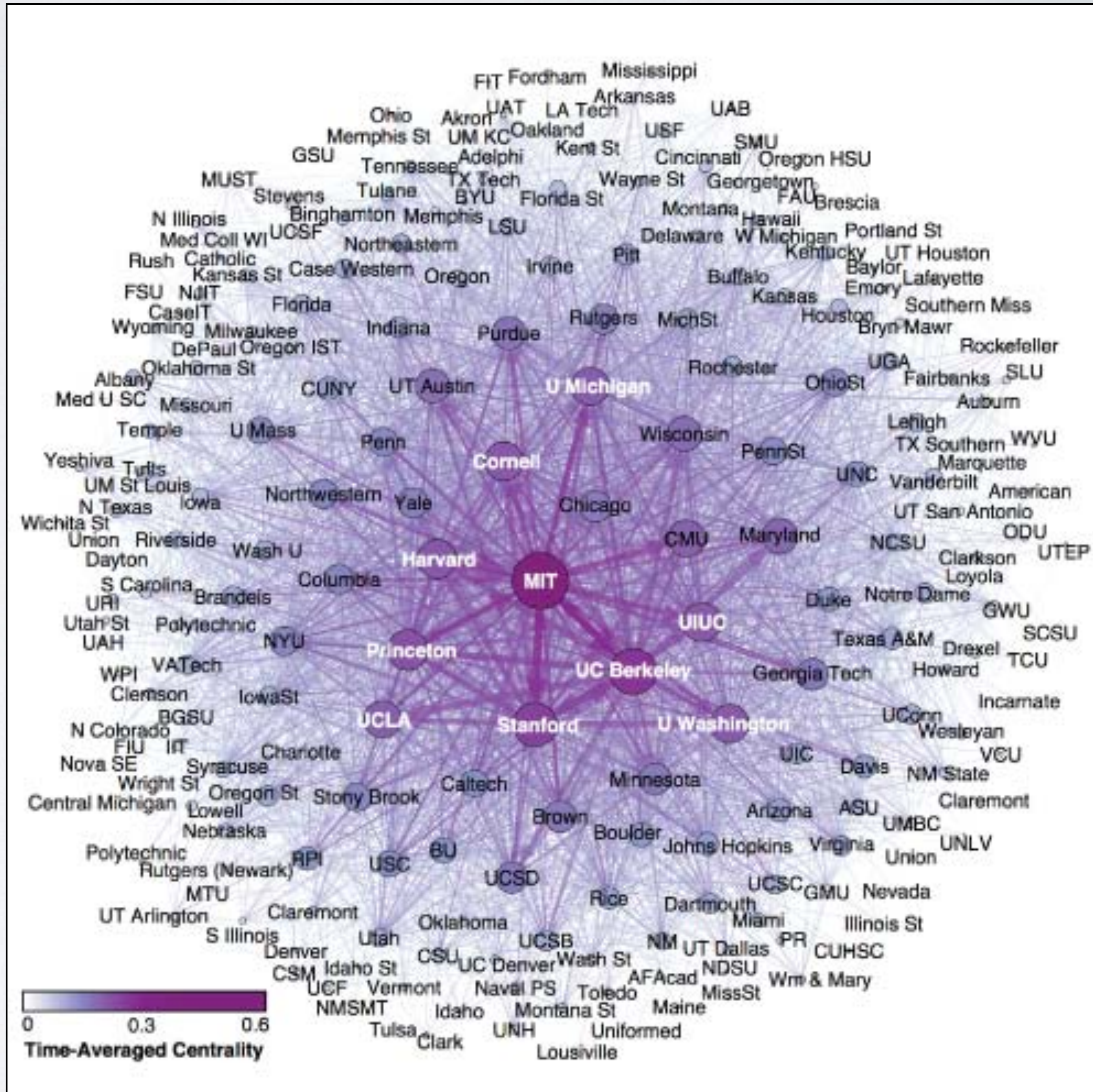
- The top actor, Clark Gable vs Groucho Marx, is too close to call using only time-average centrality
- We plot their conditional centrality trajectories for a few values of the layer coupling strength ϵ



- We find that Groucho was initially top ranked, but Clark had a longer career

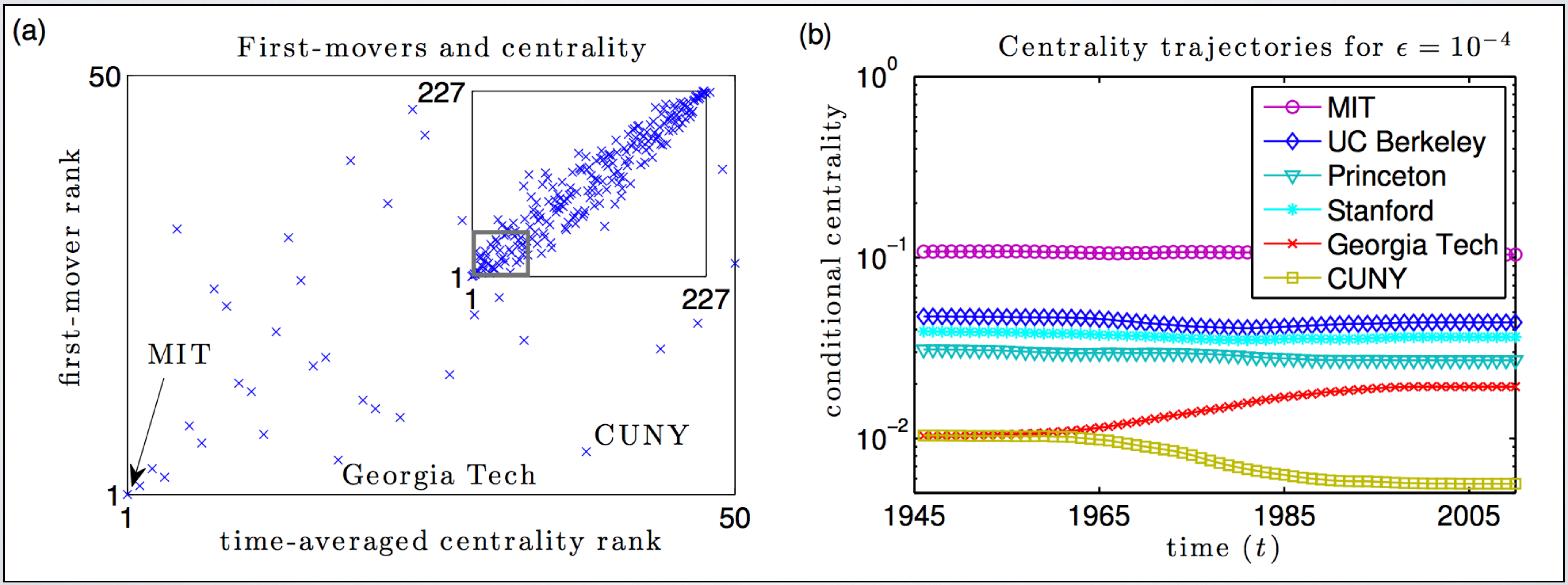
CASE STUDY 2: FLOW OF MATH PHDS

- Network encodes graduation and hiring of math PhDs using data from the Mathematical Genealogy Project
- Edges reflect the number of PhD students that graduate from university i and then later teach at university j
- Color indicates time-averaged centrality



OUTLIER MATH DEPARTMENTS

- Outliers have unusually large *first-order-mover scores* versus their *time-averaged centrality* (Georgia Tech and CUNY)



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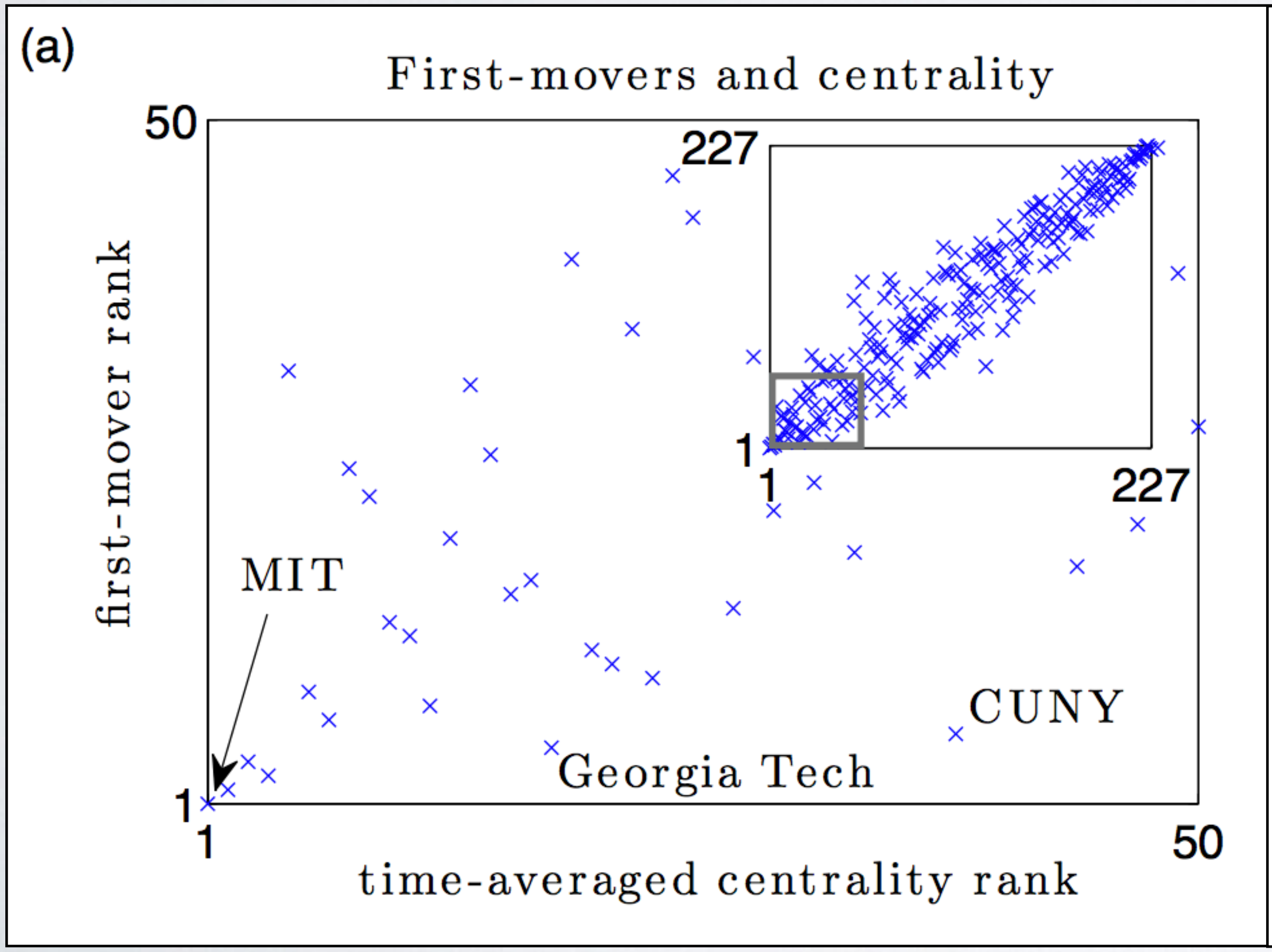

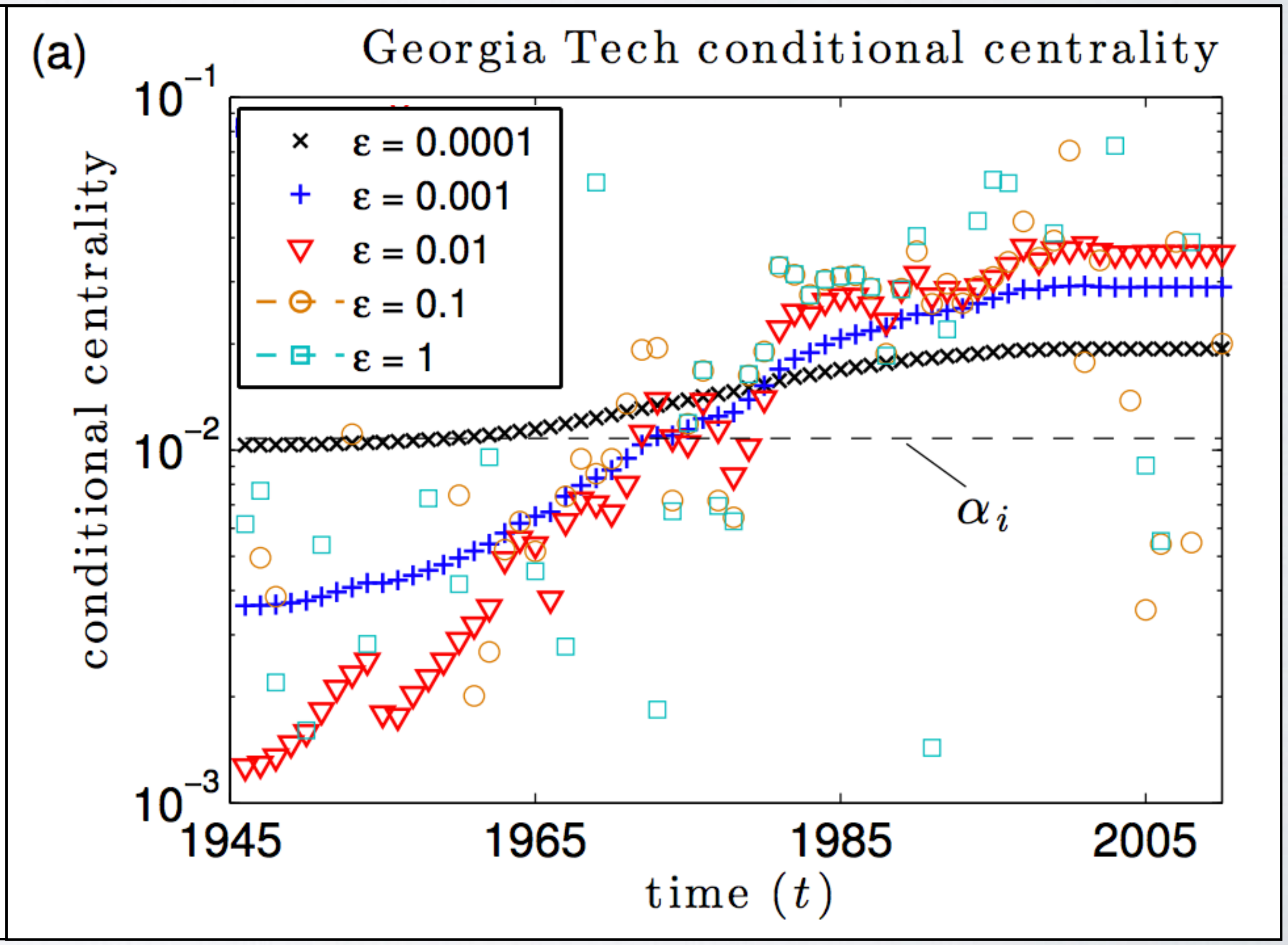

LAYER AGGREGATION

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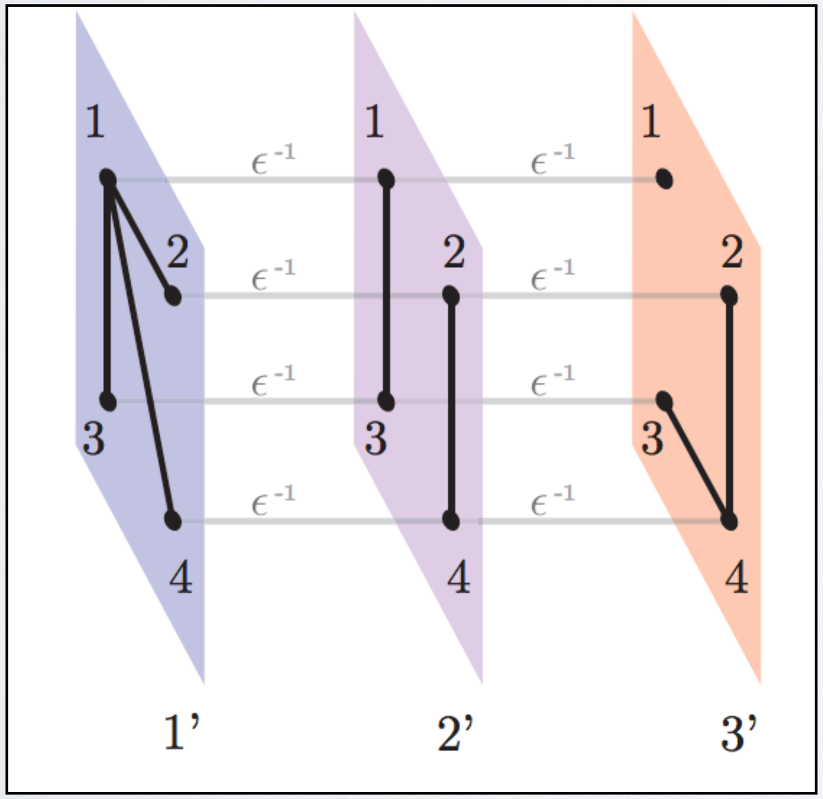

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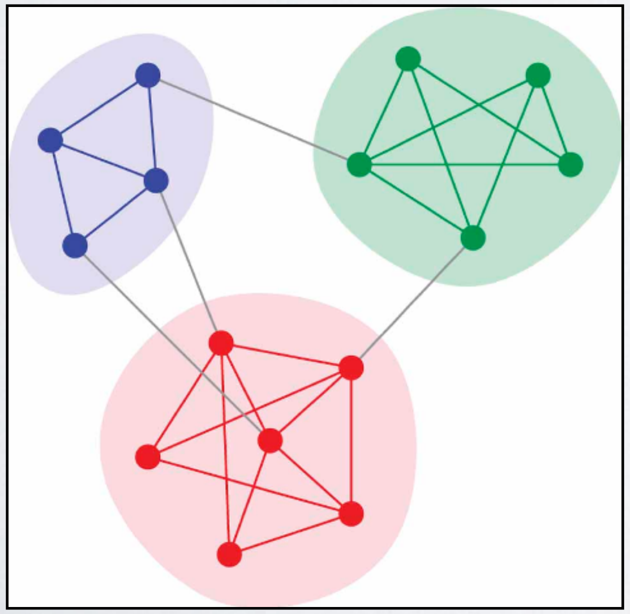
- We now consider layer aggregation and its affects on community detection

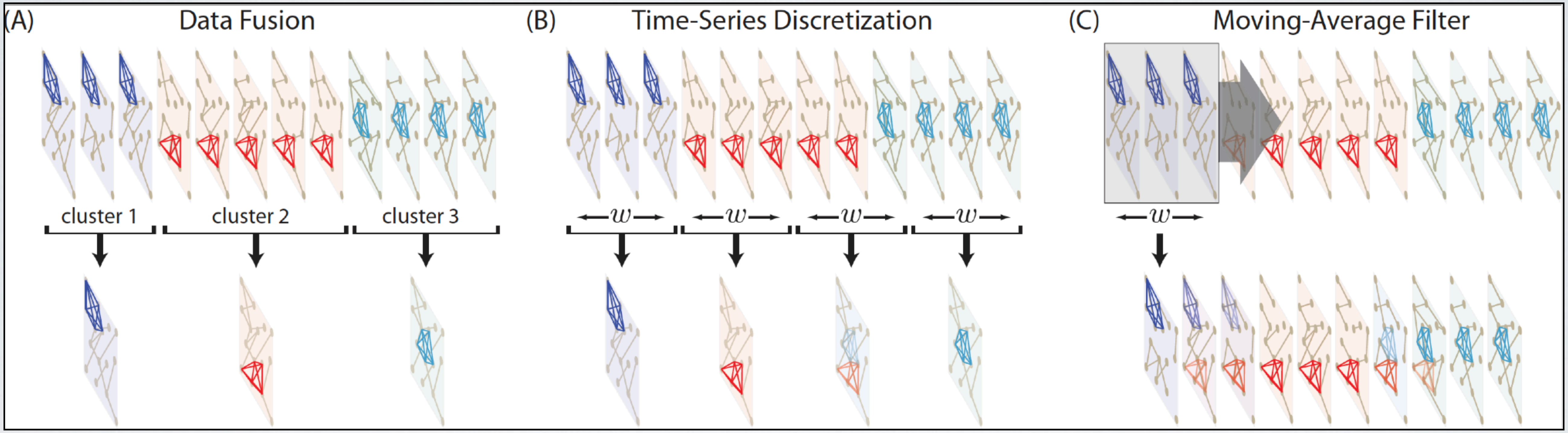


- Community detection is akin to data clustering and aims to identify groups of well-connected nodes

- **Survey of applications:**
S Shai, N Stanley, C Granell, D Taylor & PJ Mucha (2017) arXiv:1705.02305

DETECTABILITY OF COMMUNITIES IN PREPROCESSED TEMPORAL NETWORKS

- What are the effects of <u>layer-aggregation</u> on the detectability of communities?



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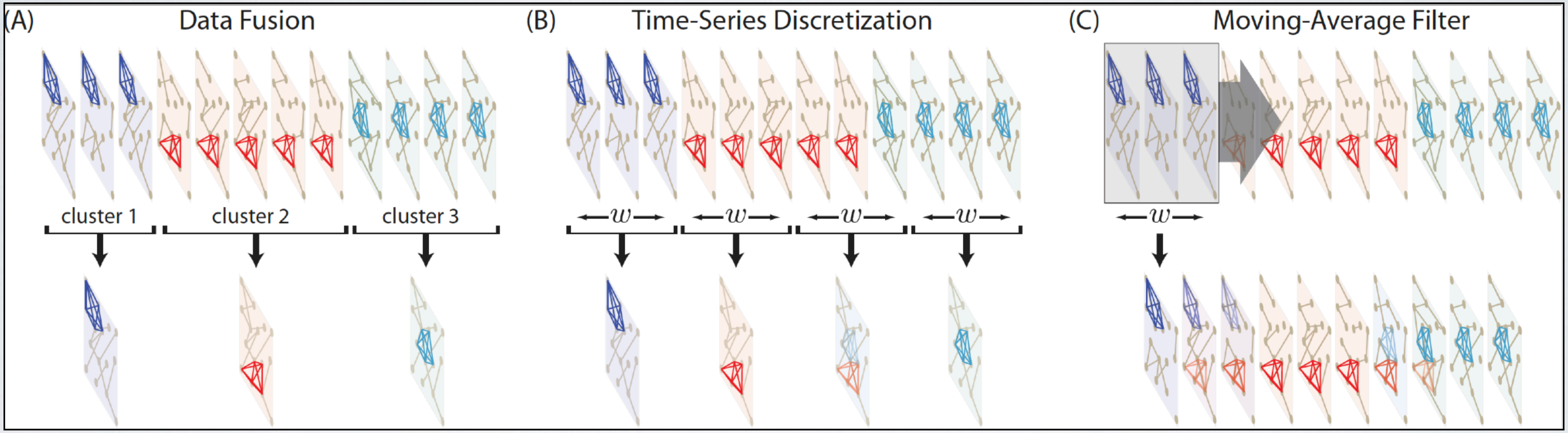

- Can we optimally preprocess network data to maximally enhance community detection?
  - We approach these questions using random-matrix theory to analyze fundamental limits for community detection.

2-COMMUNITY NETWORK MODEL

- *L* network layers are drawn from one stochastic block model (SBM)
  - Create edge (i, j) with probability

$$P_{ij} = \begin{cases} p_{in} & \text{if } c_i = c_j \\ p_{out} & \text{if } c_i \neq c_j \end{cases}$$

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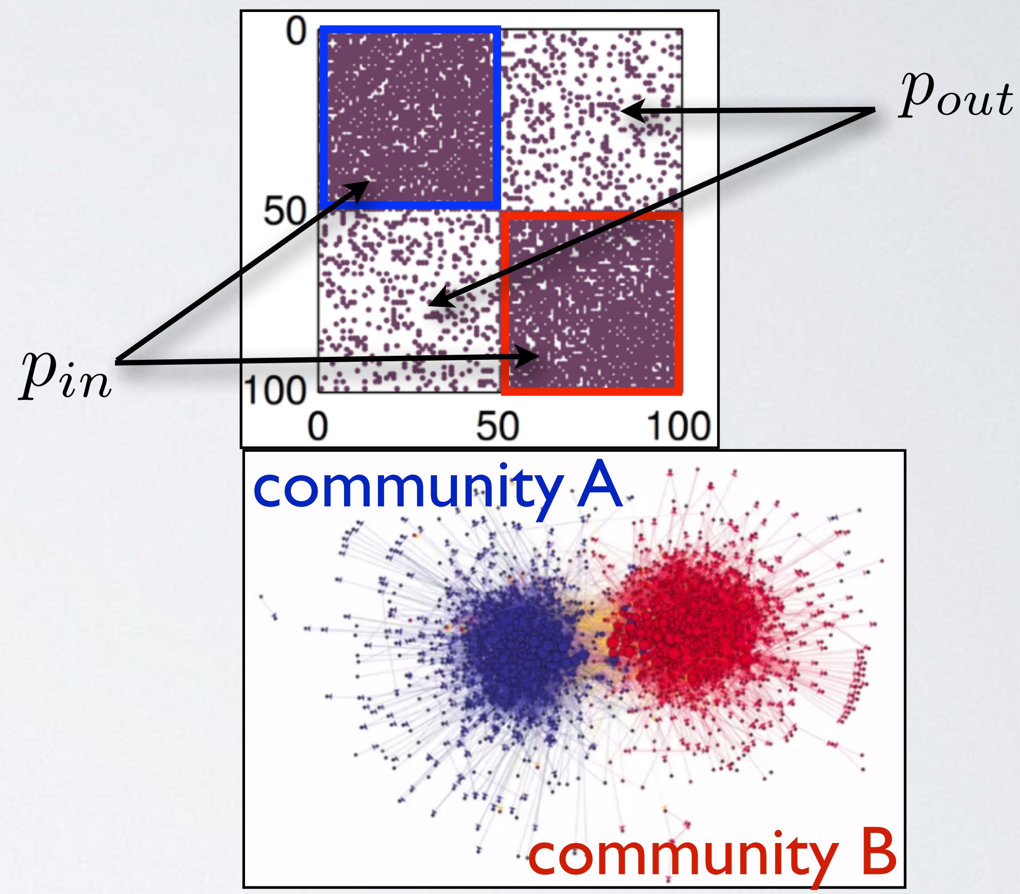

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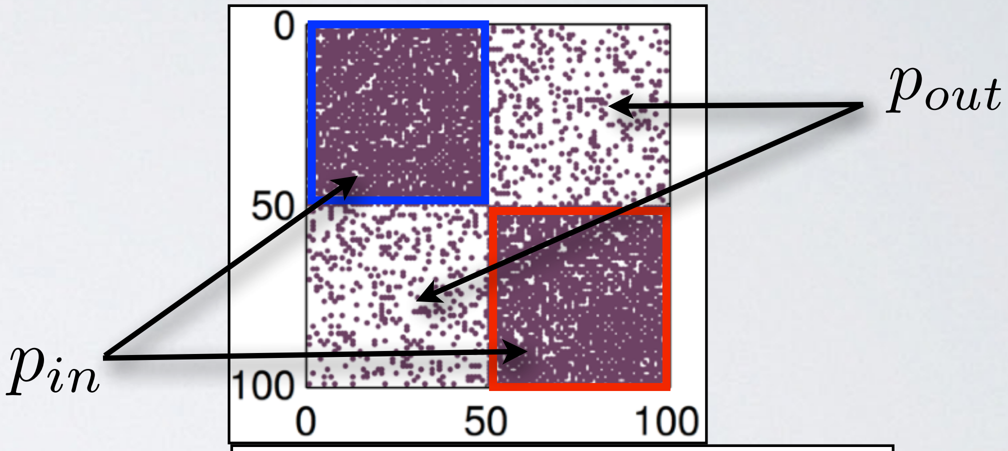

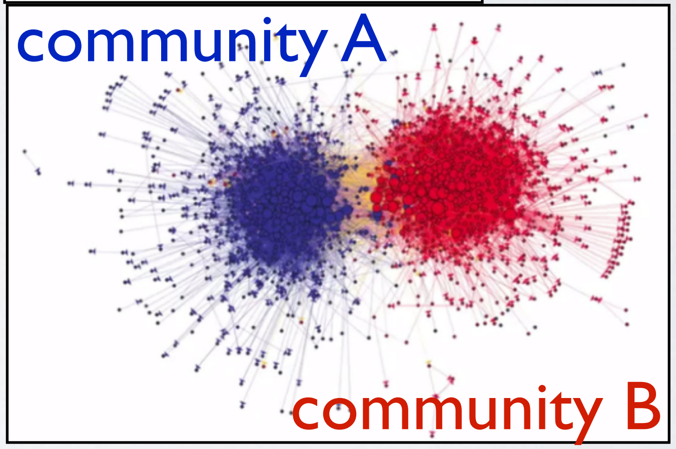

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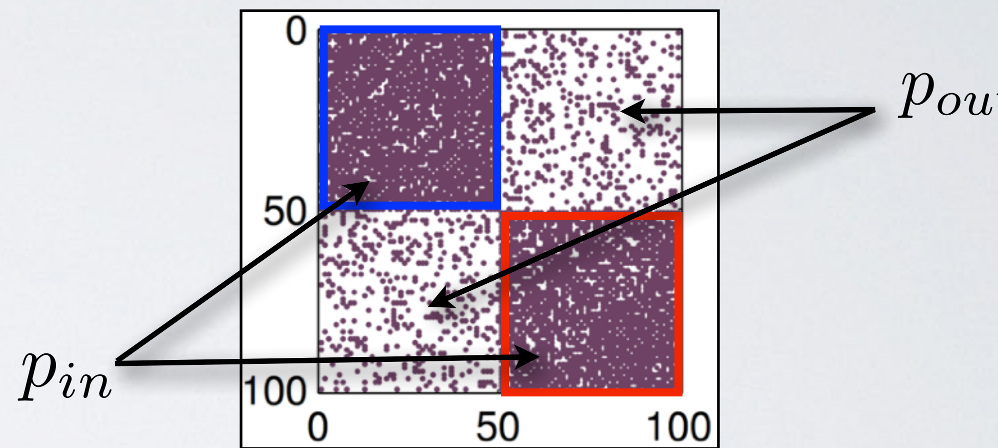

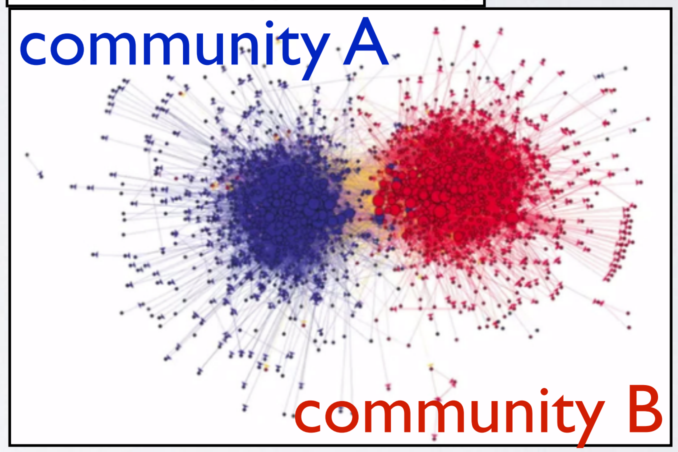

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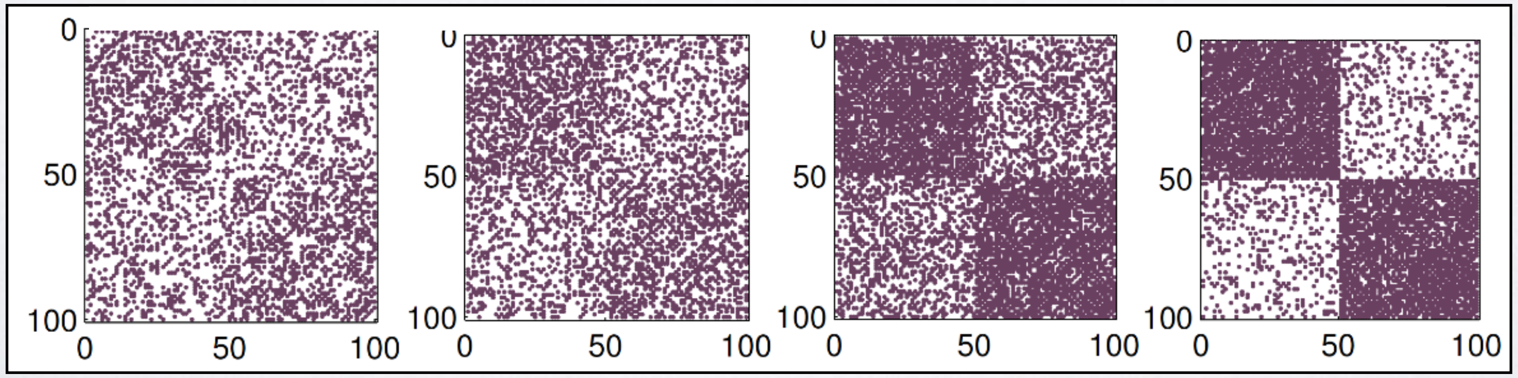
- **Detectability phase transition at $\Delta^* > 0$**

undetectable

small Δ



detectable

large Δ

ANALYSIS WITH RANDOM-MATRIX THEORY

- **We develop random matrix theory for the modularity matrix based on** Nadakuditi & Newman, PRL 2012 **and** T Peixoto, PRL 2013
- **We analyze the distribution of eigenvalues for the modularity matrix in the large *N* limit**

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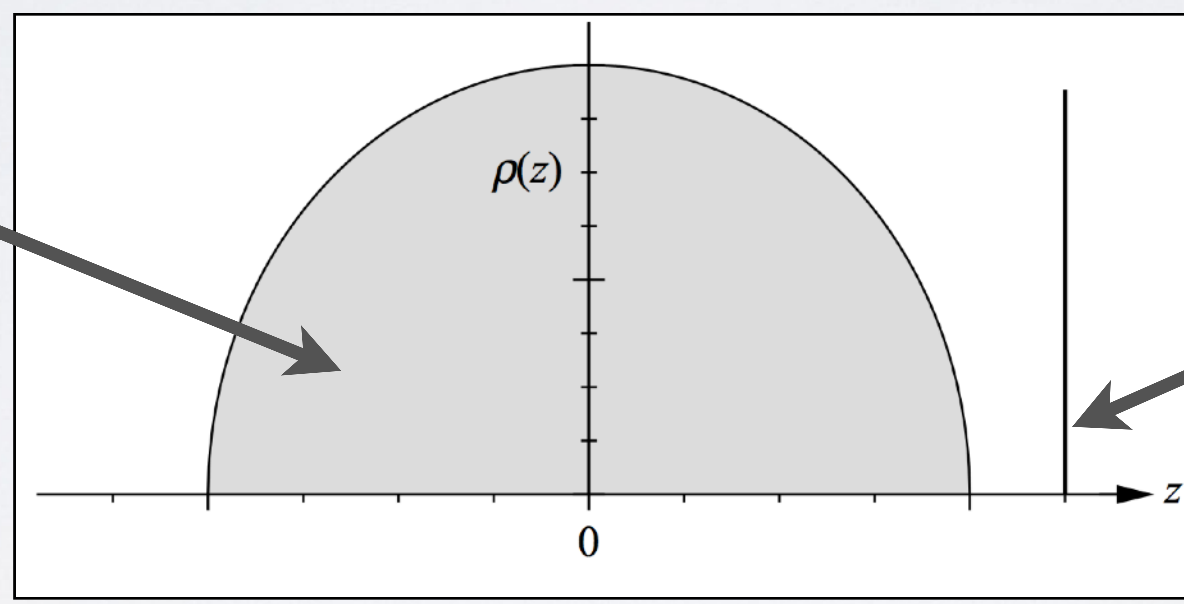
bulk eigenvalues



isolated eigenvalue whose eigenvector encodes community structure

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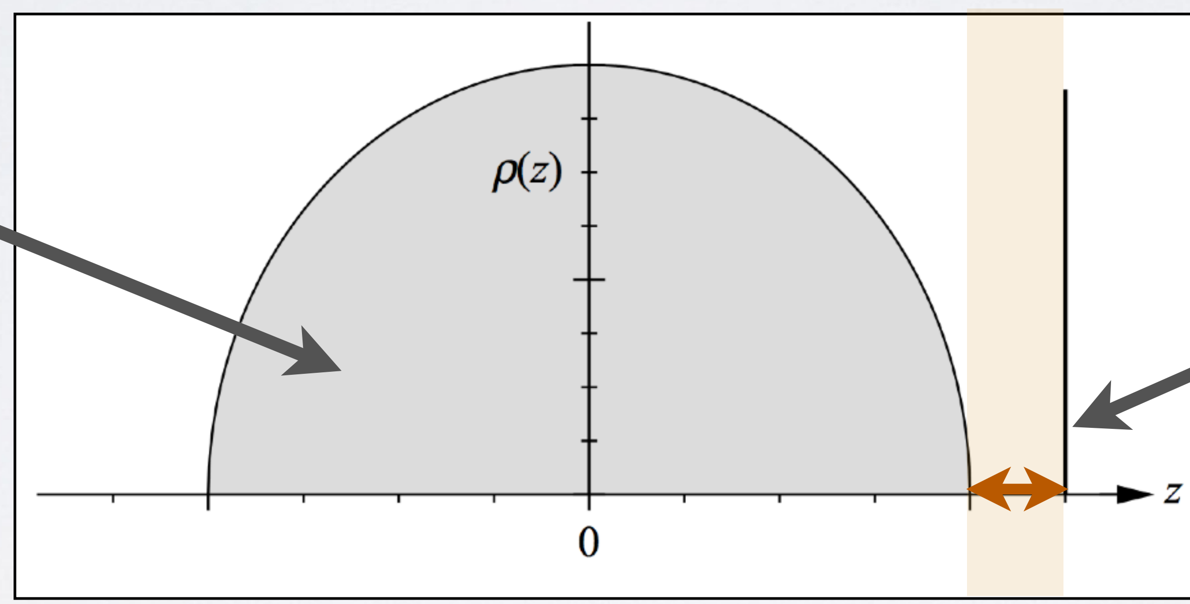
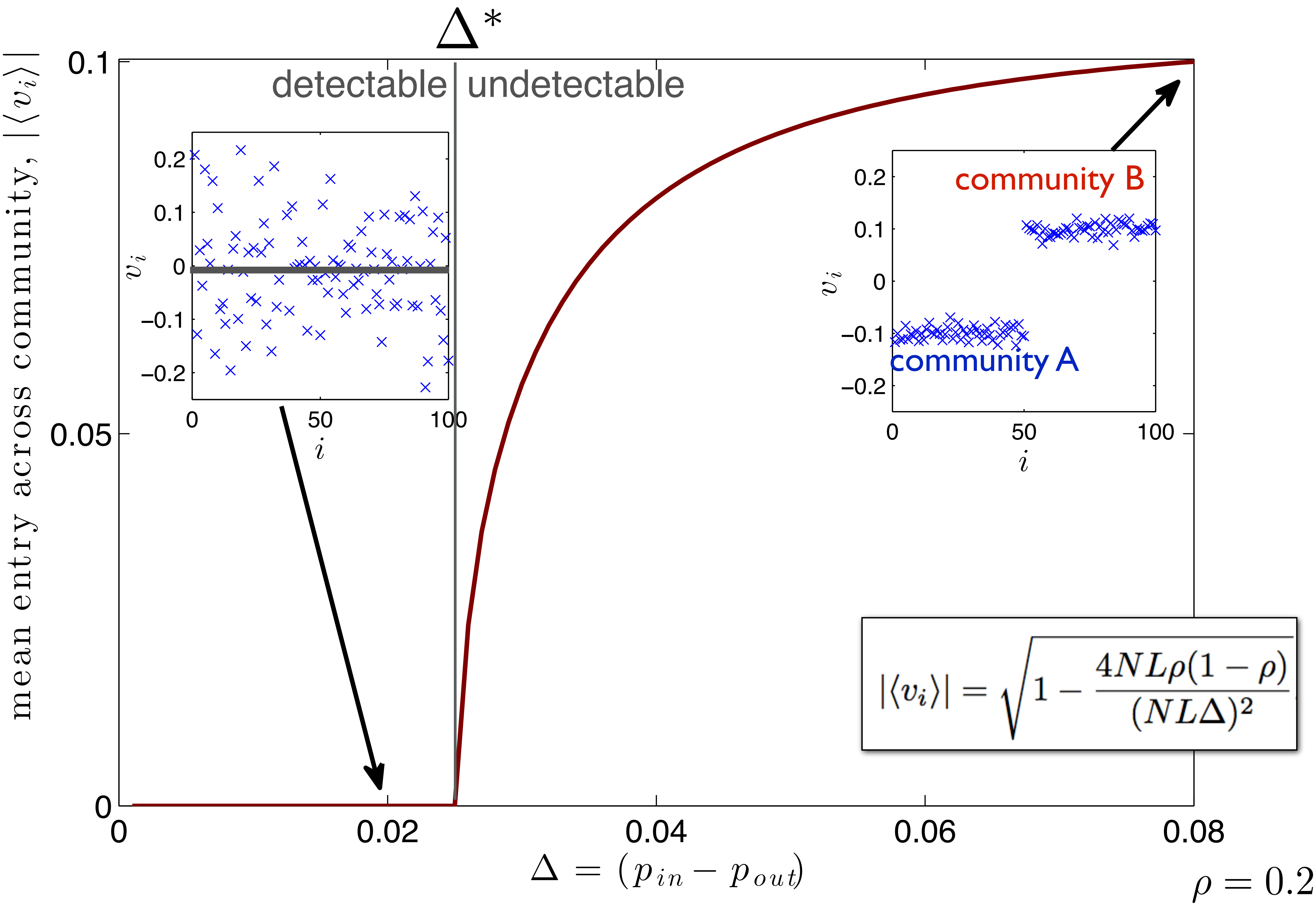
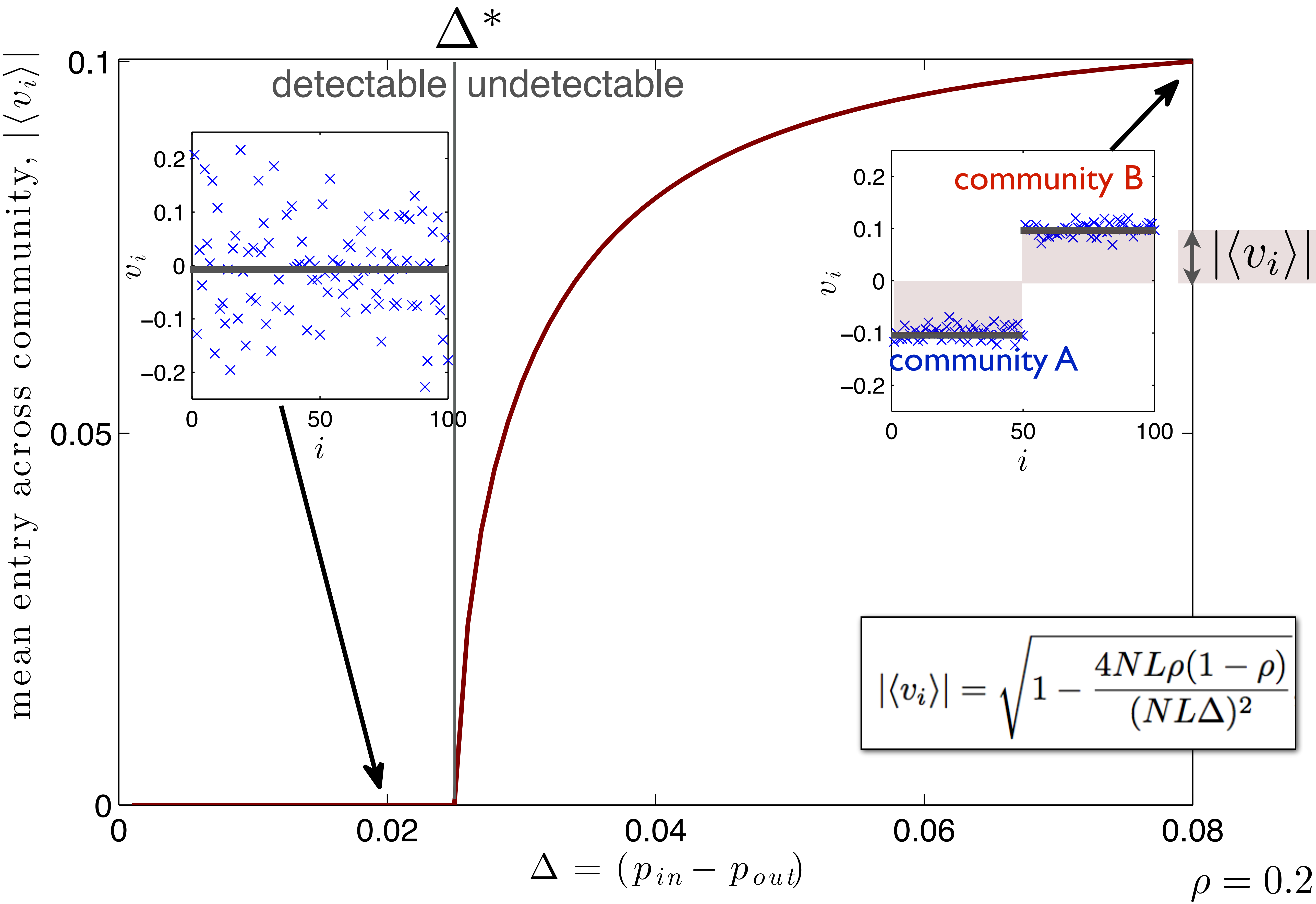


- As the gap disappears, the dominant eigenvector v becomes a random vector

PHASE TRANSITION FOR THE DOMINANT EIGENVECTOR

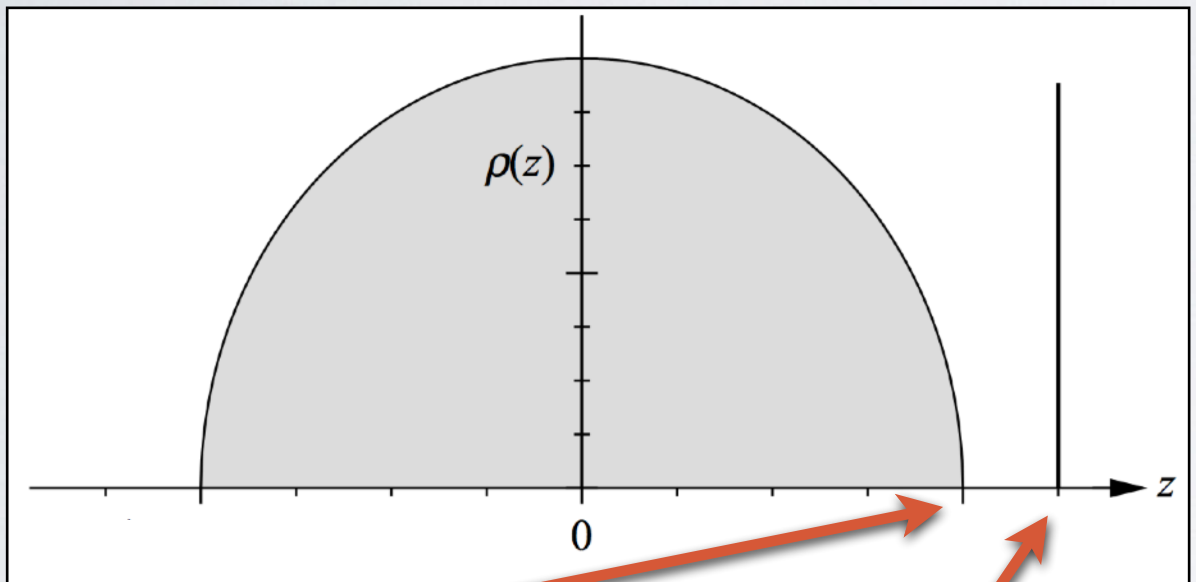


PHASE TRANSITION FOR THE DOMINANT EIGENVECTOR



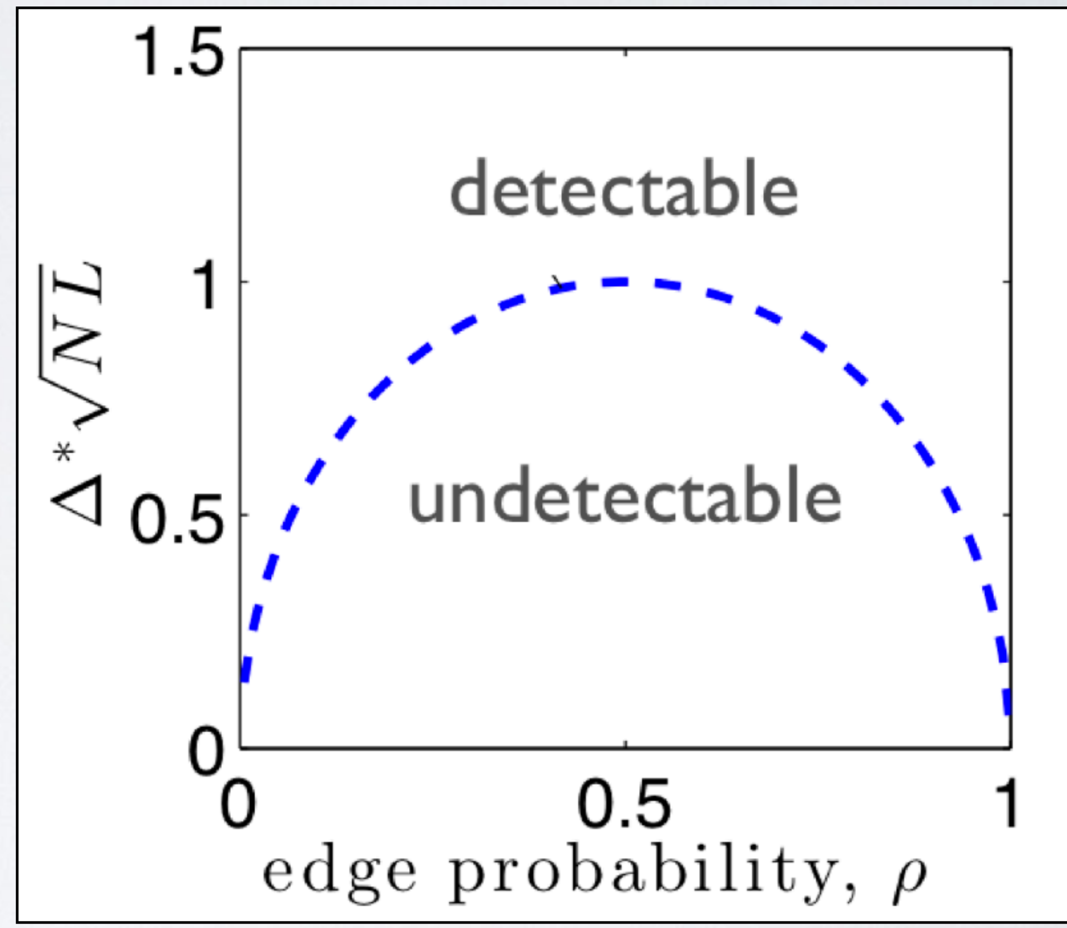
DETECTABILITY LIMIT VANISHES WITH INCREASING NUMBER OF LAYERS

Detectability equation: $\Delta^* = \sqrt{\rho(1-\rho)/NL}$



$$\lambda_2 = \sqrt{4NL\rho(1-\rho)}$$

$$\lambda_1 = NL\Delta/2 + 2\rho(1-\rho)/\Delta$$



Detectability limit vanishes as $\mathcal{O}(1/\sqrt{L})$

CONCLUSION

- **Part 1: layer-coupling and centrality**
  - While myriad real-world networks are multilayer, network analyses have traditionally been developed for single-layer networks
  - We extended eigenvector centrality to temporal networks by coupling together centrality matrices
  - We introduced concepts of joint, marginal and conditional centrality, allowing us to study centrality "trajectories" across time
  - The strong coupling limit recovers layer aggregation as a weighted sum of the layers adjacency matrices

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- **Part 1: layer-coupling and centrality**
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- **Part 2: layer aggregation and community detection**
  - Aggregating together similar layers can enhance the detection of structural patterns such as communities
  - We are developing random-matrix theory to study this phenomenon, which includes studying fundamental limits for community detection
  - We obtain $\mathcal{O}(1/\sqrt{L})$ as the scaling behavior for how the detectability limit vanishes with increasing number of layers L

ACKNOWLEDGMENTS

- Papers
  - D Taylor, SA Meyers, A Clauset, MA Porter & PJ Mucha (2017) Eigenvector-based centrality measures for temporal networks. Multiscale Modeling and Simulation, 15(1), 537-574.
  - D Taylor, RS Caceres & PJ Mucha (2017) Super-resolution community detection in layer-aggregated multilayer networks. Physical Review X:1609.04376.
  - D Taylor, S Shai, N Stanley & PJ Mucha (2016) Enhanced detectability of community structure in multilayer networks through layer aggregation. Physical Review Letters 116, 228301.
  - N Stanley, S Shai, D Taylor & PJ Mucha (2016) Clustering network layers with the strata multilayer stochastic block model. IEEE Transactions on Network Science 3, 95-105.

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