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Formal ontology of space, time, and physical entities in Classical Mechanics

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Abstract. Classical (i.e., non-quantum) mechanics is the foundation of many models of dynamical physical phenomena. As such those models inherit the ontological commitments inherent in the underlying physics. Therefore, building an ontology of dynamic phenomena requires a clear understanding of the ontology of classical mechanics. The axiomatic theory presented here in conjunction with the specification of its intended interpretation in the underlying physics aim to provide a formal framework that is general enough to formalize the ontological commitments of classical mechanics in a way that is consistent with various underlying spacetime ontologies.

Keywords: ontology of spacetime, classical mechanics, differential geometry, modal logic, Isabelle/HOL

1. Introduction

Following a long tradition in philosophy (Sider, 2001; Zemach, 1970; Simons, 1987; Lowe, 2002; Hawley, 2001), formal ontologies such as Basic Formal Ontology (Smith, 2016) and DOLCE (Gangemi et al., 2003) explicitly distinguish between the categories of continuants and occurrents (Grenon and Smith, 2004; Masolo et al., 2003). Continuants are entities that persist self-identically through time while undergoing changes by having different parts and different qualities at different times. By contrast, occurrents evolve over time and never exist in full at a given moment in time. Examples of continuants are objects such as my body, my heart, etc. Examples of ocurrants are processes such as my life, the beating of my heart, etc. The logical properties of these categories as well as the logic of their interrelations are relatively well understood (Hawley, 2001; Smith and Grenon, 2004). On this basis current ontologies provide means for systematically and consistently recording facts about continuants having different properties at different times, or facts about different relations holding at different times, processes occurring across certain time intervals, and so on. In addition current ontologies provide means for distinguishing changes that are merely logically and combinatorially possible from changes that are possible in virtue of being consistent with metaphysical laws such as the laws of mereology (e.g., transitivity of parthood and weak/strong supplementation property of parthood (Simons, 1987; Goodday and Cohn, 1994), etc.).

1.1. Ontology, physics and dynamic systems

Unfortunately, current ontologies lack the capabilities to formally characterize change over time by distinguishing changes and processes that are logically, combinatorially, and *meta*physically possible from changes and processes that are *physically* possible. By contrast, Physics as the science of the dynamic character of ever-changing physical reality provides conceptual, formal and computational means to single out physical possibilities by identifying *dynamically possible processes* and *dynamically possible sequences of instantaneous states of affairs*. An important conceptual pillar of modern physics is that dynamic constraints are expressed *geometrically* (Earman, 2002; Redhead, 2002). The methodology of geometrizing physics was introduced by Klein, Hilbert, Noether, and others (Klein, 1872; Hilbert, 1924, 2005; Brading and Castellani, 2002a,b; Brown and Brading, 2002) in the *Erlangen program* and has been fundamental to modern physics since. The central idea is that dynamic phenomena can be studied geometrically by considering the properties of a space (and the phenomena 'in' it) that are *invariant* under a given group of transformations. Groups of transformations formally capture the idea that there are processes and *sequences of* processes that change some properties while leaving others unchanged. To understand the geometric nature of dynamic reality in this sense is critical to understanding what is physically possible.

Given the ubiquity and the foundational role of the physical sciences in all branches of science and engineering, it ought to be at the core of every ontology of dynamic reality to conceptually explicate what it means for a process to be physically/dynamically possible or for a sequence of instantaneous states to be physically/dynamically possible. The lack of such capabilities in current formal ontologies is an expression of a *conceptual* gap between physical theories and formal ontologies.

In addition to the conceptual gap between physical theories and formal ontologies there is a *formal* gap. Physical theories are usually stated semi-formally in the language of differential geometry. Reasoning is based on solving differential equations using methods of differential calculus combined with other means of algebraic reasoning (vector algebra or geometric algebra) or numerical computations (Arnold, 1992; Arfken et al., 2005; Hestenes, 2002). By contrast, formal ontologies are based on the explicit axiomatic representation of geometric/topological/mereological relations within the formal calculus of predicate logic. While reasoning using differential calculus heavily uses structural properties implicit in the geometry of the underlying spaces, formal ontologies seek to explicate such structural aspects within the framework of a fully axiomatized formal theory.

To help bridging the conceptual as well as the formal gap between physical theories and formal ontologies this paper provides a geometry-based analysis of ontologically and conceptually relevant structural aspects of classical mechanics and then develops an axiomatic theory that captures the notion of ‘dynamic possibility’ in the context of a mereology-based axiomatic theory. The resulting formal theory in conjunction with its computational realization in Isabelle/HOL (Nipkow et al., 2002; Paulson and Nipkow, 2017) are intended to serve as a framework to relate conceptual and formal structures of physical theories to conceptual and formal structures in formal ontologies in a way that supports the computational verification of the resulting formalisms.

Ideally, a formal ontology is an axiomatic theory that is strong enough to make underlying commitments explicit by excluding the models that are inconsistent with those commitments. Unfortunately, to express many of the ontological commitments inherent in classical mechanics requires highly expressive languages including the language of differential geometry (Fecko, 2006). In order to deal with the fact that a formal ontology expressed in (a language that in essence is equivalent to) the language of first order logic will not be strong enough to make explicit all the ontological commitments that are explicit or implicit in physical theories, the following intermediate path is taken: In the first part of the paper a class of models that captures ontologically relevant aspects of classical mechanics is specified semi-formally in the language of differential geometry. This class of models is general enough to capture a significant number of ontologically relevant classical mechanics.

The axiomatic theory that is developed in the second part focuses on the logic of parthood, instantiation and location but is tightly linked to this class of models through the explicit specification of the intended interpretation of the primitives of the formal theory. To make explicit what is meant by ‘physically or dynamically *possible*’ a modal predicate logic is used (Sec. 4).[1] In this way the presented formal ontology provides means to give precise and formal formulations of specifications of primitives that in many currently existing ontologies are realized only as informal elucidations. In BFO (Smith, 2016), for example, there are informal elucidations of the form: "ELUCIDATION: A spatial region is a continuant entity that is a continuant_part_of space$_R$ as defined relative to some frame R." (Smith, 2016, 035-001). The formal ontology presented here provides a precise interpretation of primitives such as ’Spatial Region’ and ’continuant_part_of space$_R$’ in a way that directly corresponds to the understanding of such notions in classical mechanics. A summary of how informal elucidations of the current version of BFO can be given a precise formulation is given in Sec. 7.2.

[1]Some features of second order logic are used in order to avoid axiomatic schemata which would be difficult to represent in the HOL-based computational representation of the formal theory.

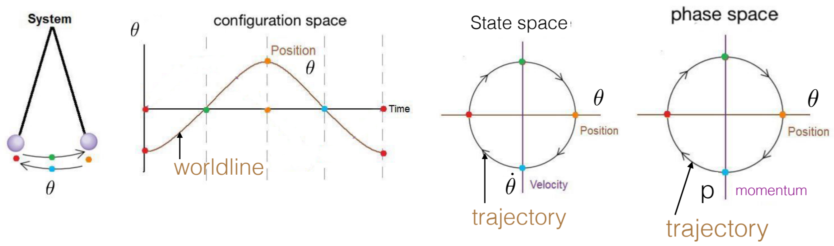


Fig. 1. From left to right: A simple pendulum *system* Π consisting of the bob mounted to the ceiling via a massless rod. The location of the bob is represented by its center of mass. Since the pendulum swings in a fixed plane, the spatial location of the bob is determined by the angle θ; the *configuration* space of Π with Π's cosine-shaped worldline; the *state* space of Π with its circular trajectory of sequences of states determined by location and velocity; the *phase* space of Π with its trajectory of sequences of states determined by location and momentum. (Adopted from Wikimedia Commons (2016).)

1.2. Classical mechanics from an ontological perspective

The term 'Classical mechanics' here is understood as *non-quantum* mechanics. Within the realm of classical mechanics the focus is on the mechanics of systems consisting of finitely many particles that can be described without the machinery of dynamic field theories (Marsden and Ratiu, 1999) and statistical mechanics (Frigg, 2012). Although nowadays classical mechanics is superseded by quantum mechanics and does include dynamic field theories as well as statistical mechanics, the scope of the paper is still rich enough to address ontologically relevant problems at the intersection of logic, metaphysics, and physics.

Classical mechanics in the sense understood in this paper is based on a number of fundamental metaphysical, logical, and formal assumptions (Hestenes, 2002, p. 121):

(1) Every physical object/system is a composite of (atomic) particles.
(2) The behavior of particles is governed by their interactions with other particles. Interactions in turn are processes in which particles participate.
(3) The properties of physical wholes are determined by the properties of their parts.[2] The aim of mechanics is it to describe wholes based on a classification of particles and a classification of interactions between particles.
(4) There exists a four-dimensional spacetime in which particles are located and in which interactions take places.[3]

Based on (1-4) particles can be characterized in classical mechanics geometrically if the following conditions are satisfied: (i) Every particle is located at a unique region of space at every time of its existence; (ii) Particles exclusively occupy their location at any given time; (iii) Continuity of existence manifests itself in terms of continuous change of location, i.e., continuous *motion*; (iv) The *state of motion* of a particle at a given point in time is uniquely determined by the location of the particle at that time in conjunction with the rate of change of location (the velocity) of the particle at that time. If (i-iv) obtain then (a) every particle is uniquely identified by the (sequence of) location(s) it occupies during its existence – the particle's *worldline* in spacetime and (b) every particle is uniquely identified by the (sequence of) its states of motion – *trajectories* in spaces of possible states. Worldlines and trajectories are illustrated for a simple pendulum in Fig. 1.

There are at least three frameworks for specifying possible behaviors of physical systems (Abraham and Marsden, 1978; Arnold, 1997): the *Lagrangian* framework, the *Newtonian* framework, and the *Hamiltonian* framework. Roughly, in the Lagrangian framework the focus is on expressing physical possibilities in terms of geometric constraints on possible worldlines in spacetime (e.g, constraints that result in cosine-shaped worldlines like the one depicted in the image labeled 'configuration space' of Fig. 1). In the other

[2]This is not to imply that set of properties of the parts is a subset of the set of properties of the whole.

[3]Classical theories differ in their assumptions about the structure of spacetime as well as to whether spacetime *is* a substantive manifold or whether spacetime *is* a metric field and the manifold is only a descriptive vehicle to label locations and to express smoothness conditions (Rovelli and Vidotto, 2015).

two frameworks the focus is on geometric constraints on possible trajectories in more abstract spaces (e.g, constraints that result in circular trajectories like the ones depicted in the images labeled 'state space' and 'phase space' of Fig. 1). Each of the frameworks has strengths and weaknesses when considered as tools for the working physicist: The Lagrangian formulation is particularly useful (in the sense of efficient to work with) for constrained systems (roughly systems for which the laws that constrain possible changes of location take the form $F = m\ddot{x}$). On the other hand the Hamiltonian formulation is particularly useful (again, in the sense of efficient to work with) for closed systems (roughly, systems in which energy and momentum are conserved). Due to space limitations the discussion here will focus on the Lagrangian framework. In addition to the Lagrangian, Newtonian, and Hamiltonian *frameworks*, approaches to classical mechanics differ in the *spacetime structure* they presuppose. In this context it will be necessary to distinguish Newtonian spacetime as well as the Minkowski spacetime of special and general relativity (Sec. 2).

Despite the differences of the various approaches to classical mechanics they all share a number of ontological commitments which can be analyzed by explicating the following structural features at the intersection between metaphysics and physics:

(I) Commitments regarding the structure of spacetime in which actual physical particles exist (and form mereological wholes) and physical processes actually occur (Sec. 2);
(II) Commitments regarding the structure of spaces of possibilities (configuration/state/phase spaces) which constrain *geometrically* what is physically possible (Sec. 3);
(III) Commitments regarding constraints on the actualization of logically, metaphysically, and physically possible entities. One way of expressing such constraints and commitments is in terms of restrictions on their instantiation of physical entities at regions of spacetime (Sec. 6).

The axiomatic theory presented in the second part of this paper aims to provide a formal framework that is general enough to address (I-III) in a way that is (a) consistent with the various conceptions of spacetime in classical mechanics and (b) lends itself to computational representations that facilitate formal validation.

For readers trained in formal and applied ontology but without much background in physics and differential geometry it may be beneficial to skim through the development of the formal ontology in Sec. 4 – 6 and the discussion in Sec. 7 before going through the details of the formal models and their differential geometry in the first part of the paper.

2. The geometry of spacetime in classical mechanics

In this section the language of the differential geometry of manifolds is used to explicate some of the ontological commitments underlying classical physical theories. Manifolds are mathematical structures capable of capturing important topological and geometric aspects of spacetime and of physical systems. In particular they provide formal means to spell out what smoothness and continuity of change means. This is because their geometry is characterized not only by points, curves, and relations between them but also by their tangent spaces and the geometric objects that populate these spaces (vectors, vector fields, etc.) in conjunction with relations between them. In addition, manifolds establish the links between such geometric objects and their numeric representations that are critical for the working physicist in order to perform actual computations. For a brief review of differential geometry see Appendix A. A summary of the notations introduced in Appendix A is given in Table 1. The mathematical structures introduced in this and the next two sections will provide the intended interpretation of the axiomatic theory developed in the second part of the paper.[4]

[4]There is an ongoing debate in the philosophy of physics and the philosophy of quantum mechanics about the reality of the geometric structures invoked by the physical theories. The positions range from the claim that the geometric spaces discussed in the next two sections are purely mathematical in nature and what is real are spacetime in conjunction with the geometric features of spacetime that are picked out using the mathematical apparatus of differential geometry (one of the most common example of this view is the instrumentalist view of the wave function in Quantum mechanics (Ney, 2012; North, 2012)). On the other end

symbolic expression	description	defined in
M	Manifold	A.1
$M_1 \sqsubseteq M_2$	M_1 is a submanifold of M_2	A.1
$\bigsqcup S$	Union of a set S of manifolds such that the result is a manifold.	
$T_x M$	Tangent space on manifold M at point $x \in M$	A.1
TM	Tangent bundle on M. TM is the disjoint union of the tangent spaces $T_x M$ for all $x \in M$	A.1
$\gamma : \Re \to M$	parametric curve on M	A.2
$\gamma \sqsubset M$	$\gamma =_{df} \{\gamma(\tau) \in M \mid \tau \in \Re\}$ is the curve $\gamma \subset M$ represented by the parametric curve $\gamma(\tau)$ with $\tau \in \Re$	A.2
$\xi = \frac{d}{d\tau}\gamma(\tau)\vert_x$	ξ is the tangent on γ at point $x \in \gamma$	A.2
$H : M \to \Re$	H is a scalar field on M	A.2
$X : M \to TM$	X is a vector field on M such that $X_x = X(x) \in T_x M$ for all $x \in M$	A.2
$\gamma_{X,x} : \Re \to M$	integral curve through $x \in M$ with respect to the vector field $X : M \to TM$. I.e., if $\gamma(\tau) = y$ then $X(y)$ is the tangent on γ at $y \in M$	Eq. 23

Table 1

Summary of Appendix A: Basic notions of differential geometry

2.1. Geometric structure

The topological structure of spacetime in classical mechanics is identified with the structure of a n-dimensional Hausdorff (Alexandroff, 1961) manifold[5] with the topology $\mathcal{ST} = (\Re \times M)$ for some $(n-1)$-dimensional manifold M (See Appendix A). The topology of time is identified with the topology of the real numbers and the topology of space is identified with the topology of some Hausdorff manifold M. In classical mechanics the dimension of M is usually 3. The geometric structure of the spacetime manifold $\mathcal{ST}$ is induced by a symmetric bilinear functional g on $\mathcal{ST}$ (Arnold, 1997) – the *metric field*. Roughly, a metric field g on a manifold $\mathcal{ST}$ is a symmetric mapping $g_x : (T_x\mathcal{ST} \times T_x\mathcal{ST}) \to \Re$, that, at every point $x \in \mathcal{ST}$, maps pairs of vectors ξ, η in the tangent space $T_x\mathcal{ST}$ of *ST* at point x (Table 1) to a real number such that $g_x(\xi, \eta) = g_x(\eta, \xi)$ (g is symmetric at all points x of the underlying manifold). The expression $|\xi|^2 =_{df} g_x(\xi, \xi)$ defines the square of the length $|\xi|$ of the vector $\xi \in T_x\mathcal{ST}$ according to the metric g of $\mathcal{ST}$ at point x (see Arnold (1997) for details). Thus, if the tangent space $T_x\mathcal{ST}$ sufficiently closely approximates the neighborhood of x in M then if the vector $\xi \in T_x\mathcal{ST}$ begins at x and ends at the point $y \in M$ then $|\xi|$ is just the distance between the points x and y according to the metric field g in the neighborhood of x. Differential geometry then provides means to combine local linear approximations to determine the length of smooth curves in possibly curved spacetime manifolds (Arnold, 1997).

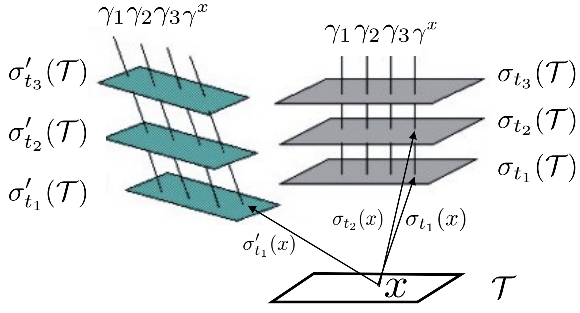
Classical mechanics includes the following postulate (illustrated in the left of Fig. 2):

Postulate 1 (adopted from the work of Belot (2007))**.** *The geometry g of the spacetime manifold $(\mathcal{ST}, g)$ singles out: (a) a distinguished class $\sigma(\mathcal{T})$ (see below) of hyper-surfaces that correspond to* instants *of time (or* time-slices*) and (b) a distinguished class Γ of curves that correspond to (geometrically) possible* worldlines *of particles.*

Let $\mathcal{ST}$ be a $n+1$ dimensional manifold with topology $(\Re \times M)$ where M is a manifold of dimension n (usually 3). In addition, let $\mathcal{T}$ be a n-(usually 3) dimensional manifold $(\mathcal{T}, g_{\mathcal{T}})$ carrying a Riemanian geometry (i.e., $g_{\mathcal{T}}$ is required to be symmetric, definite positive, and may vary smoothly) (Arnold, 1997):

of the spectrum there are positions that hold that the structures discussed below are as real as spacetime itself (examples include the view of wave function realism in Quantum mechanics (Ney, 2012; North, 2012) and views of structural realism (Ladyman, 2016)). The presentation here aims to remain neutral with respect to the various interpretational choices. The aim is to provide a framework in which such choices can be made explicit.

[5]Roughly, in a Haunsdorff manifold there are for any distinct points $x, y \in M$ disjoint (open) neighborhoods $U_x, U_y \subset M$ such that $x \in U_x, y \in U_y$, and $U_x \cap U_y = \emptyset$.



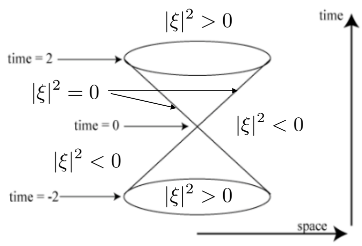


Fig. 2. $\mathcal{T}$-slicings σ' and σ of a spacetime with worldlines $\gamma_1 - \gamma_3, \gamma^x$ (left); Partition of spacetime into regions of $|\xi|^2 > 0$, $|\xi|^2 = 0$, and $|\xi|^2 < 0$ (right).

Definition 1 ($\mathcal{T}$-slicing – adopted from the work of Belot (2007, Def. 27))**.** *A $\mathcal{T}$-slicing of $(\mathcal{ST}, g)$ is a smooth map (diffeomorphism) $\sigma : \Re \times \mathcal{T} \to (\Re \times M)$ with the following properties (Illustration in the left of Fig. 2):*

(i) Every slice $(t, \sigma(\{t\} \times \mathcal{T})) = \{(t, \sigma_t(x)) \mid x \in \mathcal{T}\}$ of the $\mathcal{T}$-slicing σ at $t \in \Re$ is a hypersurface (an instant, a timeslice) according to the geometry g of $(\mathcal{ST}, g)$. In what follows it will be convenient to use the notation $\sigma_t(\mathcal{T})$ to refer to the timeslice $\{(t, \sigma_t(x)) \mid x \in \mathcal{T}\}$ in terms of the slicing σ;
(ii) The $\mathcal{T}$-slicing respects the worldline structure of spacetime in the sense that the set $\gamma^x =_{df} \sigma(\Re \times \{x\}) = \{(t, \sigma_t(x)) \mid t \in \Re\}$, for any $x \in \mathcal{T}$, is a possible worldline of a particle through $(t, \sigma_t(x)) \in \mathcal{ST}$ according to the geometry g of $(\mathcal{ST}, g)$, i.e., $\gamma^x \in \Gamma$.
(iii) The $\mathcal{T}$-slicing σ is such that for every $t \in \Re$ the mapping $\sigma_t : \mathcal{T} \to \sigma_t(\mathcal{T})$ is an isomorphism between $\mathcal{T}$ and $\sigma_t(\mathcal{T})$.

In the configuration space of Fig. 1 time slices are the vertical lines that are parallel to the θ axis. Def. 1 gives rise to the following naming conventions:

Definition 2. *The manifold $\mathcal{T}$ is called the* abstract instant *of the $\mathcal{T}$-slicing σ and each $\sigma_t(\mathcal{T})$ is called a* concrete *time instant of the slicing σ. The parameter $t \in \Re$ of σ_t is called coordinate time associated with σ. Σ is the set of all $\mathcal{T}$-slicing of a given underlying spacetime.*

The time axis in the configuration space of Fig. 1 represents coordinate time. Similarly, in the left of Fig. 2 the parameters t_1, t_2, t_3 are coordinate times

Def. 1 is used to further constrain what is geometrically possible:

Postulate 2. *For every kinematically possible spacetime $(\mathcal{ST}, g)$ there exists a $\mathcal{T}$-slicing, i.e., $\Sigma \neq \emptyset$.*

In physical theories Postulates 1 and 2 are complemented additional kinematic and dynamic constraints that restrict what is physically possible. This is discussed in what follows.

2.2. *Classical spacetimes*

Postulates 1 and 2 allow for a wide range of possible spacetime geometries including Newtonian spacetime, the global Minkowski spacetime of special relativity (Einstein, 1951; Minkowsk, 1908), and the locally Minkowskian spacetime of general relativity (Einstein, 1951).

Newtonian spacetime. Newtonian spacetime has the geometric structure of an Euclidean manifold, i.e., the geometry of $\mathcal{ST}$ is isomorphic to the geometry of $\Re^4$: $(\Re^4, \iota) \cong (\mathcal{ST}, g)$. The metric ι is a functional that is symmetric, definite positive, and the same at all points of spacetime. In such a geometry there is a *unique* slicing σ of spacetime into timeslices, i.e., $\Sigma = \{\sigma\}$. All timeslices are equipped with an Euclidean geometry that is isomorphic to the geometry of $\Re^3$. Newtonian spacetime does not place restrictions on the rate of change of location (velocity) of physically possible entities. This puts Newtonian spacetime in conflict with Classical electrodynamics where there is a maximum for the speed of light. (Norton, 2012)

symbolic expression	description
$(\mathcal{ST}, g)$	Spacetime manifold of topology $(\Re \times M)$ and geometry (metric field) g
g_x	Metric field: $g_x : T_x\mathcal{ST} \times T_x\mathcal{ST} \to \Re$ at $x \in \mathcal{ST}$, defines the length $\lvert\xi\rvert$ of a vector $\xi \in T_x\mathcal{ST}$ via $\lvert\xi\rvert^2 = g_x(\xi, \xi)$
$(\mathcal{T}, g_\mathcal{T})$	Abstract time slice with geometry $g_\mathcal{T}$. If $\mathcal{ST}$ has the topology $\Re \times M$ then $\mathcal{T}$ has the dimension of M
$\mathcal{T}$-slicing σ	A $\mathcal{T}$-slicing σ of $(\mathcal{ST}, g)$ is a smooth map $\sigma : \Re \times \mathcal{T} \to (\Re \times M)$
$\sigma_t(\mathcal{T})$	Concrete timeslice (time instant, hyperplane of simultaneity) of spacetime according to the $\mathcal{T}$ slicing σ. I.e., $\sigma_t(\mathcal{T}) = \{(t, \sigma_t(x)) \mid x \in \mathcal{T}\}$. σ_t is a isomorphism from $\mathcal{T}$ to $\sigma_t(\mathcal{T})$.
$\sigma(\mathcal{T})$	A particular slicing of spacetime $\mathcal{ST}$ into hyperplanes of simultaneity
γ^x	worldline γ consistent with the slicing σ identified by the point $x \in \mathcal{T}$, i.e., $\gamma^x = \{(t, \sigma_t(x)) \mid t \in \Re\}$
Γ	The set of geometrically possible worldlines (Postulate 1)
Σ	The set of $\mathcal{T}$-slicings (Def. 2) of a given underlying spacetime

Table 2

Spacetime ontology in terms of differential geometry

Global Minkowski spacetime. According to the theory of Special Relativity (Einstein, 1951; Minkowsk, 1908), spacetime $(\mathcal{ST}, g)$ has the structure of a manifold with topology $(\Re \times \Re^3)$ and a constant pseudo-Riemanian geometry induced by the metric η. That is, $(\mathcal{ST}, g) \cong ((\Re \times \Re^3), \eta)$. In a constant pseudo-Riemanian geometry the time-slices have an Euclidean geometry, i.e., the geometry of space is isomorphic to $\Re^3$. By contrast, *spatio-temporal* distances may be positive, zero, or negative. At every point $x \in \mathcal{ST}$ the metric $\eta(x)$ partitions spacetime in regions of positive, negative and zero distance with respect to x – the so-called light cone at x (right image of Fig. 2). More precisely, the metric field η of $(\mathcal{ST}, \eta)$ is symmetric and indefinite but the same at all points of spacetime. A spacetime curve γ is *time-like* if and only if the square of the length all of the tangent vectors of γ is positive[6]. The set of all time-like worldlines of a Minkowskian spacetime is:

$$\begin{aligned}\Gamma_M =_{df} \{\gamma \in \Gamma \mid \forall x \in M : \forall \tau' \in \Re : x = \gamma(\tau') \to \\ \forall \xi \in T_x M : \xi = \tfrac{d}{d\tau}\gamma(\tau)|_{\tau=\tau'} \to \eta_x(\xi, \xi) > 0\}\end{aligned} \tag{1}$$

The restriction to time-like curves in Minkowski spacetime thereby geometrically encodes the postulate of Special relativity that there is maximal velocity for particles – the speed of light.

Postulate 3. *The kinematically possible worldlines of particles in Minkowski spacetime are the time-like curves of* Γ_M.

Definition 3 (Proper time)**.** *The length of a time-like curve* $\gamma \in \Gamma_M$ *according to the metric* η *is called proper time.*

The topology $\mathcal{ST} = (\Re \times \Re^3)$ in conjunction with the metric η does not fix a unique $\mathcal{T}$-slicing σ of spacetime. That is, there are many distinct $\mathcal{T}$-slicings of $\mathcal{ST}$ in Σ. Proper time (Def. 3) is considered more fundamental than coordinate time (Def. 2) since it is directly linked to the underlying spacetime geometry and does not depend on a particular slicing of spacetime.

The spacetime of general relativity. According to the general theory of relativity, spacetime has the structure of a pseudo-Riemanian manifold with topology $\mathcal{ST} = (\Re \times M)$, where M is a three dimensional Riemanian manifold – a smooth but possibly curved manifold. In contrast to the spacetime of special relativity, the metric structure of spacetime according to general relativity is such that in the neighborhood of every point $x \in \mathcal{ST}$ the Minkowski metric η is only a linear approximation of the spacetime metric g_x at that point. In addition, the metric field g_x may vary (smoothly) from point to point.

[6]Of course, the sign is pure convention which depends on the specifics of the definition of the Minkowski metric η (Minkowsk, 1908).

	Newtonian ST	global Minkowski ST	local Minkowski ST
ST geometry metric field is	global Euclidean constant	global Minkowski constant	locally Minkowski varies smoothly
distance function $\|\xi\|^2$ is	symmetric $\|\xi\|^2 > 0$	symmetric $\|\xi\|^2 > 0$ or $\|\xi\|^2 = 0$ or $\|\xi\|^2 < 0$	symmetric
geometry of timeslices	euclidean constant	euclidean constant	locally euclidean varies smoothly
worldlines	Γ	$\Gamma_M \subset \Gamma$	$\Gamma_M \subset \Gamma$

Table 3

Classical spacetimes that are consistent with Postulates 1 and 2.

From the perspective of formal ontology, the main result of the general theory of relativity (Einstein, 1951) is that the metric field *is* spacetime (Rovelli, 1997). The abstract (mathematical) structure of the underlying manifold only provides means to label positions and identify positions across the various fields that constitute physical reality. By contrast, the metric field is part of physical reality and interacts with other fields.

3. Geometry and physical possibilities

In addition to spacetime, classical mechanics also presupposes additional spaces (manifolds) which *geometrically* encode what is *physically* possible in the spirit of the Erlangen program (Klein, 1872). In the Lagrangian framework the space that geometrically encodes the physical possibilities is the *configuration space*.

3.1. Configuration spaces

In Lagrangian mechanics, the space of *geometric* possibilities of a *single* particle (system) arises from a manifold with the topology $(\Re \times M)$ which is structurally identical to the spacetime manifold $\mathcal{ST}$. By Postulates 1 and 2 the manifold $((\Re \times M), g)$ gives rise to the class of worldlines Γ, which constitutes the space of geometrical possibilities of a single particle system. Additional constraints then further restrict what is geometrically possible to what is kinematically possible (e.g., Postulate 3).

Example 1. Consider the configuration space of the simple pendulum in Fig. 1. The system consists of a single massive particle (the bob) mounted to the ceiling via a massless rod. For many physical systems – like the simple pendulum – it is possible to constrain geometric possibilities to certain sub-manifolds of spacetime. In the case of the simple pendulum possible spatial locations are constrained to a class of curves determined by the plane of movement and the length of the rod. The location in spacetime (and therefore in configuration space) can be described by a time and an angle. The configuration space of the simple pendulum is a manifold $\mathcal{Q}_\Pi$ with the topology $\Re \times \Re$ and a geometry such that $\mathcal{Q}_\Pi \sqsubset \mathcal{ST}$. That is, $\mathcal{Q}_\Pi$ is a two-dimensional euclidean space with one spatial dimension, θ, and one temporal dimension, t, as displayed in Fig. 1. The temporal dimension is the coordinate time of the unique $\mathcal{T}$-slicing of the underlying Newtonian spacetime.

The restriction of possible locations to certain submanifolds of spacetime leads to the notion of *generalized coordinates* (Abraham and Marsden, 1978) in Lagrangian and Hamiltonian mechanics. To simplify the presentation in this paper the notion of generalized coordinates is omitted outside the sequence of examples regarding the simple pendulum.

$\triangledown$

In contrast to the configuration space of a single particle system, the space that gives rise to the geometric and kinematic possibilities of a m-particle system is a manifold $\mathcal{Q}(\mathcal{ST})$ with the topology $(\Re \times (M_1 \times \ldots \times M_m))$. That is, $\mathcal{Q}(\mathcal{ST})$ is constituted of the product of m manifolds $M_1, \ldots, M_m$ and the real numbers $\Re$. Every submanifold of the form $(\Re \times M_i)$ with $1 \leq i \leq m$ is an isomorphic copy of the underlying spacetime manifold $\mathcal{ST}$. In addition every submanifold $((\Re \times M_i), g)$ gives rise to the class of possible worldlines Γ_i – the space of geometric possibilities of the ith particle of the underlying m-particle system. Every geometric possibility of an m-particle system as a whole is a combination of m worldlines $(\gamma_1, \ldots, \gamma_m) \in \Gamma_1 \times \ldots \times \Gamma_m$ along which respectively each of the m particles can evolve within realm of geometric possibilities of the system as a whole. The dimension of the configuration space $\mathcal{Q}(\mathcal{ST})$ may be relatively small as in the case of an (insulated) two-particle system or gigantic as in the case of the observable universe as a whole with its approximately 10^{86} particles.[7] Postulate 1 is assumed to generalize to the higher-dimensional spaces $\mathcal{Q}(\mathcal{ST})$ of geometric possibilities of multiple particle systems in the obvious ways:

Postulate 4 (Geometry of configuration space)**.** *For a configuration space $\mathcal{Q}(\mathcal{ST})$ the geometry g of the underlying spacetime manifold $\mathcal{ST}$ singles out: (a) a distinguished class $\sigma(\mathcal{Q}(\mathcal{T}))$ (see below) of hypersurfaces of $\mathcal{Q}(\mathcal{ST})$ that correspond to* instants *of time and (b) a distinguished class $\Gamma_1 \times \ldots \times \Gamma_m$ of curves that correspond to possible worldlines of systems of m particles.*

Let $\mathcal{Q}(\mathcal{ST}) = (\Re \times (M_1 \times \ldots \times M_m))$ be a configuration space that satisfies Postulate 4. Let $\mathcal{T}^{\mathcal{Q}} = (\mathcal{T}_1 \times \ldots \times \mathcal{T}_m, g)$ be a system of m isomorphic copies of abstract time slices with geometry g such that there is a submanifold $(\mathcal{T}_i, g) \sqsubseteq \mathcal{T}^{\mathcal{Q}}$ for every of the $1 \leq i \leq m$ particles.

Definition 4 ($\mathcal{T}^{\mathcal{Q}}$-slicing)**.** *A $\mathcal{T}^{\mathcal{Q}}$-slicing of $\mathcal{Q}(\mathcal{ST})$ is a smooth map $\sigma : \Re \times (\mathcal{T}_1 \times \ldots \times \mathcal{T}_m) \to (\Re \times (M_1 \times \ldots \times M_m))$ with the obvious generalizations of the properties specified in Def. 1:*

(i) Every slice $\bigsqcup_{1 \leq i \leq m} \{(t, \sigma_t(x)) \mid x \in \mathcal{T}_i\}$ of the $\mathcal{T}^{\mathcal{Q}}$-slicing σ at $t \in \Re$ is a hypersurface according to the geometry g of $(\mathcal{Q}(\mathcal{ST}), g)$;
(ii) The $\mathcal{T}^{\mathcal{Q}}$-slicing respects the worldline structure of the configuration space: For every $(x_1, \ldots, x_m) \in \mathcal{T}_1 \times \ldots \times \mathcal{T}_m$ the curves $(\gamma_1^{x_1}, \ldots, \gamma_m^{x_m}) \in \{(t, \sigma_t(x_1)) \mid t \in \Re\} \times \ldots \times \{(t, \sigma_t(x_m)) \mid t \in \Re\}$ form a possible m-tuple of worldlines of a system of m particles through $(t, (\sigma_t(x_1), \ldots, \sigma_t(x_m))) \in \mathcal{Q}(\mathcal{ST})$ if and only if $(\gamma_1^{x_1}, \ldots, \gamma_m^{x_m}) \in \Gamma_1 \times \ldots \times \Gamma_m$.
(iii) The $\mathcal{T}^{\mathcal{Q}}$-slicing σ is such that for every $t \in \Re$ the mapping $\sigma_t : \mathcal{T}_i \to \sigma_t(\mathcal{T}_i)$ is an isomorphism between $\mathcal{T}_i$ and $\sigma_t(\mathcal{T}_i)$.

Classical mechanics then requires :

Postulate 5. *For every kinematically possible configuration space $\mathcal{Q}(\mathcal{ST})$ there exists a $\mathcal{T}^{\mathcal{Q}}$-slicing.*

Not all combinations of m single particle worldlines that are possible according to Postulate 5 are possible for composite systems. For example, kinematically possible worldlines of classical particles must not overlap because distinct particles cannot occupy the same spacetime location. In addition there are further constraints on kinematic possibilities that arise from the underlying spacetime structure. For example, in Minkowski spacetime kinematically possible worldlines in configuration space are time-like.

In what follows $\Gamma^{\mathcal{Q}} \subset \Gamma_1 \times \ldots \times \Gamma_m$ is the classes of kinematically ($\Gamma^{\mathcal{Q}}$) possible worldlines in the configuration space of a m particle system. The context will disambiguate between the kinematically possible worldlines in configuration spaces that arise from Newtonian spacetime or from Minkowski spacetime.

3.2. Lagrangian mechanics and dynamic possibilities

Only a subset of the kinematically possible worldlines are dynamically, i.e., physically, possible. To specify the dynamics of a physical system in the Lagrangian framework is to identify worldlines along which physically possible processes can occur and along which physically possible particles can evolve. The left part of Figure 3 depicts kinematically possible worldlines of a free particle p in a two-dimensional configuration space, i.e., $(\Re \times \Re) = \mathcal{ST} = \mathcal{Q}(\mathcal{ST})$. The paths from 'start' to 'finish' represent (parts of)

[7] In concrete coordinate based descriptions of an m-particle system in a configuration space $\mathcal{Q}(\mathcal{ST})$ of dimension $3m+1$ – one dimension for coordinate time (Def. 2) and three spatial coordinates of each of the m particles. Complex systems of worldlines $(\gamma_1, \ldots, \gamma_m)$ are described by a single curve in the $3m + 1$ dimensional space.

symbolic expression	description
$\mathcal{Q}(\mathcal{ST})$	configuration space of a m-particle system with topology $(\Re \times (M_1 \times \ldots \times M_m))$ where every M_i is associated with the space of possible locations of the i-th particle of a m particle system
$((\Re \times M_i), g)$	Subspace of the kinematic possibilities of the i-th particle of the configuration space of a m particle system – an isomorphic copy of the underlying spacetime manifold $(\mathcal{ST}, g)$
$(\mathcal{T}^{\mathcal{Q}}, g_{\mathcal{T}})$	Abstract time slice of a configuration space $\mathcal{Q}(\mathcal{ST})$ with geometry $g_{\mathcal{T}}$ such that $\mathcal{T}^{\mathcal{Q}} = (\mathcal{T}_1 \times \ldots \times \mathcal{T}_m, g_{\mathcal{T}})$ is a system of isomorphic copies of $\mathcal{T}$-slices $(\mathcal{T}_i, g_{\mathcal{T}})$ with geometry $g_{\mathcal{T}}$
$\mathcal{T}^{\mathcal{Q}}$-slicing σ	A $\mathcal{T}^{\mathcal{Q}}$-slicing σ of $(\mathcal{Q}(\mathcal{ST}), g)$ is a smooth map $\sigma : \Re \times \mathcal{T}^{\mathcal{Q}} \to \mathcal{Q}(\mathcal{ST})$
$\sigma_t(\mathcal{T}^{\mathcal{Q}})$	Hyperplane of simultaneity in configuration space according to the $\mathcal{T}^{\mathcal{Q}}$ slicing Σ. I.e., $\sigma_t(\mathcal{T}^{\mathcal{Q}}) = \{(t, (\sigma_t(x_1), \ldots, \sigma_t(x_m))) \mid (x_1, \ldots, x_m) \in \mathcal{T}^{\mathcal{Q}}\}$, such that for $1 \leq i \leq m$: σ_t is an isomorphic map from $\mathcal{T}_i$ to $\sigma_t(\mathcal{T}_i)$.
$\sigma(\mathcal{T}^{\mathcal{Q}})$	A particular slicing of configuration space $\mathcal{Q}(\mathcal{ST})$ into hyperplanes of simultaneity
Γ_i	The geometrically possible worldlines of the ith particle in the submanifold $((\Re \times M_i), g)$ of $\mathcal{Q}(\mathcal{ST})$
$\Gamma^{\mathcal{Q}}$ / $\Gamma^{\mathcal{L}}$	Set of kinematically / dynamically possible systems of worldlines of a m particle system with configuration space $\mathcal{Q}(\mathcal{ST})$: $\Gamma^{\mathcal{L}} \subset \Gamma^{\mathcal{Q}} \subset \Gamma_1 \times \ldots \times \Gamma_m$.
Prj$^{\mathcal{Q}}$ / *Prj*$^{\mathcal{L}}$	Set of kinematically / dynamically possible worldlines of spacetime along which a m-particle system can evolve

Table 4

The differential geometry of possibilities in a configuration space of a m particle system

kinematically possible woldlines, i.e., members of $\Gamma^{\mathcal{Q}}$. The curve labeled 'path taken' is the dynamically possible curve from 'start' to 'finish' and is a member of $\Gamma^{\mathcal{L}} \subset \Gamma^{\mathcal{Q}}$.

The essence of the Lagrangian framework is to identify the dynamically, i.e., physically, possible worldlines within the larger class of kinematically possible worldlines using a scalar field that is called *The Lagrangian* $\mathcal{L}$. Intuitively, at each point in configuration space the Lagrangian field describes the possible interactions of a particle at that point with other particles in the system. There is the potential (in the sense of a disposition (Choi and Fara, 2016)) for interactions at every point of spacetime and this gives rise to continuous potential fields in configuration space. The possible interactions of a particle p at a given location x with the other particles in the system are constrained by the potential fields in the neighborhood of x in conjunction with p's state of motion. The information about all possible interactions at x in conjunction with all possible states of motion of a particle at x is encoded in the value of the Lagrangian field at x.

More precisely, the Lagrangian $\mathcal{L}$ is a scalar field on the tangent bundle $T\mathcal{Q}(\mathcal{ST})$ of the configuration space $\mathcal{Q}(\mathcal{ST})$. That is, $\mathcal{L}$ is a function that takes a point $x \in \mathcal{Q}(\mathcal{ST})$ and a vector $\xi \in T_x\mathcal{Q}(\mathcal{ST})$ to the real numbers while taking into account the potential fields in the neighborhood of x. Consider the free particle p in the left of Fig. 3 and let $\gamma \sqsubset \mathcal{Q}(\mathcal{ST})$ be one of the kinematically possible worldlines displayed in the picture. In addition, let $\xi \in T_x(\mathcal{Q}(\mathcal{ST}))$ be a tangent vector on γ at point $x \in \mathcal{Q}(\mathcal{ST})$. In most classical systems the value of the Lagrangian $\mathcal{L}(x, \xi)$ is identical to the difference of the kinetic energy (K) and the potential energy (U) of p at x. The kinetic energy of a particle p at x is usually determined by p's velocity ξ at x and p's mass. The potential energy of p at x is determined by the interaction of p with (fields in) the environments near x. For the free particle in the left of Fig. 3 the potential field is zero.

Example 2 (Cont. from Example 1). As discussed above, the location of the bob of the simple pendulum is a function of the angle θ. Assume that $\theta = 0$ in the equilibrium position (the position of minimal height). The configuration space $\mathcal{Q}_\Pi$ is a two-dimensional euclidean space with one spatial dimension θ and one temporal dimension t as displayed in Fig. 1. The tangent bundle (Appendix A) $T\mathcal{Q}_\Pi$ of $\mathcal{Q}_\Pi$ consists of points $((\theta, t), \dot{\theta})$ such that $\dot{\theta} \in T_{(\theta,t)}\mathcal{Q}_\Pi$. The Lagrangian field $\mathcal{L}_\Pi$ is a mapping of type $\mathcal{L}_\Pi : T\mathcal{Q}_\Pi \to \Re$. As in most classical systems the Lagrangian field of the simple pendulum varies across space but not across time and therefore the temporal coordinate is omitted in the specification of the Lagrangian field. The Lagrangian field of the simple pendulum is defined as:

$$\mathcal{L}_\Pi(\theta, \dot{\theta}) = K(\dot{\theta}) - U(\theta) = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) \tag{2}$$

Here m is the mass of the bob, g is the gravitational constant which specifies the gravitational interaction of the Earth with massive objects near its surface and l is the length of the massless rod.

∇

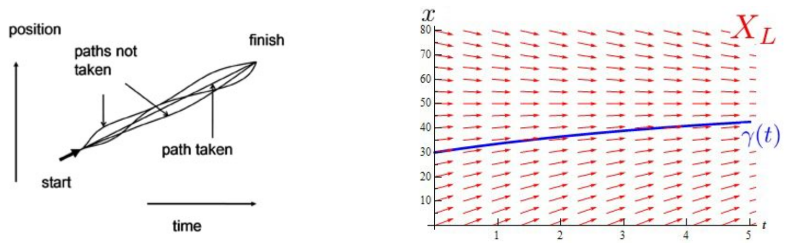


Fig. 3. Kinematically and dynamically possible worldlines (left)(Quincey, 2006); Dynamically possible worldline and the underlying Lagrangian vector field (right) (Dawkins, modified).

To determine the space of dynamically, i.e., physically, possible worldlines of a physical system with configuration space $\mathcal{Q}(\mathcal{ST})$ is to identify kinematically possible curves that are *stationary* with respect to the Lagrangian field $\mathcal{L}$. The formal expression of this this statement is known as *Hamilton's principle* (Postulate 7 of Appendix B). Consider, again, the left part of Fig. 3. Intuitively, the curve labeled 'path taken' is the dynamically possible curve from 'start' to 'finish' because infinitesimal changes of the curve in the interval between 'start' to 'finish' do not change the 'sum' (i.e., the integral) of all values for $\mathcal{L}$ along the curve within this interval (See Eq. 25 of Appendix B). An infinitesimal change of the curve means that the system takes a slightly different path in the configuration space to get from 'start' to 'finish' (Def. 6 of Appendix B).

The Lagrangian (scalar) field $\mathcal{L}$ in conjunction with Hamiltons principle gives rise to a *Lagrangian vector field* $X^{\mathcal{L}}$ on the configuration space $\mathcal{Q}(\mathcal{ST})$. The dynamically possible worldlines $\gamma \in \Gamma^{\mathcal{L}}$ are the integral curves (Eq. 23 of Appendix A.2) of the vector field $X^{\mathcal{L}}$ (Def. 5 of Appendix B). The image in the right of Fig. 3 displays the dynamically possible worldline $\gamma \in \Gamma^{\mathcal{L}}$ as an integral curve of the Lagrangian vector field $X^{\mathcal{L}}$. Roughly, the image is a graphic representation of the geometry underlying an equation of motion of the form $\vec{F}(\gamma(t)) = m\ddot{\gamma}(t)$, where $\vec{F}$ is the Lagrangian vector field and the curves $\gamma \in \Gamma^{\mathcal{L}}$ that make the equation true are the integral curves and therefore the dynamically possible worldlines (Appendix B). $\ddot{\gamma}(t)$ is the rate of change of the rate of change at $\gamma(t)$ for all $t \in \Re$, i.e., the acceleration on a particle moving along γ at $\gamma(t)$.

Example 3 (Cont. from Example 2). The Lagrangian field of Eq. 2 of Example 2 gives rise to the Lagrangian vector field. This field maps every point in configuration space to the gradient (direction of steepest slope) of the potential field U at that point:

$$X^{\mathcal{L}}_{\Pi} : (\theta, t) \in \mathcal{Q}_{\Pi} \mapsto \frac{\delta \mathcal{L}_{\Pi}}{\delta \theta}|_{(\theta,t)} \in T_{(\theta,t)}\mathcal{Q}_{\Pi} \tag{3}$$

Since the potential field (like the Lagrangian field) varies across space but not across time the gradient is a vector along the θ-axis of length $mgl(1 - \sin\theta)$. The dynamically possible worldlines $\gamma \in \Gamma^{\mathcal{L}}_{\Pi}$ are the integral curves of this vector field. The integral curves are the parametric curves $\gamma(t) = \theta(t)$ that make the equation of motion

$$\ddot{\theta}(t) + \frac{g}{l}\sin\theta(t) = 0 \tag{4}$$

true. The equation of motion is obtained from the Lagrangian field via Hamilton's principle as described in Appendix B. A particular curve γ^x among the set of dynamic possibilities (a specific solution) is picked out by specifying a point $x = (\theta, t)$ in conjunction with the rate of change $\dot{\theta}$ of γ at x – the *initial state* of the particular system.

∇

3.3. Projecting onto spacetime

Every kinematic and therefore every dynamic possibility in the space of kinematic and dynamic possibilities $\Gamma^{\mathcal{L}} \subset \Gamma^{\mathcal{Q}}$ is a system of worldlines $(\gamma_1, \ldots, \gamma_m) \in \Gamma^{\Gamma^{\mathcal{Q}}}$ along which a system with m particles can evolve according to the laws of kinematics and dynamics. In the presented geometric framework possibilities in configuration space need to be projected back into spacetime:

Postulate 6. *The kinematic possibilities of a m-particle system in configuration space correspond to mereological sums of worldlines in spacetime. The m-tuples of possibilities $(\gamma_1, \ldots, \gamma_m) \in \Gamma^{\mathcal{Q}}$ are related to worldlines $\gamma \sqsubseteq \mathcal{ST}$ via projections of the form:*

$$prj^{\mathcal{Q}}_m =_{df} (\gamma_1, \ldots, \gamma_m) \in \Gamma^{\mathcal{Q}} \mapsto \left(\bigsqcup \{\gamma_1, \ldots, \gamma_m\} \right) \sqsubseteq \mathcal{ST}, \tag{5}$$

where the union of manifolds $\bigsqcup\{\gamma_1, \ldots, \gamma_m\}$ represents mereological sums of worldlines.

The set of kinematically possible worldlines of spacetime along which kinematically possible m-particle systems can evolve is:

$$Prj^{\mathcal{Q}}_m =_{df} \{prj^{\mathcal{Q}}_m(\gamma_1, \ldots, \gamma_m) | (\gamma_1, \ldots, \gamma_m) \in \Gamma^{\mathcal{Q}}\} \tag{6}$$

Similarly $prj^{\mathcal{L}}_m$ is a map that take m-tuples of dynamically possible worldlines of configuration space to submanifolds of spacetime. $Prj^{\mathcal{L}}_m$ is the corresponding set of spacetime (sub)manifolds. The mappings $prj^{\mathcal{Q}}_m$ and $prj^{\mathcal{L}}_m$ a play a critical role in constraining the interpretation of the predicates of the formal theory.

4. A logic of metaphysical, kinematic, and dynamic possibilities and their actualization in spacetime

At the ontological level a three-dimensionalist (Lowe, 2002) (BFO- or DOLCE-like (Smith, 2016; Gangemi et al., 2003)) top-level ontology is assumed. In such an ontology there is a fundamental distinction between the categories of continuants and occurrents. Physical particles and fields are continuants and physical interactions and movements are occurrents. To specify the classes of kinematic and dynamic possibilities in the framework of a formal ontology a two-dimensional modal predicate logic is used. The modal logic is two- dimensional because there are two kinds of modalities in the formal ontology corresponding to two kinds of accessibility relations that characterize the underlying physics (details are discussed in Sec. 4.2 and Sec. 7.1). The computational representation is realized using the HOL-based theorem proving environment Isabelle (Paulson and Nipkow, 2017; Paulson, 1994). (Details in Sec. 7.3.) The second order features of Isabelle/HOL are essential to the encoding of the modal aspects of the formal theory (Benzmüller, 2015; Benzmüller and Woltzenlogel Paleo, 2015). In this context it then makes sense to employ the resources of the underlying higher order logic also in the object language of the formal theory. Second order features are, however, used only in a very limited way to (a) avoid the kinds of axiomatic schemata that are often used for formalizing mereological sums (Sec. 5.1) and (b) for expressing that there are finitely many atomic entities (Sec. 6.3).

In the first two subsections the syntax and the semantics of the formal language are specified in the standard ways (Gabbay, 2003; Hughes and Cresswell, 2004). A specific class of models, $\mathcal{KS}$-structures, are introduced in the third subsection. $\mathcal{KS}$-structures are intended to connect the physics encoded in the differential geometry of the first part of the paper with the formal ontology in the second part.

4.1. Syntax

The formal language includes three disjoint sets of variable symbols: *Var*$_{ST}$, *VAR*$_{ST}$, and *Var*$_E$. *Var*$_{ST}$ contains variables denoted by letters u, v, w, possibly with subscripts (u_1, v_2, etc.). *VAR*$_{ST}$ contains variables denoted by capital letters A, B, etc. *Var*$_E$ contains variables x, y, z, possibly with subscripts. *VAR*$_E$ contains variables denoted by capital letters X, Y, etc. *Var* is the union *Var*$_{ST} \cup$ *VAR*$_{ST} \cup$ *Var*$_E \cup$ *VAR*$_E$. *Pred* is a set of predicate symbols. If F is a n-ary predicate symbol in *Pred* and $t_1, \ldots, t_n$ are variables in *Var* then $F\,t_1 \ldots t_n$ is a well-formed formula. Complex, non-modal formulas are formed inductively in the usual ways, i.e., if α and β are well-formed formulas, then so are $\neg\alpha$, $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \rightarrow \beta$, $(x)\alpha$, $(\exists x)\alpha$ (Gabbay, 2003; Hughes and Cresswell, 2004). All quantification is restricted to a single sort of variables.[8] If not marked explicitly, restrictions on quantification are understood by conventions on variable usage.

Finally, the modalities $\Box^\Gamma$, $\Box^\Sigma$, $\Diamond^\Gamma$ and $\Diamond^\Sigma$ are included in the formal language, i.e., if α is a well-formed formula, then so are $\Box^i\alpha$ and $\Diamond^i\alpha$ with $i \in \{\Gamma, \Sigma\}$.

4.2. Semantics

A model of such a multi-dimensional sorted modal language is a structure $\langle \mathcal{D}_{ST}, \mathcal{D}_E, \mathcal{K}, \mathsf{V}\rangle$. $\mathcal{D}_{ST}$ and $\mathcal{D}_E$ are non-empty domains of quantification. $\mathcal{K}$ is a non-empty set of possible worlds. V is the interpretation function: if $F \in$ *Pred* is a n-ary predicate then $\mathsf{V}(F)$ is a set of $n+1$-tuples of the form $(\mathsf{d}_1, \ldots, \mathsf{d}_n, \kappa)$ with $\mathsf{d}_1, \ldots, \mathsf{d}_n \in \mathcal{D}$ and $\kappa \in \mathcal{K}$, where $\mathcal{D} = \mathcal{P}(\mathcal{D}_{ST}) \cup \mathcal{D}_{ST} \cup \mathcal{D}_E$. ($\mathcal{P}(\mathcal{D}_{ST})$ is the set of all subsets of $\mathcal{D}_{ST}$). In all possible worlds $\kappa \in \mathcal{K}$ the variables respectively range over all the members of $\mathcal{D}_{ST}$ and $\mathcal{D}_E$. A variable assignment μ is a function such that (i) for every variable $u \in$ *Var*$_{ST}$, $\mu(u) \in \mathcal{D}_{ST}$, (ii) for every variable $x \in$ *Var*$_E$, $\mu(x) \in \mathcal{D}_E$, (iii) for every variable $A \in$ *VAR*$_{ST}$, $\mu(A) \in \mathcal{P}(\mathcal{D}_{ST})$, and (iv) for every variable $X \in$ *VAR*$_E$, $\mu(X) \in \mathcal{P}(\mathcal{D}_E)$.

Every well-formed formula has a truth value which is defined as follows:

$$
\begin{array}{l}
\mathsf{V}_\mu(F\,t_1 \ldots t_n, \kappa) = 1 \text{ if } \langle \mu(t_1), \ldots, \mu(t_n), \kappa\rangle \in \mathsf{V}(F) \text{ and 0 otherwise;}\\
\mathsf{V}_\mu(\alpha \wedge \beta, \kappa) = 1 \text{ if } \mathsf{V}_\mu(\alpha, \kappa) = 1 \text{ and } \mathsf{V}_\mu(\beta, \kappa) = 1 \text{ and 0 otherwise;}\\
\mathsf{V}_\mu(\alpha \vee \beta, \kappa) = 1 \text{ if } \mathsf{V}_\mu(\alpha, \kappa) = 1 \text{ or } \mathsf{V}_\mu(\beta, \kappa) = 1 \text{ and 0 otherwise;}\\
\mathsf{V}_\mu(\neg\alpha, \kappa) = 1 \text{ if } \mathsf{V}_\mu(\alpha, \kappa) = 0 \text{ and 0 otherwise;}\\
\mathsf{V}_\mu(\alpha \rightarrow \beta, \kappa) = 1 \text{ if } \mathsf{V}_\mu(\alpha, \kappa) = 0 \text{ or } \mathsf{V}_\mu(\beta, \kappa) = 1 \text{ and 0 otherwise;}\\
\mathsf{V}_\mu((t)\alpha, \kappa) = 1 \text{ if } \mathsf{V}_\rho(\alpha, \kappa) = 1 \text{ for every } t\text{-alternative } \rho \text{ of } \mu \text{ and 0 otherwise,}\\
\qquad \text{where a } t\text{-alternative } \rho \text{ of } \mu \text{ is a variable assignment that assigns the same domain}\\
\qquad \text{members to all variables except for } t;\\
\mathsf{V}_\mu(\Box^\Gamma\alpha, \kappa) = 1 \text{ if } \mathsf{V}_\mu(\alpha, \kappa') = 1 \text{ for all } \kappa' \in \mathcal{K} \text{ such that } R^\Gamma(\kappa, \kappa') \text{ and 0 otherwise,}\\
\qquad \text{where } R^\Gamma \text{ is the accessibility relation on } \mathcal{K} \text{ for } \Box^\Gamma.\\
\mathsf{V}_\mu(\Box^\sigma\alpha, \kappa) = 1 \text{ if } \mathsf{V}_\mu(\alpha, \kappa') = 1 \text{ for all } \kappa' \in \mathcal{K} \text{ such that } R^\Sigma(\kappa, \kappa') \text{ and 0 otherwise,}\\
\qquad \text{where } R^\Sigma \text{ is the accessibility relation on } \mathcal{K} \text{ for } \Box^\Sigma.
\end{array}
\tag{7}
$$

A well-formed formula α is true in $\langle \mathcal{D}_{ST}, \mathcal{D}_E, \mathcal{K}, \mathsf{V}\rangle$, i.e. $\mathsf{V}_\mu(\alpha) = 1$, if and only if $\mathsf{V}_\mu(\alpha, \kappa) = 1$ for all $\kappa \in \mathcal{K}$ and all assignments μ. Formula α is valid if α is true in all models. To simplify the presentation, the explicit distinction between V and V_μ will be omitted. Variables in the object language are written in *italics* and for corresponding domain members the Sans Serif font is used.

$\mathcal{K} = \Gamma^\mathcal{L} \times \Sigma$ is a set of possible worlds which has the internal structure of a product of two sets $\Gamma^\mathcal{L}$ and Σ. $\mathcal{K}$ gives rise to a product frame of a two-dimensional modal logic (Gabbay, 2003). The accessibility relations R^Γ and R^Σ of the resulting frame structure are defined as:

$$
\begin{array}{l}
R^\Gamma =_{df} \{\langle\langle\gamma_1, \sigma\rangle, \langle\gamma_2, \sigma\rangle\rangle \mid \langle\gamma_1, \sigma\rangle, \langle\gamma_2, \sigma\rangle \in \mathcal{K}\}\\
R^\Sigma =_{df} \{\langle\langle\gamma, \sigma_1\rangle, \langle\gamma, \sigma_2\rangle\rangle \mid \langle\gamma, \sigma_1\rangle, \langle\gamma, \sigma_2\rangle \in \mathcal{K}\}
\end{array}
\tag{8}
$$

R^Γ and R^Σ are both reflexive, symmetric, and transitive. That is, both, $(\Gamma^\mathcal{L}, R^\Gamma)$ and (Σ, R^Σ) form equivalence frames. In addition the two accessibility relations are compositionally related as folows (Fig. 4):

[8] In the computational realization distinct quantifiers are introduced for each sort of variables.

$$\sigma^1 \xrightarrow{R^{\Sigma}} \sigma^2 \qquad \mathcal{F}_{\Sigma}$$

$$\begin{tikzcd} \gamma^2 \\ \gamma^1 \arrow[u, "R^{\Gamma}"] \end{tikzcd} \qquad \mathcal{F}_{\Gamma}$$

$$\begin{tikzcd} \langle \gamma^2, \sigma^1 \rangle \arrow[r, "R^{\Sigma}"] & \langle \gamma^2, \sigma^2 \rangle \\ \langle \gamma^1, \sigma^1 \rangle \arrow[u, "R^{\Gamma}"] \arrow[r, "R^{\Sigma}"] & \langle \gamma^1, \sigma^2 \rangle \arrow[u, "R^{\Gamma}"'] \end{tikzcd} \qquad \mathcal{F}_{\Sigma} \times \mathcal{F}_{\Gamma}$$

Fig. 4. Two accessibility relations in distinct frames (left) and two accessibility relations in a two dimensional product frame (right) (adapted from the work of Gabbay (2003, p. 125))

$$\begin{array}{l} \text{If there are } \gamma_1, \sigma_1, \gamma_2, \sigma_2 \text{ such that } R^{\Gamma}(\langle \gamma_1, \sigma_1 \rangle, \langle \gamma_2, \sigma_1 \rangle) \text{ and } R^{\Sigma}(\langle \gamma_2, \sigma_1 \rangle, \langle \gamma_2, \sigma_2 \rangle) \\ \text{then the pair } \langle \gamma_1, \sigma_2 \rangle \in \mathcal{K} \text{ is such that } R^{\Sigma}(\langle \gamma_1, \sigma_1 \rangle, \langle \gamma_1, \sigma_2 \rangle) \text{ and } R^{\Gamma}(\langle \gamma_1, \sigma_2 \rangle, \langle \gamma_2, \sigma_2 \rangle); \\ \text{If there are } \gamma_1, \sigma_1, \gamma_2, \sigma_2 \text{ such that } R^{\Sigma}(\langle \gamma_1, \sigma_1 \rangle, \langle \gamma_1, \sigma_2 \rangle) \text{ and } R^{\Gamma}(\langle \gamma_1, \sigma_2 \rangle, \langle \gamma_2, \sigma_2 \rangle) \\ \text{then the pair } \langle \gamma_2, \sigma_1 \rangle \in \mathcal{K} \text{ is such that } R^{\Gamma}(\langle \gamma_1, \sigma_1 \rangle, \langle \gamma_2, \sigma_1 \rangle) \text{ and } R^{\Sigma}(\langle \gamma_2, \sigma_1 \rangle, \langle \gamma_2, \sigma_2 \rangle). \end{array} \tag{9}$$

The ways in which these abstract and formal definitions are related to the underlying (meta)physics are discussed in the course of the development of the formal theory and summarized in Sec. 7.

The formal theory includes the rules and axioms of Isabelle/HOL[9] as well as the S5-axiom schemata $K_{\Box^i}$, $T_{\Box^i}$, and $5_{\Box^i}$ for $i \in \{\Gamma, \Sigma\}$ (Hughes and Cresswell, 2004). $\Diamond^i$ is defined in the usual way as the dual of $\Box^i$ for $i \in \{\Gamma, \Sigma\}$ ($D_{\Diamond^i}$). The Barcan formula and its converse are true in all models ($BC_{\Box^i}$).

$$\begin{array}{llll} D_{\Diamond^i} & \Diamond^i \alpha \equiv \neg \Box^i \neg \alpha & K_{\Box^i} & \Box^i(\alpha \to \beta) \to (\Box^i \alpha \to \Box^i \beta) \\ T_{\Box^i} & \Box^i \alpha \to \alpha & BC_{\Box^i} & (x)\Box^i \alpha \leftrightarrow \Box^i (x)\alpha \\ 5_{\Box^i} & \Diamond^i \alpha \to \Box^i \Diamond^i \alpha & MS_{\Box} & \Box^{\Gamma}\Box^{\Sigma}\alpha \leftrightarrow \Box^{\Sigma}\Box^{\Gamma}\alpha \end{array}$$

Lemma 1 ((Hughes and Cresswell, 2004, p. 249))**.** *The system S5 (K+T+5) + BC is sound for all equivalence frames.*

Both modal operators are independent and the order of their application is immaterial ($MS_{\Box}$).

Lemma 2. *$MS_{\Box}$ is true in all product frames $\mathcal{K} = (\Gamma^{\mathcal{L}} \times \Sigma, R^{\Gamma}, R^{\Sigma})$.*

The proof follows immediately from Fig. 4 and Eq. 9. In what follows the notation $\Box\alpha$ will be used as an abbreviation for $\Box^{\Gamma}\Box^{\Sigma}\alpha$ or equivalently $\Box^{\Sigma}\Box^{\Gamma}\alpha$. Similarly for $\Diamond\alpha$.

All axioms of the formal theory below are true in all possible worlds have an implicit leading $\Box$ operator. In addition, leading universal quantifiers are omitted.[10] Axioms $BC_{\Box^i}$ and $MS_{\Box}$ ensure that the order of leading universal quantifiers and leading $\Box$ operators is immaterial.

4.3. $\mathcal{KS}$ structures

According to classical mechanics the number and kinds of fundamental particles is constant and so is the number of spatiotemporal regions.[11][12] In what follows the number of particles that exist is a metaphysical parameter of the formal theory – the parameter m in Eq. 10. Within such metaphysical constraints of

[9] Again, none of the second order features of Isabelle/HOL that are used in the object language of the formal theory (D_{Sum} and A19) are essential. The theory could be expressed easily in a regular two-dimensional first order modal predicate logic with identity.

[10] In the computational representation all quantifiers and modal operators are are stated explicitly.

[11] Particle creation and annihilation is the subject of quantum field theory (Teller, 1997; Brown and Harré, 1996).

[12] At this point it is assumed that in the context of a classical framework it is consistent to believe that the expansion of the universe leads to the expansion of existing regions but not to the creation of new regions.

what particles exist, the kinematic component of classical mechanics constrains along which worldlines the metaphysically given particles can evolve and at which worldlines physical processes that involve those particles can be located. Within the class of kinematic possibilities there is the class of dynamic possibilities which, according to Hamilton's principle, are the stationary curves of a system that is characterized by a Lagrangian field $\mathcal{L}$ (Sec. 3.2, Appendix B). The specific Lagrangian fields that determine the dynamic possibilities of specific physical systems are often determined empirically (North, 2009) and are treated logically as parameters of the formal theory. The relationships between metaphysical, geometrical, kinematic, and dynamic possibilities are explicated in the metalanguage of the formal ontology using $\mathcal{KS}$-structures of the form

$$\mathcal{KS}(m, \mathcal{L}) =_{df} \langle \mathcal{D}_{ST}, \mathcal{D}_E, \mathcal{K}, \mathsf{V}, \sqcup, \overline{\sqcap}, \mathsf{TS}, \mathsf{InstST}, \mathsf{AtE} \rangle. \tag{10}$$

The properties of $\mathcal{KS}$-structures are discussed in the remainder of this subsection and in the course of the development of the axiomatic theory.

4.3.1. Regions of spacetime

The members of $\mathcal{D}_{ST}$ include the non-empty sub-manifolds of spacetime $\mathcal{ST}$. In particular, $\mathcal{ST} \in \mathcal{D}_{ST}$. Similarly, the set of all non-empty sub-manifolds of the members of $Prj^{\mathcal{Q}}$ (the set of all the mereological sums of worldlines along which processes that involve systems constituted of m particles can possibly evolve according to the geometry and the kinematics of spacetime (Eq. 5)) is a subset of $\mathcal{D}_{ST}$, i.e., $\{\gamma \in \mathcal{D}_{ST} \mid \exists \gamma' \in Prj^{\mathcal{Q}} \wedge \gamma \sqsubseteq \gamma'\} \neq \emptyset$. In addition, for all slicings $\sigma \in \Sigma$, the set of non-empty sub-manifolds of the concrete time slices on the slicing σ are members of $\mathcal{D}_{ST}$, i.e., $\{\mathsf{u} \in \mathcal{D}_{ST} \mid \exists \sigma \in \Sigma : \exists t \in \Re : \mathsf{u} \sqsubseteq \sigma_{\mathrm{t}}(\mathcal{T})\} \neq \emptyset$.

4.3.2. Physical possibilities

$\mathcal{K} = \Gamma^{\mathcal{L}} \times \Sigma$ is a set of physical possibilities. $\Gamma^{\mathcal{L}}$ is a set of dynamically possible worldlines along which worlds/systems with m particles can evolve in a configuration space $\mathcal{Q}(\mathcal{ST})$ with a Lagrangian field $\mathcal{L}$ (Eq.27 of Appendix B). Σ is a set of $\mathcal{T}^{\mathcal{Q}}$-slicings σ (Sec.3). The accessibility relations R^{Γ} and R^{Σ} and their properties of reflexivity, symmetry, and transitivity in conjunction with the fact that R^{Γ} and R^{Σ} commute in the sense of Eq. 9 explicate a number of metaphysical commitments (to be discussed in more detail in Sec. 7): (1) there are no logical distinctions among the dynamically possible worldlines in $\Gamma^{\mathcal{L}}$, i.e., all physically possible worlds are equivalent from a logical perspective; (2) there are no logical distinctions among the possible $\mathcal{T}^{\mathcal{Q}}$-slicings in Σ – in particular, there is no preferred slicing of spacetime; and (3) there is no logical primacy of one accessibility relation over the other. That is, the logic of the relation between physically possible worlds is independent of the logic of slicings of spacetime and vice versa. In the object language this is expressed by the axiom $MS_{\Box}$.

4.3.3. The domain of entities

On the intended interpretation $\mathcal{D}_E$ is the domain of possible entities (particulars and universals) in a world with m atoms – the set AtE with $\mathsf{AtE} \subseteq \mathcal{D}_E$. While the number and kinds of atomic particles that exist are fixed, whether and which complex continuants are formed by the given atomic entities is a contingent matter. Whatever complex entities can exist, however, must obey the laws of mereology in a way that is consistent with the mereology of the underlying spacetime. The domain $\mathcal{D}_E$ of possible entities and the domain $\mathcal{D}_{ST}$ of regions of spacetime are linked via the relation of instantiation $\mathsf{InstST} \subseteq \mathcal{D}_E \times \mathcal{D}_E \times \mathcal{D}_{ST} \times \mathcal{K}$.

As indicated in Eq. (10), the sets $\sqcup$, $\overline{\sqcap}$, TS, InstST, AtE of $\mathcal{KS}(m, \mathcal{L})$ serve as the interpretations of the axiomatic primitives of the formal theory in the context of the worlds in $\mathcal{K}$. A summary is given in Table 5.

4.4. An example model

For illustrative purposes and to check the consistency of the formal theory it will be useful to use a simple and finite set-theoretic model to illustrate some important aspects of the $\mathcal{KS}$-structures that were introduced above. Due to its simplistic nature this model falls short of capturing many of the topological,

symbolic expression	description	see
$\mathcal{K}$	Product frame with possible worlds of the form $\langle\gamma,\sigma\rangle \in \Gamma^{\mathcal{L}}\times\Sigma$	Eqs. 8, 9, Lemma 2
R^{Γ}	accessibility relation $R^{\Gamma} \subseteq (\Gamma^{\mathcal{L}} \times \Sigma) \times (\Gamma^{\mathcal{L}} \times \Sigma)$ relating the members of $\Gamma^{\mathcal{L}}$	Eq. 8, Fig. 4
R^{Σ}	accessibility relation $R^{\Gamma} \subseteq (\Gamma^{\mathcal{L}} \times \Sigma) \times (\Gamma^{\mathcal{L}} \times \Sigma)$ relating the members of Σ	Eq. 8, Fig. 4
$\mathcal{KS}(m,\mathcal{L})$	Intended class of models parametrized by a fixed number m of atomic particles and a Lagrangian $\mathcal{L}$.	Eq. 10
$\mathcal{D}_{ST}$	domain of regions formed by sub manifolds of spacetime in including spacetime itself, kinematically possible worldlines, time slices, etc.	Sec. 4.3.1
$\sqsubseteq$	the submanifold relation between members of $\mathcal{D}_{ST}$, i.e., $\sqsubseteq \subset \mathcal{D}_{ST} \times \mathcal{D}_{ST}$	
$\mathcal{D}_E$	domain of entities formed by particulars (continuants and occurrents) and universals	Sec. 4.3.3
V	interpretation function	
$\underline{\sqcup}$	set serving as interpretation for the predicate signifying the mereological union of two regions of spacetime; $\underline{\sqcup} \subseteq (\mathcal{D}_{ST} \times \mathcal{D}_{ST} \times \mathcal{D}_{ST} \times \mathcal{K})$	Eq. 12
$\overline{\sqcap}$	set serving as interpretation for the predicate signifying the mereological intersection of two regions of spacetime; $\overline{\sqcap} \subseteq (\mathcal{D}_{ST} \times \mathcal{D}_{ST} \times \mathcal{D}_{ST} \times \mathcal{K})$	Eq. 12
TS	set serving as interpretation for the primitive predicate for time slices; $\mathsf{TS} \subset \mathcal{D}_{ST} \times \mathcal{K}$	Eq. 14
InstST	set serving as interpretation for the primitive instantiation predicate; $\mathsf{InstST} \subset \mathcal{D}_E \times \mathcal{D}_E \times \mathcal{D}_{ST} \times \mathcal{K}$	Sec. 6.1
AtE	set serving as interpretation for the primitive predicate for atomic entities; $\mathsf{AtE} \subset \mathcal{D}_E \times \mathcal{K}$	Sec. 6.3

Table 5

The class of intended models $\mathcal{KS}(m,\mathcal{L}) =_{df} \langle \mathcal{D}_{ST}, \mathcal{D}_E, \mathcal{K}, \mathsf{V}, \underline{\sqcup}, \overline{\sqcap}, \mathsf{TS}, \mathsf{InstST}, \mathsf{AtE}\rangle$. (Overview)

geometric and differential structures that were discussed in the first part of the paper. Nevertheless it will make it easier to link the first part of the paper to the more logic and ontology orientated second part.

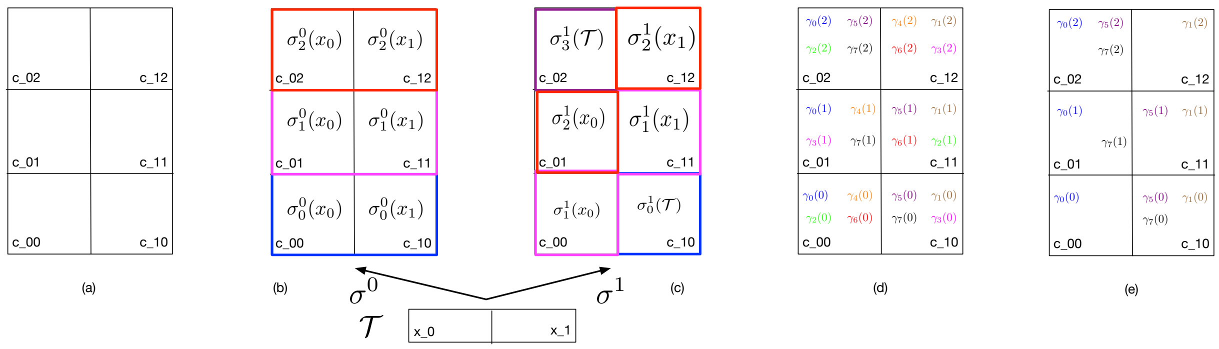


Fig. 5. A simple toy model.

Consider a two-dimensional 'spacetime' with six distinct locations as indicated in Fig. 5(a), i.e., $\mathcal{ST} = \{c_{00}, c_{10}, c_{01}, c_{11}, c_{02}, c_{12}\}$ according to the labeling in the figure. In this spacetime the sub-manifold relation simplifies to the subset relation and thus the domain of spacetime regions is the set of non-empty subsets of $\mathcal{ST}$, i.e., $\mathcal{D}_{ST} = \{r \subseteq \mathcal{ST} \mid r \neq \emptyset\}$.

Let $\mathcal{T} = (\{x_0, x_1\}, g)$ be an abstract time slice (Def. 1, Fig. 2 (left)) with a geometry g defined as: $g(u,v) = \begin{cases} 0 \; if \; u = v \\ 1 \; if \; u \neq v \end{cases}$. Suppose further that there are two slicing of spacetime $\Sigma = \{\sigma^0, \sigma^1\}$ such that the slicing σ^0 is defined as $\sigma^0_0(x_i) = c_{i0}$, $\sigma^0_1(x_i) = c_{i1}$, $\sigma^0_2(x_i) = c_{i2}$ (Fig. 5(b)) and the slicing σ^1 is defined as $\sigma_0^1(x_i) = c_{10}$, $\sigma_1^1(x_0) = c_{00}$, $\sigma_1^1(x_1) = c_{11}$, $\sigma_2^1(x_0) = c_{01}$, $\sigma_2^1(x_1) = c_{12}$, $\sigma_3^1(x_i) = c_{02}$ for $i \in \{0, 1\}$ (Fig. 5(c)). That is, σ_i^0 for $i = 0, 1, 2$ are the three time slices that compose the slicing σ^0, and σ_i^1 for $i = 0, 1, 2, 3$ are the four time slices that compose the slicing σ^1. Clearly, unlike σ^0, σ^1 is not an isomorphism as required in Def. 1 because its inverse does not always exist. This is an artifact of the finite nature of $\mathcal{ST}$.

The worldlines that are kinematically possible with respect to the slicing σ^0 are visualized in Fig. 5(d). The worldlines that are kinematically possible with respect to the slicings σ^0 and σ^1 are visualized in Fig. 5(e) and listed in Eq. 11. The temporal parameter of the γ_i is understood to correspond to the (coordinate) time according to the slicing σ^0 of $\mathcal{ST}$, i.e., $\tau \in 0 \ldots 2$. The aim here is to illustrate that, according to Postulate 3 which imposes a maximal velocity, it impossible for a particle to move from one position in a timeslice to another position in the very same timeslice.

$$\begin{array}{ll} \gamma_0(\tau) = c_{0\tau},\ \tau \in 0 \ldots 2; & \gamma_5(0) = c_{10}, \gamma_5(1) = c_{11}, \gamma_5(2) = c_{02}; \\ \gamma_1(\tau) = c_{1\tau},\ \tau \in 0 \ldots 2; & \gamma_7(0) = c_{10}, \gamma_7(1) = c_{01}, \gamma_7(2) = c_{02}. \end{array} \tag{11}$$

If one demands, in accordance with classical mechanics, that distinct particles cannot occupy the same location in spacetime then worldlines of distinct particles cannot intersect. In a world with two atomic particles the worldlines γ_0 and γ_1 are the only kinematically possible particle worldlines. The only kinematically possible complex worldline in this world is (γ_0, γ_1) such that $\Gamma^{\mathcal{Q}} = \{(\gamma_0, \gamma_1)\}$. The projection of (γ_0, γ_1) from the configuration space $\mathcal{Q}$ onto the spacetime manifold $\mathcal{ST}$ is $prj_m^{\mathcal{Q}}((\gamma_0, \gamma_1)) = \bigsqcup\{\gamma_0, \gamma_1\} =_{df} \gamma_0^2$.

The corresponding laws of physics that are encoded in the Lagrangian field $\mathcal{L}$ are such that neither of the two atomic particles can change its spatial location , i.e., $\dot{x} = 0$. Suppose that neither particle has potential energy. In such a world the underlying Lagrangian is of the form $\mathcal{L}(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 = 0$ and $\Gamma^{\mathcal{Q}} = \Gamma^{\mathcal{L}} = \{(\gamma_0, \gamma_1)\}$ is the set of dynamically possible worldlines in configuration space. The projection on the spacetime manifold is γ_0^2.

In what follows the two atoms are called At_0 and At_1. Respectively At_0 and At_1 evolve along the worldlines γ_0 and γ_1. In addition it is assumed that there exists a complex object Compl_0 that is constituted by the atoms At_0 and At_1. The worldline of Compl_0 is γ_0^2. On these assumptions $\mathcal{K} \subseteq \Gamma^{\mathcal{L}} \times \Sigma$, the set of physical possibilities, is $\mathcal{K} = (\{(\gamma_0, \gamma_1)\}, \{\sigma^0, \sigma^1\})$.

Within the realm of physical possibilities in $\mathcal{K}$ an ontology that commits to the existence of continuant particulars, occurrent particulars as well as to universals which are instantiated by such particular entities (e.g., BFO (Smith, 2016), DOLCE (Gangemi et al., 2003), etc.) then is committed to acknowledging the existence of at least the following entities:

(i) The continuants At_0, At_1, and Compl_0;
(ii) The occurrents Occ_0, Occ_1, and Occ_0 (the respective lives of the above continuants);
(iii) At least two universals (UC_0 and UO_0) which are respectively instantiated by the continuants and occurrents.

On the given assumptions the set of physically possible entities is

$$\mathcal{D}_E = \{\mathsf{At}_0, \mathsf{At}_1, \mathsf{Compl}_0, \mathsf{Occ}_0, \mathsf{Occ}_1, \mathsf{Occ}_0, \mathsf{UC}_0, \mathsf{UO}_0\}.$$

This example is constructed to minimize the number of physical possibilities without being trivial. This simplicity of the example model greatly reduces the complexity of the (mostly brute force and case-based) proofs that establish that the axioms of the formal ontology are satisfied in this toy model. In general, within the framework of highly expressive languages, the more skilled the developer of the of the computational realization of an ontology the more realistic and sophisticated the models that are realized will be.

5. Mereology of spacetime

In this section a formal theory of regions of spacetime that is based on work by Champollion and Krifka (2015); Bittner and Donnelly (2004, 2006a) is developed. The theory is mereological in nature and thereby mirrors the fundamental assumptions (1) and (2) of Sec. 1.2. As a mereological theory it falls short to explicate many of the topological and geometric aspects of the manifold structures that are critical for capturing the commitments underlying the kinematics and dynamics of classical mechanics. Those aspects are captured in the meta-language which specifies the class of intended models – $\mathcal{KS}$-structures. They are linked to the mereology-based object language via the intended interpretation of the primitives of the formal theory.

5.1. Mereology

To capture the mereological structure of spacetime regions the primitive binary operation $\sqcup : \mathcal{D}_{ST} \times \mathcal{D}_{ST} \to \mathcal{D}_{ST}$ is introduced in the object language of the formal theory. On the intended interpretation $\sqcup$ is the mereological union of regions u_1 and u_2. More precisely, $\sqcup$ is interpreted as an operation that yields the least upper bound $\bigsqcup : \mathcal{P}(\mathcal{D}_{ST}) \to \mathcal{D}_{ST}$ of the set $\{\mathsf{u}_1, \mathsf{u}_2\}$ with respect to the ordering imposed on $\mathcal{D}_{ST}$ by $\sqsubseteq$. (See e.g., Champollion and Krifka (2015).) As specified in Eq. 12, the mereological sums are the same at all possible worlds. This explicates at the level of the interpretation of the formal theory that the mereological structure of spacetime is *absolute* in the sense that it is the same on all physical possibilities and slicings.

The second primitive of the formal theory is the ternary functional relation $\sqcap$. On the intended interpretation $\sqcap$ is the mereological intersection that holds between regions u_1, u_2 and u_3 if and only if the greatest lower bound $\bigsqcap : \mathcal{P}(\mathcal{D}_{ST}) \to (\mathcal{D}_{ST} \cup \emptyset)$ is non-empty and $\mathsf{u}_3 = \bigsqcap\{\mathsf{u}_1, \mathsf{u}_2\}$. (See e.g., Champollion and Krifka (2015).) As in the case of mereological unions, the mereological intersections are the same at all possible worlds (Eq. 12).

$$\begin{array}{lll} \mathsf{V}(\sqcup) & = \underline{\sqcup} =_{df} & \{\langle \mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, \kappa\rangle \in \mathcal{D}_{ST} \times \mathcal{D}_{ST} \times \mathcal{D}_{ST} \times \mathcal{K} \mid \mathsf{u}_3 = \bigsqcup\{\mathsf{u}_1, \mathsf{u}_2\}\} \\ \mathsf{V}(\sqcap) & = \overline{\sqcap} =_{df} & \{\langle \mathsf{u}_1, \mathsf{u}_2, \mathsf{u}_3, \kappa\rangle \in \mathcal{D}_{ST} \times \mathcal{D}_{ST} \times \mathcal{D}_{ST} \times \mathcal{K} \mid \mathsf{u}_3 = \bigsqcap\{\mathsf{u}_1, \mathsf{u}_2\}\} \end{array} \tag{12}$$

The binary predicate of parthood, $P\,uv$, is defined to hold if and only if the union of u and v is identical to v (D_P).[13] Proper parthood (*PP*), overlap (*O*), and summation are defined in the standard ways (Simons, 1987). The predicate *ST* holds of a region which has all regions as parts (D_{ST}).

$$\begin{array}{ll} D_P & P\,uv \equiv u \sqcup v = v \\ D_O & O\,uv \equiv (\exists w)(P\,wu \wedge P\,wv) \\ D_{Sum} & Sum\,xA \equiv (\forall w)(O\,xw \leftrightarrow (\exists z)(z \in A \wedge O\,zw)) \end{array} \qquad \begin{array}{ll} D_{PP} & PP\,uv \equiv P\,uv \wedge v \neq u \\ D_{ST} & ST\,u \equiv (v)P\,vu \end{array}$$

On the intended interpretation parthood is the submanifold relation and *ST* holds of the spacetime manifold $\mathcal{ST}$. The overlap predicate is true if the greatest lower bound (with respect to $\sqsubseteq$) of two regions ($\sqcap$) is a member of $\mathcal{D}_{ST}$. The *Sum* predicate holds of the least upper bounds of some non-empty subsets of $\mathcal{D}_{ST}$.

$$\begin{array}{lll} \mathsf{V}(P) & = & \{\langle \mathsf{u}_1, \mathsf{u}_2, \kappa\rangle \in \mathcal{D}_{ST} \times \mathcal{D}_{ST} \times \mathcal{K} \mid \mathsf{u}_1 \sqsubseteq \mathsf{u}_2\} \\ \mathsf{V}(ST) & = & \{\langle \mathcal{ST}, \kappa\rangle \in \mathcal{D}_{ST} \times \mathcal{K} \mid \mathcal{ST} = \bigsqcup \mathcal{D}_{ST}\} \\ \mathsf{V}(Sum) & \subseteq & \{\langle \mathsf{u}, \mathsf{A}, \kappa\rangle \in \mathcal{D}_{ST} \times \mathcal{P}(\mathcal{D}_{ST}) \times \mathcal{K} \mid \mathsf{A} \neq \emptyset \wedge \mathsf{u} = \bigsqcup \mathsf{A}\} \\ \mathsf{V}(O) & = & \{\langle \mathsf{u}, \mathsf{v}, \kappa\rangle \in \mathcal{D}_{ST} \times \mathcal{D}_{ST} \times \mathcal{K} \mid \exists \mathsf{w} \in \mathcal{D}_{ST} : \mathsf{w} \sqsubseteq \mathsf{u} \wedge \mathsf{w} \sqsubseteq \mathsf{v}\} \end{array} \tag{13}$$

Axioms are introduced requiring that $\sqcup$ is idempotent, associative, commutative (A1 – A3). Structures satisfying (A1 – A3) are called join semi-lattices. In the context of mereology join semi-lattices lack a

[13]In most axiomatic formulations of mereological theories the parthood relation is selected as a primitive. There are many equivalent choices of primitives and axiom systems which are all well understood (Simons, 1987; Varzi, 2003; Champollion and Krifka, 2015). The particular choice of primitives and associated axioms here is such that it best integrates in the existing infrastructure of Isabelle/HOL to facilitate the computational realization of the formal theory.

minimal element. This is reflected in axiom A4 which ensures that an intersection of two *overlapping* regions exists. The resulting structure is a lattice without minimal element. Furthermore an axiom of separation (A5) holds (Champollion and Krifka, 2015).[14] There exists a spacetime region which has all regions as parts (A6). Axioms A7 and A8 express in the object language that the mereological structure of spacetime is absolute in the sense that it is the same on all physical possibilities and slicings.

$$\begin{array}{ll} A1\; u \sqcup u = u & A5\; \mathit{PP}\; uv \rightarrow (\exists w)(\neg O\; uw \wedge v = u \sqcup w) \\ A2\; u \sqcup (v \sqcup w) = (u \sqcup v) \sqcup w & A6\; (\exists u)(\mathit{ST}\; u) \\ A3\; u \sqcup v = v \sqcup u & A7\; u \sqcup v = w \rightarrow \Box(u \sqcup v = w) \\ A4\; O\; uv \rightarrow (\exists w)(\sqcap\; uvw) & A8\; \sqcap\; uvw \rightarrow \Box(\sqcap\; uvw) \end{array}$$

One can prove that P is antisymmetric, reflexive, and transitive (T1-3). If the parthood predicate holds then it holds across all physical possibilities (T4). Region x is part of region y only if every region that overlaps x also overlaps y (T5). The spacetime region is unique (T6). Sums are unique whenever they exist (T7). Two regions overlap iff they have a non-empty mereological intersection (T8).

$$\begin{array}{ll} T1\; P\; uu & T5\; (u)(O\; ux \rightarrow O\; uy) \rightarrow P\; xy \\ T2\; P\; uv \wedge P\; vu \rightarrow u = v & T6\; \mathit{ST}\; u \wedge \mathit{ST}\; v \rightarrow u = v \\ T3\; P\; uv \wedge P\; vw \rightarrow P\; uw & T7\; \mathit{Sum}\; uA \wedge \mathit{Sum}\; vA \rightarrow u = v \\ T4\; P\; uv \rightarrow \Box P\; uv & T8\; O\; uv \leftrightarrow (\exists w)(\sqcap\; uvw) \end{array}$$

$T1 - T8$ show that $A1 - A6$ are the axioms of an extensional mereology (Simons, 1987; Varzi, 1996) with a maximal element. Additional theorems that follow from the axioms can be found in the file `Plattice.thy` at `http://www.buffalo.edu/~bittner3/Theories/OntologyCM/` (See also Table 6).

5.2. *Time slices*

A third primitive is the unary predicate *TS*. On the intended interpretation *TS* holds of time slices $\sigma_t(\mathcal{T})$ induced by the $\mathcal{T}$-slicing σ:

$$\mathsf{V}(TS) \quad = \quad \mathsf{TS} =_{df} \{\langle \mathsf{u}, \langle \gamma, \sigma \rangle \rangle \in \mathcal{D}_{ST} \times \mathcal{K} \mid \exists \mathsf{t} \in \Re : \mathsf{u} = \sigma_{\mathsf{t}}(\mathcal{T})\} \tag{14}$$

The following mereological axioms for *TS* are added: distinct time slices do not overlap (A9); there are at least two non-overlapping time slices (A10); every region overlaps some time-slice (A11).

$$\begin{array}{ll} A9\;\; \mathit{TS}\; u \wedge \mathit{TS}\; v \wedge O\; uv \rightarrow u = v & \\ A10\; (\exists u)(\exists v)(\mathit{TS}\; u \wedge \mathit{TS}\; v \wedge \neg O\; uv) & A11\;\; (\exists u)(\mathit{TS}\; u \wedge O\; uv) \end{array}$$

In terms of the primitive time slice predicate spatial and soatio-temporal regions are defined: Spatial regions are regions that are parts of some time slice (D_{SR}). Spatio-temporal regions are regions that overlap two distinct time slices D_{STR}).

$$\begin{array}{ll} D_{SR} & \mathit{SR}\; u \equiv (\exists t)(\mathit{TS}\; t \wedge P\; ut) \\ D_{STR} & \mathit{STR}\; u \equiv (\exists t_1)(\exists t_2)(\mathit{TS}\; t_1 \wedge \mathit{TS}\; t_2 \wedge O\; ut_1 \wedge O\; ut_2 \wedge \neg O\; t_1 t_2) \end{array}$$

On the intended interpretation *SR* u means: Spatial regions u are parts of spacetime which, on a given $\mathcal{T}$-slicing σ are sub-manifolds of some time slice induced by σ. On the slicing σ the region u is not extended at all in time. By contrast, on a given slicing spatio-temporal regions extend across time slices.

[14] By not requiring uniqueness of separation, Axiom (A5) is slightly weaker than the version in (Champollion and Krifka, 2015). For this reason the notions of mereological sum and the least upper bound of a set do not exactly coincide as indicated in Eq. 13.

$$\begin{aligned}
&\mathsf{V}(SR) = \{\langle \mathsf{u}, \langle \gamma, \sigma\rangle\rangle \in \mathcal{D}_{ST} \times \mathcal{K} \mid \exists t \in \Re : \mathsf{u} \sqsubseteq \sigma_t(\mathcal{T})\} \\
&\mathsf{V}(STR) = \{\langle \mathsf{u}, \langle \gamma, \sigma\rangle\rangle \in \mathcal{D}_{ST} \times \mathcal{K} \mid \mathsf{u} \sqsubseteq prj^{\mathcal{Q}}_{m}(\gamma) \wedge \\
&\qquad\qquad \exists t_1, t_2 \in \Re :\ t_1 \neq t_2 \wedge \mathsf{u} \cap \sigma_{t_1}(\mathcal{T}) \neq \emptyset \wedge \mathsf{u} \cap \sigma_{t_2}(\mathcal{T}) \neq \emptyset\}
\end{aligned} \tag{15}$$

This interpretation reflects at the level of the formal models that which submanifolds of $\mathcal{ST}$ count as spatial regions depends on the underlying slicing σ.

One can prove: time-slices are maximal spatial regions (T9). Any part of a spatial region is a spatial region (T10). Spacetime is a spatio-temporal region (T11); Spatio-temporal regions are parts of spatio-temporal regions (T12); Spatio-temporal regions are not spatial regions (T13).

$$\begin{array}{llll}
T9 & \mathit{TS}\, u \leftrightarrow \mathit{SR}\, u \wedge & T11 & \mathit{ST}\, u \rightarrow \mathit{STR}\, u \\
 & \quad (v)(\mathit{SR}\, v \wedge O\, uv \rightarrow P\, vu) & T12 & \mathit{STR}\, u \wedge P\, uv \rightarrow \mathit{STR}\, v \\
T10 & \mathit{SR}\, u \wedge P\, vu \rightarrow \mathit{SR}\, v & T13 & \mathit{STR}\, u \rightarrow \neg \mathit{SR}\, u
\end{array}$$

Two regions are *simultaneous* if and only if they are parts of the same time-slice (D_{SIMU}).

$$D_{\mathit{SIMU}}\ \mathit{SIMU}\, uv \equiv (\exists w)(\mathit{TS}\, w \wedge P\, uw \wedge P\, vw)$$

On the intended interpretation:

$$\mathsf{V}(\mathit{SIMU}) = \{\langle \mathsf{u}, \mathsf{v}, \langle \gamma, \sigma\rangle\rangle \in \mathcal{D}_{ST} \times \mathcal{D}_{ST} \times \mathcal{K} \mid \exists t \in \Re : \mathsf{u} \sqsubseteq \sigma_t(\mathcal{T}) \wedge \mathsf{v} \sqsubseteq \sigma_t(\mathcal{T})\} \tag{16}$$

One can prove that *SIMU* is an equivalence relation (reflexive, symmetric, transitive) on the sub-domain of spatial regions. *SIMU* uv is always false if u or v is a spatio-temporal region. Of course, which spatial regions are simultaneous depends on the underlying slicing of spacetime.

5.3. Newtonian vs. Minkowski spacetimes

If $\mathcal{ST}$ is a Newtonian spacetime then *TS* holds of the timeslices of the unique $\mathcal{T}$-slicing σ, i.e., $\Sigma = \{\sigma\}$. In such structures the axiom A_N holds (trivially). One can then prove that if a region is a spatial/spatio-temporal region on some slicing then it is a spatial/spatio-temporal region on all slicings and that simultaneity is absolute ($T_N1 - T_N3$).

$$\begin{array}{llll}
A_N & \mathit{TS}\, u \rightarrow \Box^{\Sigma} \mathit{TS}\, u & T2_N & \mathit{STR}\, u \rightarrow \Box^{\Sigma} \mathit{STR}\, u \\
T1_N & \mathit{SR}\, u \rightarrow \Box^{\Sigma} \mathit{SR}\, u & T3_N & \mathit{SIMU}\, uv \rightarrow \Box^{\Sigma} \mathit{SIMU}\, uv
\end{array}$$

By contrast, if $\mathcal{ST}$ has the global or local structure of a Minkowski spacetime then there are many slicings, i.e., $\#\Sigma > 1$. In such spacetimes the axiom A_M holds requiring that simultaneity is not absolute.

$$A_M\ \mathit{SIMU}\, uv \wedge u \neq v \rightarrow \Diamond^{\Sigma} \neg \mathit{SIMU}\, uv$$

In Minkowski spacetimes some regions of spacetime are spatial regions on some slicings but not on others. Similarly for some spatio-temporal regions.

6. Instantiation in spacetime

To link the ontological categories of a formal ontology to geometric structures that capture the physical possibilities (Sec. 2 – 3 and Appendix B) a formal theory by Bittner and Donnelly (2006a) that explicitly spells out the ways in which different kinds of entities can instantiate or can be instantiated in spacetime is used. This choice allows to mirror at the formal level recent trends in physics to focus on the instantiation and co-instantiation of universals rather than taking the location of particulars as fundamental (Pooley, 2013; Maudlin, 1993; Earman and Norton, 1987). The aim, again, is to provide a framework to explicate choices and commitments rather than to defend or criticize specific choices.

6.1. Instantiation, location, and categorization

A primitive ternary relation *Inst* between two entities and a region is introduced in the object language of the formal theory. *Inst* xyu is interpreted as y is *instantiated by* x at region u (or, equivalently, x *instantiates* y at region u or x is an *instance* of y at region u). On the intended interpretation: $\mathsf{V}(\mathit{Inst}) =_{df} \mathsf{InstST} \subseteq \mathcal{D}_E \times \mathcal{D}_E \times \mathcal{D}_{ST} \times \mathcal{K}$ where the set InstST is part of the underlying $\mathcal{KS}$ structure (Eq. 10). The following axioms (some are adopted from the work of Bittner and Donnelly (2006a)) are included in the formal theory to constrain InstST: if x instantiates y at u then it is not physically possible that some z is an instance of x at some region v (A12); every entity instantiates or is instantiated on some physical possibility (A13); every entity is instantiated or instantiates at spatial regions or at spatio-temporal regions (A14); if x instantiates at a spatial region then on all slicings: x instantiates at spatial regions (A15); if x is instantiated at a spatio-temporal region then on all slicings: x is instantiated at spatio-temporal regions (A16); if x instantiates y at a spatio-temporal region u then x is uniquely located (A17); if x instantiates at two simultaneous spatial regions u and v then u and v are identical (A18).

$$
\begin{array}{l}
A12\ \mathit{Inst}\ xyu \rightarrow \neg\Diamond(\exists z)(\exists v)(\mathit{Inst}\ zxv) \\
A13\ \Diamond(\exists y)(\exists u)(\mathit{Inst}\ xyu \vee \mathit{Inst}\ yxu) \\
A14\ \mathit{Inst}\ xyu \rightarrow (\mathit{SR}\ u \vee \mathit{STR}\ u) \\
A15\ \mathit{Inst}\ xyu \wedge \mathit{SR}\ u \rightarrow \Box^{\Sigma}(z)(v)(\mathit{Inst}\ xzv \rightarrow \mathit{SR}\ v) \\
A16\ \mathit{Inst}\ yxu \wedge \mathit{STR}\ u \rightarrow \Box^{\Sigma}(z)(v)(\mathit{Inst}\ zxv \rightarrow \mathit{STR}\ v) \\
A17\ \mathit{Inst}\ xyu \wedge \mathit{Inst}\ xzv \wedge \mathit{STR}\ u \wedge \mathit{STR}\ v \rightarrow u = v \\
A18\ \mathit{Inst}\ xyu \wedge \mathit{Inst}\ xzv \wedge \mathit{SR}\ u \wedge \mathit{SR}\ v \wedge \mathit{SIMU}\ uv \rightarrow u = v
\end{array}
$$

Axiom (A12) guarantees that there is a categorical distinction between entities that instantiate and entities that are instantiated on all physical possibilities. Axioms (A14-16) ensure that entities cannot be instantiated/instantiate at different kinds of regions of spacetime. That is, the categorial distinction between entities that instantiate at spatial regions (continuants) and entities that instantiate at spatio-temporal regions (occurants) is independent on the slicing of the spacetime manifold. Axioms (A17-18) explicate the distinction between instantiation at spatial regions and instantiation at spatio-temporal regions. Axioms (A16-17) ensure that entities that are instantiated at a spatio-temporal region (ocurrants/processes) are instantiated at a single spatio-temporal region and this region must counts as a spatio-temporal region on all slicings. This requirement restricts the spacetime regions at which entities can be instantiated. It mirrors in the object language restrictions on kinematically possible worldlines as they are expressed for example in Postulate 3. For entities that instantiate at spatial regions (continuants) the slicing affects which entities instantiate simultaneously and at which spatial regions an entity is instantiated.

In terms of the instantiation primitive one can define: Entity x is *located* at region u if and only if there exists an entity y such that x instantiates y at u or x is instantiated by y at u (D_L); Entity x exists at timeslice t iff there is a region at which x is located and that overlaps t (D_E). An entity is *persistent* iff it is not confined to a single time-slice (D_{Pe}). Entity x is a *particular* if and only if x is a persistent entity that instantiates at some region (D_{Part}). Entity x is a *universal* if and only if x is a persistent entity that is instantiated at some region (D_{Uni}).

$$
\begin{array}{ll}
D_L\ L\ xu \equiv (\exists y)(\mathit{Inst}\ xyu \vee \mathit{Inst}\ yxu) & D_{Pe}\ \ \mathit{Pe}\ x \equiv (\exists u)(\exists v)(L\ xu \wedge L\ xv \wedge \neg \mathit{SIMU}\ uv) \\
D_E\ E\ xt \equiv \mathit{TS}\ t \wedge (\exists u)(L\ xu \wedge O\ ut) & D_{Part}\ \mathit{Part}\ x \equiv \mathit{Pe}\ x \wedge (\exists y)(\exists u)(\mathit{Inst}\ xyu) \\
 & D_{Uni}\ \ \mathit{Uni}\ x \equiv \mathit{Pe}\ x \wedge (\exists y)(\exists u)(\mathit{Inst}\ yxu)
\end{array}
$$

Intuitively, $L\ xu$ means: spatio-temporal entity x is *exactly located* at region u. This corresponds to the usual understanding of the location relation in formal ontology (Casati and Varzi, 1999). In other words, x takes up the whole region u but does not extend beyond u. Thus, my body is always located at a me-shaped region of space at all times at which my body exists. My life is uniquely located at my worldline. On the intended interpretation:

$$\begin{array}{lcl} \mathsf{V}(L) & = & \{\langle \mathsf{x},\mathsf{u},\kappa\rangle \in \mathcal{D}_E \times \mathcal{D}_{ST} \times \mathcal{K} \mid \\ & & \quad \exists \mathsf{y} \in \mathcal{D}_E : \langle \mathsf{x},\mathsf{y},\mathsf{u},\kappa\rangle \in \mathsf{InstST} \vee \langle \mathsf{y},\mathsf{x},\mathsf{u},\kappa\rangle \in \mathsf{InstST}\} \\ \mathsf{V}(E) & = & \{\langle \mathsf{x},\mathsf{t},\langle\gamma,\sigma\rangle\rangle \in \mathcal{D}_E \times \mathcal{D}_{ST} \times \mathcal{K} \mid \exists \tau \in \Re : \mathsf{t} = \sigma_\tau(\mathcal{T}) \wedge \\ & & \quad \exists \mathsf{u} \in \mathcal{D}_{ST} : \langle \mathsf{x},\mathsf{u},\langle\gamma,\sigma\rangle\rangle \in \mathsf{V}(L) \wedge \mathsf{u} \cap \mathsf{t} \neq \emptyset\} \\ \mathsf{V}(\mathit{Pe}) & = & \{\langle \mathsf{x},\langle\gamma,\sigma\rangle\rangle \in \mathcal{D}_E \times \mathcal{K} \mid \exists : \mathsf{u},\mathsf{v} \in \mathcal{D}_{ST} : \\ & & \quad \langle \mathsf{x},\mathsf{u},\kappa\rangle \in \mathsf{V}(L) \wedge \langle \mathsf{x},\mathsf{v},\kappa\rangle \in \mathsf{V}(L) \wedge \langle \mathsf{x},\mathsf{u},\mathsf{v},\kappa\rangle \notin \mathsf{V}(\mathit{SIMU})\} \\ \mathsf{V}(\mathit{Part}) & = & \{\langle \mathsf{x},\kappa\rangle \in \mathsf{V}(\mathit{Pe}) \mid \exists \mathsf{y} \in \mathcal{D}_E : \exists \mathsf{u} \in \mathcal{D}_{ST} : \langle \mathsf{x},\mathsf{y},\mathsf{u},\kappa\rangle \in \mathsf{InstST}\} \\ \mathsf{V}(\mathit{Uni}) & = & \{\langle \mathsf{x},\kappa\rangle \in \mathsf{V}(\mathit{Pe}) \mid \exists \mathsf{y} \in \mathcal{D}_E : \exists \mathsf{u} \in \mathcal{D}_{ST} : \langle \mathsf{y},\mathsf{x},\mathsf{u},\kappa\rangle \in \mathsf{InstST}\} \end{array} \tag{17}$$

One can prove: if an entity x is located at a spatial region then x is located at spatial regions on all possible slicings of spacetime (T14). Similarly for entities located at spatio-temporal regions (T15). Every entity is either on all slicings located at spatial regions or on all slicings located on spatio-temporal regions (T16). Particulars are particulars on all physical possibilities in which they exist (T17). Similarly for universals (T18). It is physically possible for every entity to exist at some time(slice) (T19).

$$\begin{array}{ll} T14\ L\ xu \wedge \mathit{SR}\ u \to \Box^{\Sigma}(v)(L\ xv \to \mathit{SR}\ v) & T17\ \mathit{Part}\ x \to \Box((\exists u)(L\ xu) \to \mathit{Part}\ x) \\ T15\ L\ xu \wedge \mathit{STR}\ u \to \Box^{\Sigma}(v)(L\ xv \to \mathit{STR}\ v) & T18\ \mathit{Uni}\ x \to \Box((\exists u)(L\ xu) \to \mathit{Uni}\ x) \\ T16\ (\Box^{\Sigma}(L\ xu \to \mathit{SR}\ u)) \vee (\Box^{\Sigma}(L\ xu \to \mathit{STR}\ u)) & T19\ \Diamond(\exists t)(E\ xt) \end{array}$$

Persistent entities are distinguished into continuants and occurrents. Entity x is a *continuant* iff x is persistent and x is located at some spatial region (D_{Cont}) . By contrast, x is a *occurrent* iff x is located at some spatio-temporal region (D_{Occ}).

$$D_{Cont}\ \mathit{Cont}\ x \equiv \mathit{Pe}\ x \wedge (\exists u)(L\ xu \wedge \mathit{SR}\ u) \qquad D_{Occ}\ \ \mathit{Occ}\ x \equiv (\exists u)(L\ xu \wedge \mathit{STR}\ u)$$

On the intended interpretation :

$$\begin{array}{l} \mathsf{V}(\mathit{Cont}) = \{\langle \mathsf{x},\kappa\rangle \in \mathsf{V}(\mathit{Pe}) \mid \exists \mathsf{u} \in \mathcal{D}_{ST} : \langle \mathsf{x},\mathsf{u},\kappa\rangle \in \mathsf{V}(L)\ \wedge \langle \mathsf{u},\kappa\rangle \in \mathsf{V}(\mathit{SR})\} \\ \mathsf{V}(\mathit{Occ})\ \ = \{\langle \mathsf{x},\kappa\rangle \in \mathsf{V}(\mathit{Pe}) \mid \exists \mathsf{u} \in \mathcal{D}_{ST} : \langle \mathsf{x},\mathsf{u},\kappa\rangle \in \mathsf{V}(L) \wedge \langle \mathsf{u},\kappa\rangle \in \mathsf{V}(\mathit{STR})\} \end{array} \tag{18}$$

One can prove that continuants are continuants on all slicings in which they persist (T20). Similarly for occurrents (T21). Occurrents are persistent entities (T22). Continuants are not occurrents (T23). Continuants are always located at spatial regions (T24) and occurrents are always located at spatio-temporal regions (T25).

$$\begin{array}{ll} T20\ \mathit{Cont}\ x \to \Box^{\Sigma}(\mathit{Pe}\ x \to \mathit{Cont}\ x) & T23\ \mathit{Cont}\ x \to \neg \mathit{Occ}\ x \\ T21\ \mathit{Occ}\ x \to \Box^{\Sigma}(\mathit{Pe}\ x \to \mathit{Occ}\ x) & T24\ \mathit{Cont}\ x \wedge L\ xu \to \mathit{SR}\ u \\ T22\ \mathit{Occ}\ x \to \mathit{Pe}\ x & T25\ \mathit{Occ}\ x \wedge L\ xu \to \mathit{STR}\ u \end{array}$$

One can also prove that occurrent particulars are uniquely located (T26), and continuant particulars uniquely located within time slices (T27).

$$\begin{array}{l} T26\ \mathit{Occ}\ x \wedge \mathit{Part}\ x \wedge L\ xu \wedge L\ xv \to u = v \\ T27\ \mathit{Cont}\ x \wedge \mathit{Part}\ x \wedge L\ xu \wedge L\ xv \wedge \mathit{SIMU}\ uv \to u = v \end{array}$$

Finally, an axiom is included that ensures that every persistent entity has a worldline (A19). Region u is the worldline of entity x if and only if u a spatio-temporal region that is the mereological sum of all locations at which x is located (D_{WLOf});

$$D_{WLOf}\ \mathit{WLOf}\ xu \equiv \mathit{STR}\ u \wedge u\ \mathit{Sum}\ \{v \mid L\ xv\} \qquad A19\ \mathit{Pe}\ x \to (\exists u)(\mathit{WLOf}\ xu)$$

On the intended interpretation *WLOf* is:

$$\begin{aligned} \mathsf{V}(\mathit{WLOf}) \quad = \quad & \{\langle \mathsf{x}, \mathsf{v}, \langle\gamma,\sigma\rangle\rangle \in \mathcal{D}_E \times \mathcal{D}_{ST} \times \mathcal{K} \mid \langle \mathsf{v}, \langle\gamma,\sigma\rangle\rangle \in \mathsf{V}(\mathit{STR}) \text{ and} \\ & \mathsf{v} = \bigsqcup\{\mathsf{u} \in \mathcal{D}_{ST} \mid \langle \mathsf{x}, \mathsf{u}, \langle\gamma,\sigma\rangle\rangle \in \mathsf{V}(L)\} \text{ and } \mathsf{v} \sqsubseteq \mathit{prj}^{\mathcal{L}}_m(\gamma)\} \end{aligned} \tag{19}$$

On this interpretation all entities instantiate at or along the worldlines singled out by the geometry of the underlying spacetime manifold as required in Postulates 4 and 5 in conjunction with the dynamics determined by the Lagrangian field according to Postulate 7.

6.2. Mereology of particulars

The mereological structure of the subdomain of continuants is characterized by the ternary parthood relation P_c which, holds between a time slice t and two continuant particulars x and y that are instantiated respectively at regions u_1 and u_2 such that u_1 is a part of u_2 and u_2 is part of the time slice t (D_{P_c}).

$$\begin{aligned} & D_{P_c}\ P_c\, xyt \equiv \mathit{Cont}\ x \wedge \mathit{Cont}\ y \wedge \mathit{TS}\ t \wedge \\ & \qquad (\exists u_1)(\exists u_2)(\exists z_1)(\exists z_2)(\mathit{Inst}\ xz_1u_1 \wedge \mathit{Inst}\ yz_2u_2 \wedge P\ uv \wedge P\ vt) \end{aligned}$$

On the intended interpretation P_c means:

$$\begin{aligned} \mathsf{V}(P_c) \quad = \quad & \{\langle \mathsf{x}_1, \mathsf{x}_2, \mathsf{t}, \kappa\rangle \in \mathcal{D}_E \times \mathcal{D}_E \times \mathcal{D}_{ST} \times \mathcal{K} \mid \langle \mathsf{t}, \kappa\rangle \in \mathsf{V}(\mathit{TS}) \wedge \\ & \exists \mathsf{y}_1, \mathsf{y}_2 \in \mathcal{D}_E : \exists \mathsf{u}_1, \mathsf{u}_2 \in \mathcal{D}_{ST} : \mathsf{u}_1 \sqsubseteq \mathsf{u}_2 \sqsubseteq \mathsf{t} \wedge \\ & \quad \langle \mathsf{x}_1, \mathsf{y}_1, \mathsf{u}_1, \kappa\rangle \in \mathsf{InstST} \wedge \langle \mathsf{x}_2, \mathsf{y}_2, \mathsf{u}_2, \kappa\rangle \in \mathsf{InstST}\} \end{aligned} \tag{20}$$

One can then prove: Continuant x exists at a timeslice iff x is a part of itself at that time slice (T28) and that at every timeslice P_c is transitive (T29).

$T28\ \mathit{Cont}\ x \to (E\ xt \leftrightarrow P_c\ xxt)$ $\qquad$ $T29\ P_c\ xyt \wedge P_c\ yzt \to P_c\ xzt$

In Minkowski spacetime the parthood relation among continuants (P_c) is logically linked to the underlying slicing of spacetime. This is an immediate consequence of axiom (A_M). Only continuants that exist simultaneously at a time can be parts at that time.

The mereological structure of the subdomain of occurrents is characterized by the binary parthood relation P_o defined as: x is part of y if and only if the location of x is part of the location of y and the location of x is a spatio-temporal region (D_{P_o});

$$D_{P_o}\ P_o\, xy \equiv (\exists u_1)(\exists u_2)(\exists z_1)(\exists z_2)(\mathit{Inst}\ xz_1u_1 \wedge \mathit{Inst}\ yz_2u_2 \wedge P\ uv \wedge \mathit{STR}\ u)$$

On the intended interpretation P_o means:

$$\begin{aligned} \mathsf{V}(P_o) \quad = \quad & \{\langle \mathsf{x}_1, \mathsf{x}_2, \kappa\rangle \in \mathcal{D}_E \times \mathcal{D}_E \times \mathcal{K} \mid \exists \mathsf{y}_1, \mathsf{y}_2 \in \mathcal{D}_E : \exists \mathsf{u}_1, \mathsf{u}_2 \in \mathcal{D}_{ST} : \\ & \langle \mathsf{x}_1, \mathsf{y}_1, \mathsf{u}_1, \kappa\rangle \in \mathsf{InstST} \wedge \langle \mathsf{x}_2, \mathsf{y}_2, \mathsf{u}_2, \kappa\rangle \in \mathsf{InstST} \wedge \mathsf{u}_1 \sqsubseteq \mathsf{u}_2, \\ & \langle \mathsf{u}_1, \kappa\rangle \in \mathsf{V}(\mathit{STR}), \langle \mathsf{u}_2, \kappa\rangle \in \mathsf{V}(\mathit{STR})\} \end{aligned} \tag{21}$$

On the subdomain of occurrent particulars P_o is reflexive (T30) and transitive (T31).

$T30\ \mathit{Part}\ x \to (\mathit{Occ}\ x \leftrightarrow P_o\ xx)$ $\qquad$ $T31\ P_o\ xy \wedge P_o\ yz \to P_o\ xz$

6.3. Atomic entities

The final primitive of the formal theory is the unary predicate At_e which, on the intended interpretation, holds of atomic entities ($\mathsf{V}(At_e) =_{df} \mathsf{AtE} \subset \mathcal{D}_E \times \mathcal{K}$) such that the following axioms hold (in accordance with the conception of atoms in classical mechanics): There exist finitely many atomic entities (A20). If x is an atomic entity then x is an atomic entity on all physical possibilities (A21); Atomic entities are instantiated at all physical possibilities (A22); Atomic entities are always instantiated at parts of time slices (A23). For every atomic entity x there is some slicing such that x is always instantiated at proper parts of time slices (A24).[15] Every atomic entity is instantiated at some non-simultaneous regions on all slicings of spacetime (A25). Atomic entities that are instantiated at regions where one region is part of the other are identical (A26).

$$
\begin{array}{l}
A20 \text{ finite } \{x \mid At_e\, x\} \\
A21\, At_e\, x \rightarrow \Box At_e\, x \\
A22\, At_e\, x \rightarrow \Box(\exists y)(\exists u) \mathit{Inst}\, xyu \\
A23\, At_e\, x \wedge \mathit{Inst}\, xyu \rightarrow (\exists t)(\mathit{TS}\, t \wedge P\, ut) \\
A24\, At_e\, x \rightarrow \Diamond^{\Sigma}(t)(\mathit{TS}\, t \rightarrow (\exists u)(\exists y)(\mathit{Inst}\, xyu \wedge \mathit{PP}\, ut)) \\
A25\, At_e\, x \rightarrow \Box^{\Sigma}(\exists y)(\exists z)(\exists u)(\exists v)(\mathit{Inst}\, xyu \wedge \mathit{Inst}\, xzv \wedge \neg \mathit{SIMU}\, uv) \\
A26\, At_e\, x_1 \wedge At_e x_2 \wedge \mathit{Inst}\, x_1 y_1 u_1 \wedge \mathit{Inst}\, x_2 y_2 u_2 \wedge P\, u_1 u_2 \rightarrow x_1 = x_2
\end{array}
$$

These axioms ensure that atoms cannot fail to be atoms and to instantiate in every physically possible world. One could add a stronger version of axion A22 and demand an atom to instantiate the same universal in all physically possible worlds (A22* $At_e\, x \rightarrow (\exists y)\Box(\exists u)\mathit{Inst}\, xyu$). Consider the oxygen atoms that are part of my body at this point in time. If axiom (A22*) holds then they cannot fail to be oxygen atoms. Note, however, that which wholes atoms form is contingent. Some or all of the oxygen atoms that are currently part of my body could be part of the water of Lake Erie.

One can prove: On all physical possibilities atoms are located at spatial regions (T32); For every atomic entity there is some slicing on which it exists at every time (T33); Atomic entities that are parts of one another are identical (T34); Atomic entities are particulars on all physical possibilities (T35); Atomic entities are persistent continuants on all possible slicings of spacetime (T36,T37).

$$
\begin{array}{ll}
T32\, At_e\, x \rightarrow \Box(u)(L\, xu \rightarrow \mathit{SR}\, u) & T35\, At_e\, x \rightarrow \Box \mathit{Part}\, x \\
T33\, At_e\, x \rightarrow \Diamond^{\Sigma}(\forall t)(\mathit{TS}\, t \rightarrow E\, xt)) & T36\, At_e\, x \rightarrow \Box^{\Sigma} \mathit{Pe}\, x \\
T34\, At_e\, x \wedge At_e\, y \wedge (P_c\, xyt \vee P_c\, yxt) \rightarrow x = y & T37\, At_e\, x \rightarrow \Box^{\Sigma} \mathit{Cont}\, x
\end{array}
$$

This concludes the development of the mereology of persistent physical particulars.

7. Discussion

The presented formal theory is intended to serve as a framework to relate conceptual and formal structures of classical mechanics to conceptual and formal structures in formal ontologies. In this context two important aspects of the formal theory developed above are emphasized: (i) techniques for formally capturing the notion of ‘dynamic possibility’ in a formal ontology and (ii) the ways in which the presented theory provides precise interpretations of fundamental primitives such as ’spatial-’, and ’spatio-temporal region’, ’location’, ’instantiation’, ’Atom’, etc. The former point is discussed in Sec. 7.1. To illustrate the latter point formal specifications of informal elucidations from the current version of BFO (Smith, 2016) are discussed in Sec. 7.2. The computational realization of the presented formal theory is discussed in Sec. 7.3.

[15] In non-finite spacetimes this can be demanded of for all slicings.

7.1. Physical possibilities

Ontology is the study of what can possibly exist (Lowe, 2002; Smith, 2003). An important aspect of characterizing dynamic reality from an ontological perspective is to clearly and formally distinguish processes and sequences of states of enduring entities that are logically, metaphysically, and physically possible. In classical mechanics such possibilities are encoded in the geometry of set-theoretic structures (manifolds) at several interrelated levels:

(i) The geometry of the spacetime manifold picks out possible worldlines and sequences of instantaneous states in conjunction with compatible slicings of spacetimes into hyperplanes of simultaneity (timeslices). (Postulates 1 and 2)
(ii) Kinematic constraints impose fundamental metaphysical restrictions on what is geometrically possible. For example, they constrain the causal structure by determining which worldlines are available to connect points in spacetimes via physically possible processes (Postulates 2, 3). Kinematics also constrains the ways in which the possible complex systems arise from the possibilities of simple systems (Postulate 4). Kinematic constraints manifest themselves in the specific geometry of spacetime and the associated configuration spaces.
(iii) Dynamic constraints impose physical restrictions on what is kinematically possible. Physical constraints are geometric expressions of the laws of physics in the form of equations of motion. (Appendix B, particularly Postulate 7.)

To make these various kinds of possibilities explicit and to present them in a way that can be incorporated in the meta theory of a formal ontology was the aim of the first part of this paper (Sec. 2 – 3). In the object language of the formal ontology presented in Sec. 4 – 6 these distinctions are not encoded in the geometry of some set-theoretic structure but expressed (as far as possible) explicitly in a language that is essentially equivalent to one of a modal first order predicate logic. In the formal theory the physics and the ontology are linked at the following levels:

(a) The level of the intended range of the variables of the formal ontology within the set-theoretic structures of the underlying classical mechanics. The intended ranges of variables are spelled out explicitly as part of the meta-theory of the formal ontology. (Sec. 4.3, Eq. 17, etc.)
(b) Physical possibilities are introduced via modal operators and the accessibility relations among physical possibilities. (Sec. 4.2, Fig. 4).
(c) Besides the mereological primitives, the most fundamental primitive of the axiomatic theory is the instantiation predicate. On the intended interpretation it represents the actualization of mereologically possible entities at dynamically possible worldlines and sequences of states. The instantiation predicate thereby links processes to worldlines (Sec. 6) and sequences of locations to continuant entities.

This will be discussed in some more detail in what follows.

7.1.1. Ranges of variables

The object language of the formal ontology is expressed in a modal predicate logic with two fundamental sorts of variables: variables ranging over regions of spacetime and variables ranging over mereologically possible entities.[16]

According to classical (i.e., non-quantum) mechanics the geometry of spacetime determines at what regions of space and spacetime physical entities can be instantiated. In particular the underlying geometry singles out possible worldlines and hyperplanes of simultaneity (timeslices) (Postulates 1 and 2). In the meta-language of the formal ontology $\mathcal{D}_{ST}$ includes the set of all those regions as well as distinguished classes of subregions thereof (Sec. 4.3). In the formal language region variables range over the members of this set. Important subclasses of regions are (spatio-)temporal regions and spatial regions. In the presented

[16] Again, to avoid axiomatic and definitorial schemata some second order features are used. Nevertheless, the object language is essentially a modal first order language.

framework a precise interpretation of the respective kinds of regions can be given. Spatio-temporal regions are the sub-manifolds of all kinematically possible worldlines in a world of m particles, i.e., $Prj^{\mathcal{Q}}_{m} \subset \mathcal{D}_{ST}$ (Eq. 15). Kinematically possible spatial regions are sub-manifolds of hyperplanes of simultaneity (timeslices) (Eq. 15).

According to classical mechanics the number and kinds of atomic particles is finite and constant. To make explicit that the specific number of particles is accidental, the number of particles that exist was chosen as a parameter of the formal theory. Within the formal theory only finiteness of the number of atomic particles is required explicitly. From this it follows that the domain of (physical) entities $\mathcal{D}_E$ is finite. In the formal theory axioms on instantiation and axioms for atoms constrain what kinds of complex entities can be formed within the realm of physical possibilities.

7.1.2. Modalities

Physical possibilities and their expression in the modal part of the formal ontology fall in two basic categories: physically possible worldlines and physically possible slicings of spacetime into hyperplanes of simultaneity.

Physically possible worldlines: In addition to purely mereological constraints on what complex wholes can exist (Simons, 1987, 1991; Casati and Varzi, 1994, 1999; Varzi, 1996), in physical theories there are additional geometric, kinematic, and dynamic constraints on possible complex wholes. The geometrically and kinematically possible ways in which atomic particles can or can not form complex wholes are encoded in the geometry of the configuration space (e.g., Postulates 4 and 5). These constraints are formulated in terms of restrictions on the worldlines along which complex entities can possibly evolve. Further dynamic constrains in form of the Lagrangian field and Hamilton's principle (Postulate 7) restrict the worldlines that are geometrically and kinematically possible to those that are physically possible as described in Sec. 3.2 and Appendix B. The Lagrangian field encodes the laws of physics and enters the formal ontology as a parameter of the meta-theory. Dynamically possible worldlines of configuration space, i.e., worldlines in the set $\Gamma^{\mathcal{L}}$ that satisfy Hamilton's principle (Postulate 7) are, when projected onto spacetime (Eq. 5 and Eq. 6), the worldlines along which physically possible entities can evolve and at which physically possible processes can be located. The formal structures of the meta-theory that give rise to the set $\Gamma^{\mathcal{L}}$ and the projection of its members via the mapping $prj^{\mathcal{L}}$ are summarized in Table 4.

In the modal part of the formal ontology physically possible worlds are the members of $\Gamma^{\mathcal{L}}$. A modal operator $\Box^{\Gamma}$ was introduced which is interpreted as operating on the members of $\Gamma^{\mathcal{L}}$ via the accessibility relation R^{Γ} that holds between members of $\Gamma^{\mathcal{L}}$. The relation R^{Γ} is reflexive, symmetric, and transitive, i.e., an equivalence relation. This expresses formally the thesis that all physically possible worlds are equivalent from a logical perspective.[17]

Slicings of spacetime: An important aspect that is widely ignored in current ontologies is that, according to modern physics, it is meaningless to speak of space and spatial regions without reference to a specific slicing of spacetime. In the presented theory slicings of spacetime are explicit part of the meta-language. Possible spatial regions are picked out by the *SR* predicate: $\mathsf{V}(SR) \subset \mathcal{D}_{ST} \times \mathcal{K}$ according to the underlying slicing $\sigma \in \Sigma$. In the the formal ontology the modal operator $\Box^{\Sigma}$ is interpreted as operating on the members of Σ via the accessibility relation R^{Σ} that holds between members of Σ. The relation R^{Σ} is reflexive, symmetric, and transitive, i.e., an equivalence relation.

$\mathcal{T}$-slicings $\sigma \in \Sigma$ of spacetime are associated with reference frames of distinguished particles/systems (called 'inertial observers') moving with a constant velocity along their worldlines. That R^{Σ} is an equivalence relation is an expression of the *principle of relativity* (Galilei, 1632; Einstein, 1951; Arthur, 2007). In the context of Newtonian spacetime and the global Minkowski spacetime of special relativity this is a formal expression of the claim that all inertial (non-accelerated) systems of reference are not only logically

[17]This choice precludes the interpretation of worldlines as worlds that are counterparts in the sense of David Lewis (Lewis, 1986). Lewis counterpart relation is usually understood as a similarity relation and not as an equivalence relation. A discussion of this issue is certainly warranted but goes beyond the scope of this paper. See for example work by van Inwagen (1985); Sider (2001).

but also physically equivalent. More precisely, the relation $R^{\Sigma}(\sigma_1, \sigma_2)$ has a direct physical interpretation in the sense that the slicings σ_1 and σ_2 are related by R^{Σ} if and only if they are related by a Lorentz boost[18] $\phi : \mathcal{ST} \rightarrow \mathcal{ST}$ (or are identical in Newtonian spacetime, i.e., $\phi_N = \text{id}$):[19]

$$R^{\Sigma}(\sigma_1, \sigma_2) \quad \text{iff} \quad \exists \phi : \sigma_2 = (\phi \circ \sigma_1) \tag{22}$$

Combining possible worldlines and slicings into possible worlds. In formal ontology, possible worlds are usually rather abstract. Taking into account the underlying physics allows for a more specific description of possibilities and their interrelations. In the metalanguage of the presented ontology the structure $\mathcal{K} = \Gamma^{\mathcal{L}} \times \Sigma$ is a set of physical possibilities which combines the two modal aspects discussed above:

- The dynamically possible worldlines in $\Gamma^{\mathcal{L}}$ along which worlds/systems with m particles can evolve in a configuration space $\mathcal{Q}(\mathcal{ST})$ with a Lagrangian field $\mathcal{L}$ (Eq.27 of Appendix B).
- The geometrically possible slicings Σ of spacetime into hyperplanes of simultaneity.

The accessibility relations R^{Γ} and R^{Σ} and their properties of reflexivity, symmetry, and transitivity in conjunction with the fact that R^{Γ} and R^{Σ} commute in the sense of Eq. 9 explicate the thesis that these two modal aspects are logically independent (Sec. 4).

7.1.3. Instantiation in spacetime

The axiomatic theory is based on (a) a mereology (Simons, 1987) of spatio-temporal regions in conjunction with the formal distinction between spatial regions and spatio-temporal regions, and (b) an instantiation relation holding between an entity that instantiates, an entity that is instantiated, and a region of spacetime at which the instantiating entity instantiates the instantiated entity (adapted from work by Bittner and Donnelly (2006a)). Accordingly, the fundamental primitives of the axiomatic theory are the parthood (expressed algebraically in terms of the join operation) and instantiation predicates. On the intended interpretation instantiation relates mereologically possible entities to dynamically possible worldlines. The focus on instantiation at regions of spacetime as the central formal notion mirrors in the formal ontology a fundamental aspect of physical theories – the close interrelationship between spacetime structure and physical possibilities (Sec. 2 – 3).

In terms of the instantiation predicate the categories of the formal ontology that refine the category of entities can be introduced: Particulars are entities that on all physical possibilities instantiate and universals are entities that on all physical possibilities are instantiated. An entity is located at a region of spacetime if and only if it instantiates or is instantiated at that region. Particulars are further distinguished according to the number (a single region vs. multiple regions) and the kinds of regions (spatial regions vs. spatial-temporal regions). Continuants are particulars that are located at multiple spatial regions in different timeslices. By contrast, occurrents are located at unique spatio-temporal regions (worldlines). Corresponding to the commitments of classical mechanics the mereology of (instantiating) entities has the structure of an atomic mereology.

All axioms of the formal theory are true in all physically possible worlds and on all geometrically possible slicings of spacetime. This explicates that metaphysical distinctions are more fundamental than physical distinctions.

7.2. Elucidations in BFO-like ontologies

The presented formal theory aims to provide precise interpretations of fundamental primitives such as 'spatial-', 'temporal-', and 'spatio-temporal region', 'location', 'instantiation' and others more, many of which in current versions of BFO (Smith, 2016) and similar ontologies only have informal specifications in the form of elucidations. In what follows some elucidations of BFO that rely on notions that were specified in the ontology presented above and for which the interpretation in this framework may be particularly beneficial are discussed.

[18]Transformations that map changes in the velocity of distinguished particles to changes in the slicing of spacetime (Sklar, 1977).

[19]Transformations of the form $\phi : \mathcal{ST} \rightarrow \mathcal{ST}$ on spacetime induce corresponding transformations of the $\phi^{\mathcal{Q}} : \mathcal{Q}(\mathcal{ST}) \rightarrow \mathcal{Q}(\mathcal{ST})$ on configuration space $\mathcal{Q}(\mathcal{ST})$.

ELUCIDATION (Smith, 2016, 035-001): "A spatial region is a continuant entity that is a continuant_part_of space$_R$ as defined relative to some frame R". In the context of the above theory, the term 'frame' seems to be better interpreted as the weaker notion of a $\mathcal{T}$-slicing σ of spacetime (Def. 1). This is because different frames can give rise to the same $\mathcal{T}$-slicing of the spacetime manifold. 'Space$_R$' seems to refer to some time slice $\sigma_t(\mathcal{T})$ and 'spatial region' seems to refer to the members of the set $\mathsf{V}(\mathit{SR})$ restricted to a fixed slicing $\sigma \in \Sigma$ (Eq. 15). In the object language of the formal theory elucidation [035-001] is approximated by the theorem: $\Box^\Sigma(\mathit{SR}\ u \leftrightarrow (\exists v)(\mathit{TS}\ v \wedge P\ uv))$ which immediately follows from D_{ST}. In the formal theory one can explicitly distinguish between spatial regions in Newtonian and Minkowski spacetimes using the notion of simultaneity: $\mathit{SIMU}\ uv \rightarrow \Box^\Sigma \mathit{SIMU}\ uv$ ($T3_N$) vs. $\mathit{SIMU}\ uv \wedge u \neq v \rightarrow \Diamond^\Sigma \neg \mathit{SIMU}\ uv$ (A_M).

In the presented theory regions and entities are represented by disjoint sorts of variables. This representational choice does not allow to represent regions as kinds of entities. This is clearly a shortcoming of the presented theory since, according to general relativity, spacetime interacts with matter and and vice versa. Alternatively, regions could be defined as entities that are located at themselves along the lines described in work by Casati and Varzi (1995).

However, to categorize a spatiotemporal region as a continuant entity may be a mistake because in Minkowski spacetimes there are regions that count as spatial regions on some slicing and as spatiotemporal regions on other slicings. The definitions of continuants and ocurrants (D_{Cont} and D_{Occ}) then would imply that there are regions that count as a continuant entity on some slicing and as an ocurrant entity on another slicing. That is, whether or not certain regions are continuant or ocurrants ceases to be a classification that is independent on the underlying reference system. This ultimately would introduce inconsistencies at least in the type of theories presented here. The same criticism can be raised for elucidation 095-001 below.

ELUCIDATION (Smith, 2016, 095-001): "A spatiotemporal region is an occurrent entity that is part of spacetime." In the formal theory entities and regions are disjoint sorts. For this reason spacetime regions cannot be entities. As discussed above (elucidation (Smith, 2016, 035-001)), this can be overcome in a language which does not have a sortal distinction between entities and regions. Problematic is the classification of spatial/spatio-temporal regions as continuants/ocurrents for the reasons discussed above.

ELUCIDATION (Smith, 2016, 100-001): "A temporal region is an occurrent entity that is part of time as defined relative to some reference frame." This elucidation seems to be inconsistent with elucidation (Smith, 2016, 095-001) at least in cases where the underlying spacetime structure is non-Newtonian. According to modern physics there is no unique separation of spacetime into spatial and temporal components. There are slicings of spacetime such that mappings of the form $\sigma_t(\mathcal{T})$ pick out three-dimensional hyper-surfaces of simultaneity – timeslices (Def. 1). Nothing is instantiated/located at $t \in \Re$ which is just a parameter called coordinate time (Def. 2) that identifies a timeslice on a particular slicing (Def. 1).

Elucidation (Smith, 2016, 100-001) seems to call for the a definition of the notion of a reference system associated with a given slicing. The postulates and definitions of Sec. 2.1 provide the formal tools needed for the rigorous definition of the notion of 'reference frame'.

In the object language of the formal theory of Sec. 5 - 6 all formulas are evaluated with respect to the slicing associated with the 'actual world' (the actualized physical possibility $(\omega, \sigma) \in \mathcal{K}$). Thus time-slice predicate *TS* picks out a hyperplane of simultaneity (space) associated with some instant of coordinate time associated with the slicing σ and the induced frame of reference. In the light of elucidation (Smith, 2016, 100-001) all the statements of the formal ontology interpreted with respect to a clearly defined reference frame. The operators $\Box^\Sigma$ and $\Diamond^\Sigma$ respectively allow for statements that are to be interpreted with respect to frames of reference other than the current one.

ELUCIDATION (Smith, 2016, 080-003): "To say that each spatiotemporal region s temporally_projects_onto some temporal region t is to say that t is the temporal extension of s." In accordance with the discussion of elucidation (Smith, 2016, 100-001) there are two possible interpretations of 'temporally_projects_onto':

(i) Projection onto *coordinate time* means that on a given slicing σ, every spatio-temporal region $\mathsf{s} \in \mathcal{D}_{ST}$ has a corresponding interval in the range of the parameter t representing *coordinate time* in the sense of (Def. 2) such that $\mathrm{PrjT}_\sigma =_{df} \mathsf{s} \in \mathcal{D}_{ST} \mapsto \{t \in \Re \mid \sigma_t(\mathcal{T}) \cap \mathsf{s} \neq \emptyset\}$.

(ii) Projection onto *proper time* along the worldline γ_x of some distinguished particle x means that on a given slicing σ, every spatio-temporal region $\mathsf{s} \in \mathcal{D}_{ST}$ projects onto γ^x such that: $\mathrm{PrjT}_\sigma =_{df} \mathsf{s} \in \mathcal{D}_{ST} \mapsto \{\tau \in \Re \mid \exists t \in \Re : \sigma_t(\mathcal{T}) \cap \mathsf{s} \neq \emptyset \wedge \sigma_t(\mathcal{T}) \cap \gamma^x = \gamma^x(\tau)\}$.

In either case the notion of projection would be meaningful only in the context of a particular slicing of spacetime in the sense of Def. 1 or in the trivial slicing of Newtonian spacetime (Sec. 2.2). This is relevant particularly for the design of ontologies for domains in which Newtonian spacetime is an insufficient approximation of the underlying spacetime structure. This includes ontologies that are intended to support the identification, characterization, and tracking of space objects (Cox et al., 2016; Rovetto, 2016).

All elucidations of BFO that rely on the elucidation (Smith, 2016, 100-001) of 'temporal region' suffer a similar ambiguity. This includes the elucidations of 'zero-dimensional temporal region' (Smith, 2016, 103-003) [103-001], 'one-dimensional temporal region' (Smith, 2016, 103-003), 'b exists_at t' (Smith, 2016, 118-003).

ELUCIDATION (Smith, 2016, 081-003): "To say that spatiotemporal region s spatially_projects_onto spatial region r at t is to say that r is the spatial extent of s at t." As above, the notion of projection is meaningful only in the context of a particular slicing in the sense of Def. 1. Assuming that the parameter t in this elucidation ranges over coordinate time (Def. 2) with respect to a given slicing σ, spatially_projects_onto can be understood in at least two ways:

(i) A spatio-temporal region $\mathsf{s} \in \mathcal{D}_{ST}$ projects onto lower-dimensional sub-manifolds of spacetime, i.e., $\mathsf{s} \in \mathcal{D}_{ST}$ projects onto $\mathsf{r} \in \mathcal{D}_{ST}$ at t if $\mathsf{r} = \sigma_t(\mathcal{T}) \sqcap \mathsf{s}$. Thus, if $\sigma_t(\mathcal{T}) \sqcap \mathsf{s} \neq \emptyset$ then $\mathrm{PrjS}_\sigma =_{df} \mathsf{s} \in \mathcal{D}_{ST} \mapsto \sigma_t(\mathcal{T}) \sqcap \mathsf{s}$.

(ii) A spatio-temporal region $\mathsf{s} \in \mathcal{D}_{ST}$ projects onto sub-manifolds of the *abstract* time slice $\mathcal{T}$ of the underlying slicing σ, i.e., $\mathsf{s} \in \mathcal{D}_{ST}$ projects onto $\mathsf{r} \sqsubseteq \mathcal{T}$ at t such that $\mathsf{r} = \sigma_t^{-1}(\sigma_t(\mathcal{T}) \sqcap \mathsf{s})$ if $\sigma_t(\mathcal{T}) \sqcap \mathsf{s} \in \mathcal{D}_{ST}$. Thus, if $\sigma_t(\mathcal{T}) \sqcap \mathsf{s} \neq \emptyset$ then $\mathrm{PrjS}_\sigma =_{df} \mathsf{s} \in \mathcal{D}_{ST} \mapsto \sigma_t^{-1}(\sigma_t(\mathcal{T}) \sqcap \mathsf{s})$.

Purely spatial relations (Cohn et al., 1997; Egenhofer and Herring, 1990; Egenhofer et al., 1994) seem to hold between the sub-manifolds of the abstract time slice in the sense of (ii).

ELUCIDATION (Smith, 2016, 002-001): "b continuant_part_of c at $t =_{df}$ b is a part of c at t, t is a time and b and c are continuants." In the presented formal theory there is a strict distinction between the mereology of spacetime regions and the mereology of continuants. Parthood among regions of spacetime – represented by the predicate P in the object language of the formal theory – is characterized by an unrestricted extensional mereology (Sec. 5). By contrast, parthood among continuants – represented in the object language by the defined predicate P_c – is restricted to time slices (Sec. 6.2). On this view it seems to follow that the temporal parameter t in BFO is to be interpreted as coordinate time with respect to one particular slicing σ (Def. 2).

The specific mereology of continuants proposed here is different from that of BFO. The focus here was on continuants that are physical (material) entities formed by a finite number of atomic particles. In addition the aim was to be consistent with earlier publications (Bittner et al., 2004; Bittner and Donnelly, 2004, 2006b; Donnelly and Bittner, 2008) and to have a mereology that allows distinct continuants to have the same mereological makeup in the same physically possible world. The underlying reason is that even in conjunction mereological and physical properties may be insufficient to distinguish mereologically complex continuants (Thomson, 1998). This could be changed by adding some form of an antisymmetry axiom and/or a version of the strong supplementation principle to the mereology of continuant entities (Bittner and Donnelly, 2006b; Donnelly and Bittner, 2008).

The category of continuants is defined in the object language as persistent particulars that always instantiate at spatial regions (D_{Cont}). On the intended interpretation is mirrored in the metalanguage and its differential geometry as discussed in Sec. 6.1, Eq. 18.

ELUCIDATION (Smith, 2016, 077-002): "An occurrent is an entity that unfolds itself in time or ...[20] it is a temporal or spatiotemporal region which such an entity occupies_temporal_region or occupies_spatiotemporal_region." In the presented theory, the definition of the category of occurrent (D_{Occ}, Eq. 18) takes advantage of the rigorously defined notions of (a) spatio-temporal region (D_{STR}, Eq. 15), (b) instantiation of entities at regions in spacetime (Sec. 6.1), and (c) the notion of the actual world (a member of $Prj^{\mathcal{L}}$) (Eq.5 and Eq. 6). This allows to define occurrents in the object language as persistent particulars that always instantiate at spatio-temporal regions in a way that on the intended interpretation is mirrored in the metalanguage and its differential geometry as discussed in Sec. 6.1. In particular, physical occurrents need to be instantiated at spatio-temporal regions that are part of the actual world (line) (Eq. 19). As pointed out in elucidation (Smith, 2016, 080-003) the distinction between temporal and spatio-temporal region does not seem particularly helpful.

ELUCIDATION (Smith, 2016, 041-002,082-003,132-001): "b occupies_spatial_region r at t means that r is a spatial region in which independent continuant b is exactly located" ... "p occupies_spatiotemporal_ region s. This is a primitive relation between an occurrent p and the spatiotemporal region s which is its spatiotemporal extent." ... "p occupies_temporal_region t. This is a primitive relation between an occurrent p and the temporal region t upon which the spatiotemporal region p occupies_spatiotemporal_region projects."

In the presented formal theory the occupation relations of BFO directly correspond to the location relation L which is defined in terms of the primitive instantiation relation (D_L and Eq. 17). In terms of L the predicate "b occupies_spatial_region r at t" is expressed as: $L_{SR}\, brt' \equiv L\, br \wedge SR\, r \wedge TS\, t' \wedge P\, rt'$. It is important to see that in the presented formal theory t' is a time slice. It is linked to the interpretation of t in BFO as coordinate time (Df. 2) via the 'actual world' $\langle \gamma, \sigma \rangle$ at which the location predicate is interpreted in conjunction with expressions of the form $\sigma_t(\mathcal{T}) = t'$ in the intended interpretation. Similarly, "p occupies_spatiotemporal_region s" can be expressed as $L_{STR}\, bs \equiv L\, bs \wedge STR\, s$. If 'spatio-temporal region' is understood as 'worldline of a distinguished reference particle' then "p occupies_temporal_region t" can be expressed as $L_{STR}\, bs \equiv L\, bs \wedge STR\, s \wedge (\exists x)(At_E\, x \wedge WLOf\, xs)$.

ELUCIDATION (Smith, 2016, 003-002): "b occurrent_part_of $c =_{df}$ b is a part of c and b and c are occurrents." In the formal theory "b occurrent_part_of c can be represented as $P_o\, bc$ (Sec. 6.2). The mereology of occurrents – expressed in terms of P_c – is relatively standard. Extensionality could be enforced by adding an axiom similar to (T4). The lack of an antisymmetry axiom allows for distinct continuants with identical mereological structure.

ELUCIDATION (Smith, 2016, 019-002): "A material entity is an independent continuant that has some portion of matter as proper or improper continuant part." The notion of physical possibility/necessity is clearly too weak to capture the notion of metaphysical dependence. Nevertheless the notion of matter seems to mean something like 'made up of physical particles'. Thus the predicate *Cont* (D_{Cont} and Eq. 18) should be helpful for providing a formal specification of this elucidation. In addition, the fact that the formal theory is presented in the language of a modal predicate logic makes it relatively easy to add modalities that support the formal specification of metaphysical dependency.

ELUCIDATION (Smith, 2016, 024-001): "b is an object means: b is a material entity which manifests causal unity ... and is of a type (a material universal) instances of which are maximal relative to this criterion of causal unity." The presented formal theory does not provide an account of causality. Nevertheless the presented formal ontology can contribute to make this elucidation more precise and to express it more rigorously: Whatever specific form causation takes, locations in spacetime at which causally related events can take place must be connected by dynamically (or at least kinematically) possible worldlines. For example Postulate 3 constrains causally accessible portions of spacetime by reference to the light cone structure induced by Minkowski spacetime.

[20]The original elucidation also included instantaneous occurrent entities (stages in the sense of (Sider, 2001)) and the regions they occupy. To simplify the formal theory stages have been excluded here. A formulation that does include the category of stages can be found in (Bittner and Donnelly, 2004).

ELUCIDATION (Smith, 2016, 138-001,XXX-001): "A history is a process that is the sum of the totality of processes taking place in the spatiotemporal region occupied by a material entity or site, including processes on the surface of the entity or within the cavities to which it serves as host." ... "b history_of c if c is a material entity or site and b is a history that is the unique history of c." ... "AXIOM: if b history_of c and b history_of d then $c = d$."

BFO's category of a history seems to overlap with the formal category of occurrents in the sense of D_{Occ} and Eq. 18. Clearly, the framework presented here does not include sites, holes, surfaces, and the processes in which they participate. Nevertheless occurrents in the sense of D_{Occ} and Eq. 18 do include histories of material entities. Consequently, the BFO-predicate history_of has the worldline_of predicate (D_{WLOf} and Eq. 17) as a sub-predicate. Thus the presented theory may help to make this elucidation more precise and rigorous.

7.3. Computational realization

The discussion so far has focussed on the relations between (i) the formal ontology and its potential for improving foundational ontologies such as BFO and (ii) the relations between the formal ontology and the physics encoded in its meta-language. A third class of issues that deserves discussion is methodological in nature and concerns the computational realization of the formal theory presented above. Clearly, it is far from obvious that the presented theory is consistent. Moreover, even if it is consistent it is far from obvious that the class of intended models – the $\mathcal{KS}$-structures of Sec. 4.3 – are among the models of the formal theory.

To address this class of issues a computational realization of the formal theory in the HOL-based framework of the theorem proving environment *Isabelle* (Paulson and Nipkow, 2017; Paulson, 1994) was built with the aims to formally verify (a) the consistency of the formal theory, (b) whether or not specific mathematical representations of physical systems are models of the formal theory, and (c) that all the theorems of the formal theory presented in Sec. 5 and Sec. 6 are derivable from the axioms of the theory. HOL is a framework of higher order logic which combines predicate logic (Copi, 1979) with lambda calculus (Church, 1941) in a way that is based on Church's Theory of types (Church, 1940). This highly expressive formal framework as it is realized within the Isabelle theorem proving environment (referred to as Isabelle/HOL) was chosen because, it provides all the tools needed to address the points (a) – (c). Moreover, at least in principle, this framework provides means to formally capture the mathematics required to express the physics as discussed in the first part of this paper.

To achieve the goals (a) – (c) the computational representation employs a formal infrastructure of the Isabelle/HOL system that is called *Locales* (Kammüller et al., 1999; Ballarin, 2004). Locales are generalizations of axiomatic type classes (Jones and Jones, 1997; Wenzel, 2005) that originally were introduced as part of the functional language Haskell (Thompson, 1999). In the computational realization of the formal theory presented here Locales are used to link the predicates of the formal theory to the $\mathcal{KS}$-structures in which they are interpreted. Intuitively, every predicate has an implicit argument that ranges over $\mathcal{KS}$ structures (similar to the implicit 'this' pointer in C^{++}). The computational realization of the formal theory has three fundamental levels: (I) the level of axiomatization; (II) the level of model instantiation; and (III) the level of presentation.

(I) The level of axiomatization: At the level of axiomatization the axioms A1-26 are expressed in a non-modal language with explicit reference to $\mathcal{KS}$-structures. The latter include the sets that determine the domains of quantification, the accessibility relations, as well as the relations that serve as the interpretation of the primitives of the formal theory. In essence, in the computational representation the axiomatization is realized at a level that corresponds to the level of interpretation in the presentation of the formal theory of Sec. 4 – 6. All theorems listed in the paper are proved at this level and then 'lifted' to the level of the presentation in the modal logic. (See level III.)

(II) The level of model instantiation: The Isabelle system also provides an infrastructure for instantiating abstract model-theoretic structures by specific models and for creating proof obligations that, when fulfilled, ensure that the specific model satisfies all the axioms (A1–A26) that are associated with the abstract model-theoretic structures ($\mathcal{KS}$-structures in this case). The specification of the concrete models is purely definitional within Isabelle's implementation of the HOL framework. Therefore no additional axioms that could lead to inconsistencies are introduced. (See (Nipkow et al., 2002) for a discussion of the definitional approach in Isabelle/HOL.) The infrastructure provided by Isabelle/HOL and its locales thereby provides formal means to verify the consistency of a set of axioms as well as means to verify that the intended models are among the actual models of a formal theory.

In principle Isabelle/HOL has the expressive power that is required for the formalization of the differential geometry that provided the language for expressing the physics in the first part of the paper. Unfortunately, presently no comprehensive formalization of differential geometry in Isabelle/HOL exists. To illustrate the methodology, to provide a proof of the consistency of the axioms A1-26, and to show that the axioms A1-26 have a model that at least to some degree resembles important aspects of the physics that is encoded in the differential geometry in the first part of the paper, the example of Sec. 4.4, understood as a specific model, is instantiated and thereby shown to satisfy the axioms of the formal theory.

(III) The level of presentation: For the presentation of the formal theory in Sec. 5 – 6 a modal predicate logic was selected. The choice of a modal language greatly simplifies the presentation and hides specifics of the underlying interpretation. The modal operators provide means to talk about physical possibility/necessity as well as means to talk about slicings and properties that are dependent on, or independent of, the underlying slicing of the spacetime manifold. Using a modal language facilitates conceptual clarity while maintaining formal rigor.

At this level all axioms and theorems of the level of axiomatization are 'lifted' to the modal language described in Sec. 4 using the encoding of modal logic into Isabelle/HOL (or any other HOL framework) by Benzmüller (2015); Benzmüller and Woltzenlogel Paleo (2015). This lifting heavily uses features of the underlying lambda calculus. The lifting ensures that all axioms and theorems of in Sec. 5 and Sec. 6 are theorems in computational realization of the modal language.

A more detailed discussion of the computational realization goes beyond the scope of this paper. The formal implementation in Isabelle/HOL in conjunction with the underlying methodological and design choices are presented in detail in (Bittner, 2017). The Isabelle/HOL theory files can be accessed at `http://www.buffalo.edu/~bittner3/Theories/OntologyCM/`. The links between the formalism presented here and the specific Isabelle/HOL constructs are summarized in Table 6.

8. Conclusion

The aim of this paper was to develop a formal framework that explicates important ontological commitments of classical mechanics. To be consistent with the various formulations and the various possible spacetime ontologies that underly classical mechanics, a class of models based on $\mathcal{T}$-slicings serve as the intended interpretation of the formal theory. Thus, the focus of the metalanguage were *geometric* aspects of spacetime and associated structures. By contrast, the focus of the object language were on the notion of '*physical possibility*' in the context of the logical interrelations between mereological, instantiation, and location relations.

The presented formal theory in conjunction with the development of its meta-theory and its computational representation is intended to illustrate the following points: Firstly, the clear separation of the semi-formal treatment of geometric aspects in the meta-language and the strictly formal treatment of ontological relations in the object language may help to deal with the tension between the limited expressivity of first order logic and the high complexity of the languages that are needed to describe the fundamental geometric structures that govern the kinematics and dynamics of the underlying physics. Secondly, the

level	content	theory file	axioms and definitions
(I)	Setup of the modal 'lifting', definition of $\mathcal{KS}$-structures, basic type declarations	S5_2D_base	Eq. 7, $D_{\Diamond^i}$, $T_{\Box^i}$, $5_{\Box^i}$, $K_{\Box^i}$, $BC_{\Box^i}$, $MS_{\Box}$
(I)	Mereology	PLattice	Eq. 12, A1–A7, T1–T7, D_P, D_O, D_{Sum}
(I)	Time slices, Newtonian and Minkowski spacetimes	TS_mereology	Eq. 14, A8–A10, D_{SR}, D_{STR}, D_{SIMU}, T8–T12, A_N, A_M, T_N1–T_N3
(I)	Instantiation, location, particulars, universals, continuants, occurrents, Mereology of particulars	Inst_TS_mereology	A11 – A18, D_L, D_{Pe}, D_{Part}, D_{Uni}, D_{Cont}, D_{Occ}, D_{WLOf}, D_{P_c}, D_{P_o}, T13 -T30
(I)	Atomic entities	AtE_Inst_TS_mereology	A20 – A26, T31 – T35
(II)	Definitions for the model of Sec. 4.4	ST_model_base	
(II)	Proof that the model of Sec. 4.4 satisfies the axioms A1 – A26	ST_model_proof	
(III)	Presentation of the formal theory in the modal language	S5_2D_lifted_theory	A1 – A26 as theorems, T1 – T35

Table 6

Overview of the computational representation of the formal theory in Isabelle/HOL. (Accessible at: `http://www.buffalo.edu/~bittner3/Theories/OntologyCM/`.)

development of this paper has indicated the importance of formalizing both aspects of a formal ontology: the axiomatic development in the object language and the specification of the structures that are intended to serve as interpretations. Thirdly, the development of a computational realization of the ontology in Isabelle/HOL provides strong indications that computational tools for developing axiomatic theories have reached a degree of maturity that allows for the development of large scale ontologies in highly expressive languages.

The description of physical possibilities in this paper focussed on physically possible *micro* states of systems of m particles. What was not addressed was how a m-particle system as a whole can be (permanently or temporarily) in a solid (*macro*) state, in a liquid (*macro*) state or in the (*macro*) state of a gas. The emergence of the macroscopic states of our experience in which certain mereological sums of microscopic particles give rise to solid objects is beyond the scope of this paper. Future work needs to address the distinction between micro- and macro states and the ways in which the emergence of the latter are studied for example in *statistical* mechanics (Frigg, 2012; Redhead, 1995). This may lead to interesting insights of the formal ontology of matter, stuff, and material objects.

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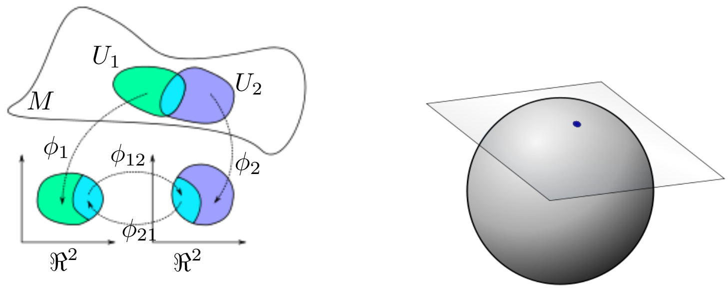

Fig. 6. Charts on a manifold (left) (Wikimedia Commons, 2015d); Tangent space $T_x\mathcal{S}$ at $x \in \mathcal{S}$ on the manifold $\mathcal{S}$ (right) (Wikimedia Commons, 2015a).

Appendix

A. Differential geometry

This section reviews some notions of differential geometry that are needed to develop the formal models for the ontology to be developed in this paper. Firstly, manifolds are introduced. They are critical for capturing geometrically notions of smoothness and continuity. Secondly, geometric objects such as scalar fields, vectors, vector fields on manifolds are introduced. Thirdly, curves on a manifold and derivatives establish relationships between scalar fields and vector fields via the notion of integral curves. Those three classes of geometric objects and their interrelations geometrically encode what is physically possible.[21]

A.1. Manifolds

A differentiable manifold is a topological manifold with a globally defined differential structure. A topological manifold is a topological space that is locally homeomorphic to a linear (i.e. vector) space. Formally, this local structure is given by local homeomorphisms – the charts ϕ_i, mapping open subsets U_i of M to subsets of $\Re^n$ which are n-dimensional vector spaces (Fig. 6 (left)). The (global) differential structure of a manifold is built up by combining the local linear structures, local charts, to a system of atlases that cover the whole manifold such that on can reach any chart from any other chart by means of a smooth transformation. A smooth transformation or *diffeomorphism* is an invertible map that takes smooth curves to smooth curves, where a smooth curve is a curve that has derivatives of all orders everywhere. Where distinct charts overlap they must be compatible. (Fig. 6 (left)).

At every point x of a differentiable manifold M, there is a linear space T_xM 'attached' to M at x (Fig. 6 (right)), i.e., T_xM is the *tangent space* of M at x. For all $x \in M$, T_xM has the same dimension as the manifold M at x. In planar (non-curved) manifolds like the Euclidean space $M = \Re^n$, the vectors in the tangent space T_xM at every point $x \in M$ span the whole manifold M. That is, every point $y \in M$ can be represented using a vector $\xi \in T_xM$ such that ξ begins at x and ends at y. By contrast, in curved manifolds like the surface of a sphere $\mathcal{S} \subset \Re^3$ only points in the immediate neighborhood U_x of $x \in \mathcal{S}$ can be represented by vectors in the tangent space $T_x\mathcal{S}$ (Fig. 6 (right)).

The disjoint union of all tangent spaces T_xM of M gives rise to the *tangent bundle* TM of M, i.e., $TM = \bigcup_{x \in M} (\{x\} \times T_xM)$. A point in the tangent bundle TM is a pair (x, ξ) with $\xi \in T_xM$. The tangent bundle of a differentiable manifold of dimension d is a differentiable manifold of dimension $2 * d$.

Between manifolds the submanifold relation $\sqsubseteq$ holds. Roughly, $M_1 \sqsubseteq M_2$ if and only if M_1 is a subset of M_2, M_1 and M_2 are manifolds, and the tangent spaces of M_1 are subspaces of the tangent spaces of M_2.

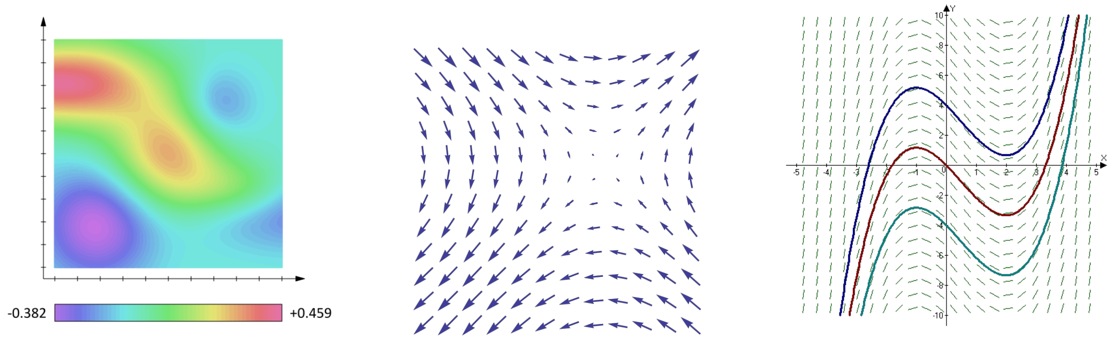

Fig. 7. A scalar field (left) (Wikimedia Commons, 2015b); a portion of the vector field $(\sin\ y, \sin\ x)$ (middle) (Wikimedia Commons, 2015c). Three integral curves for the slope field corresponding to the differential equation $dy/dx = x^2 - x - 1$. (right) (Wikimedia Commons, 2013)

A.2. *Scalar fields, vector fields, and curves*

A *curve* γ on a manifold M is a mapping $\gamma : \Re \to M$ from the real numbers to points of M. In what follows the letter 'γ' will be used to refer to parametric curves, i.e., functions of type $\gamma : \Re \to M$ as well as to the curves $\{\gamma(\tau) \in M \mid \tau \in \Re\}$ themselves. The context will disambiguate. If the curve γ is *smooth* then at every point $x = \gamma(\tau)$ there is a unique vector ξ in the tangent space T_xM such that ξ is tangent to γ at the point x (i.e., $\xi = \frac{d}{d\tau}\gamma(\tau)|_x$). If γ is a smooth curve on manifold M then $\gamma \sqsubseteq M$. Intuitively, the tangent space T_xM contains all possible "directions" along which a curve on M can tangentially pass through x. That is, tangent spaces arise naturally as structures formed by equivalence classes of curves on the underlying manifold.

A *scalar field* $H : M \to \Re$ on a manifold M is a smooth mapping from M to the domain of scalars (real numbers $\Re$ for measurable qualities) (Fig. 7 (left)). A *vector field* $X : M \to TM$ on a manifold M is a smooth mapping from M into the tangent bundle TM so that every point $x \in M$ maps to exactly one vector $\xi \in T_xM$ of the tangent space T_xM (Fig. 7 (middle)). [22] There is a close relationship between the smooth curves of a manifold and the vector fields on that manifold. The smooth parametric curve $\gamma_{X,x} : \Re \to M$ is the *integral curve* of the vector field $X \in \mathcal{X}(M)$ through the point $x \in M$ if and only if for all $\tau \in \Re$:

$$\frac{d}{d\tau}\gamma_{X,x}(\tau) = X(\gamma_{X,x}(\tau)) \text{ and } \gamma_{X,x}(0) = x. \tag{23}$$

That is, at all points $y = \gamma_{X,x}(\tau)$ the tangent to the curve $\gamma_{X,x}(\tau)$ at y is the vector $X(y) \in T_yM$. This is illustrated in the right of Fig. 7. In standard presentations of classical mechanics integral curves appear as the specific solutions of the differential equations that constitute the laws of physics – the equations of motion of the underlying physical system (Appendix B).

B. Lagrangian mechanics and dynamic possibilities

To specify the dynamics of a physical system in the Lagrangian framework is to identify worldlines along which physically possible processes can occur and along which physically possible particles can evolve. The essence of the Lagrangian framework is to identify the dynamically, i.e., physically, possible worldlines within the larger class of kinematically possible worldlines using a scalar field that is called *The Lagrangian*.

[21]The presentation of this subject here must remain brief and rather selective. For details see for example overviews by Arnold (1997); Butterfield (2007).

[22]In modern physics dynamic fields which change over time are studied (Abraham and Marsden, 1978; Teller, 1997; Wayne, 2000). For the purpose of this paper it will be sufficient to study static fields, i.e., fields that vary in space but do not change over time.

B.1. The Lagrangian vector field

Consider a physical system with configuration space $\mathcal{Q}(\mathcal{ST})$ and a set $\Gamma^{\mathcal{Q}}$ of kinematically possible worldlines. Let $\Gamma^{\mathcal{L}} \subset \Gamma^{\mathcal{Q}}$ be the set of dynamically, i.e., physically, possible worldlines. The members $(\gamma_1, \ldots, \gamma_m)$ of $\Gamma^{\mathcal{L}}$ are m-tuples of worldlines of a m particle system. Consider the i-th particle p_i. Since the space of kinematic possibilities of p_i, $(\Re \times M_i) \sqsubseteq \mathcal{Q}(\mathcal{ST})$, is a manifold and the kinematically possible worldlines of p_i are smooth curves in $(\Re \times M_i)$, there is a vector field $X_i^{\mathcal{L}}$ on $(\Re \times M_i)$ that is formed by all the tangent vectors of all the dynamically possible worldlines in $\gamma_i \in \Gamma_i^{\mathcal{L}}$:

Definition 5 (Lagrangian vector field $X^{\mathcal{L}}$ (Libermann and Marle, 1987))**.** *The Lagrangian vector field $X_i^{\mathcal{L}}$ is the vector field, i.e., a mapping of type $X_i^{\mathcal{L}} : (\Re \times M_i) \to T(\Re \times M_i)$, on $(\Re \times M_i) \sqsubseteq \mathcal{Q}(\mathcal{ST})$. The integral curves of the vector field $X_i^{\mathcal{L}}$ are the dynamically possible worldlines of the i-th particle of the underlying m particle system, i.e., the members of $\Gamma_i^{\mathcal{L}}$:*

$$\forall \gamma \in \Gamma_i^{\mathcal{L}}, \tau' \in \Re : X_i^{\mathcal{L}}(\gamma(\tau')) = \frac{d}{d\tau}\gamma(\tau)|_{\tau=\tau'}. \tag{24}$$

The Lagrangian vector field $X^{\mathcal{L}}$ on the configuration space $\mathcal{Q}(\mathcal{ST})$ as a whole is $X^{\mathcal{L}}$. At every point $((t,(x_1,\ldots,x_m)) \in \mathcal{Q}(\mathcal{ST})$ the vector field $X^{\mathcal{L}}$ is determined by the vectors $X_1^{\mathcal{L}}(t,x_1),\ldots,X_m^{\mathcal{L}}(t,x_m)$.

B.2. Lagrangians and actions

For many physical systems the Lagrangian vector field $X_i^{\mathcal{L}}$ and its integral curves $\Gamma_i^{\mathcal{L}}$ are uniquely determined by a scalar field $\mathcal{L}$ – *the Lagrangian field* – which is a function of type $\mathcal{L}_i : T(\Re \times M_i) \to \Re$ on the tangent bundle $T(\Re \times M_i)$ the space of kinematic possibilities $(\Re \times M_i)$ of the i-th particle. That is $(x,\xi) \in T(\Re \times M_i) \mapsto \mathcal{L}_i(\mathsf{x},\xi) \in \Re$.

The Lagrangian scalar field $\mathcal{L}_i : T(\Re \times M_i) \to \Re$ has a scalar value $\mathcal{L}_i(X_i^{\mathcal{L}}(\gamma_i(\tau)))$ at all $X_i^{\mathcal{L}}(\gamma_i(\tau))$ along γ_i. For a given Lagrangian field $\mathcal{L}_i$ every kinematically possible curve γ_i is characterized by a quality called the *action* $A_{\mathcal{L}}(\gamma_i)$. Roughly, the action along γ_i is the 'sum' of all the values of $\mathcal{L}_i(X_i^{\mathcal{L}}(\gamma_i(\tau)))$ along γ_i. It is computed by integrating $\mathcal{L}_i$ along γ_i (Eq. 25). The action of the m particle system as a whole is the sum of the action along the worldlines of its constituting particles:

$$A_{\mathcal{L}}(\gamma_1,\ldots,\gamma_m) =_{df} (\gamma_1,\ldots,\gamma_m) \mapsto \Sigma_{1\le i\le m} \int \left(\mathcal{L}_i(X_i^{\mathcal{L}}(\gamma_i(\tau)))\right) d\tau. \tag{25}$$

B.3. Stationary curves and Hamilton's principle

$\Gamma^{\mathcal{Q}}$, the set of kinematically possible worldlines of a m particle system forms a manifold and the 'points' of this manifold are the curves γ in $\Gamma^{\mathcal{Q}}$ (Woodhouse, 1992; Belot, 2007)[23]. As a manifold $\Gamma^{\mathcal{Q}}$ has a tangent bundle $T\Gamma^{\mathcal{Q}}$. Intuitively[24], if the possible infinitesimal changes of γ are identified with the members γ' of the tangent space $T_\gamma\Gamma^{\mathcal{Q}}$ then infinitesimal changes of γ do not change the action $A_{\mathcal{L}}$ if and only if the directional derivative[25] $dA_{\mathcal{L}}$ of the scalar field $A_{\mathcal{L}}$ on $\Gamma^{\mathcal{Q}}$ is zero in all 'directions' $\gamma' \in T_\gamma\Gamma^{\mathcal{Q}}$.

Definition 6. *A curve γ is stationary for $\mathcal{L}$ if and only if the action $A_{\mathcal{L}}(\gamma)$ does not change for infinitesimal changes of γ. That is, stationary(γ) iff: for all $(\gamma,\gamma') \in T_\gamma\Gamma^{\mathcal{Q}}$: $dA_{\mathcal{L}}(\gamma,\gamma') = 0$.*

Hamilton's principle then states:

[23]This subsection illustrates the abstract nature and the power of the formalism of of differential geometry which effortlessly applies to 'real' spaces as well as to abstract 'spaces' of functions, etc.

[24]See (Belot, 2007)for a more precise presentation.

[25]Let $H : M \to \Re$ be a smooth scalar field on M. The *directional derivative* dH_x of the field H at point $x \in M$ in direction $\xi \in T_xM$ is a mapping $dH_x : T_xM \to \Re$ which returns the rate of change of H at x in direction ξ. That is, if $\gamma : [-1,1] \to M$ is a smooth curve with $\gamma(0) = x$ and $\frac{d}{d\tau}\gamma(0) = \xi$, then the directional derivative is defined by

$$dH_x =_{df} \left.\frac{d}{d\tau}(H \circ \gamma)(\tau)\right|_{\tau=0}. \tag{26}$$

Postulate 7 (Hamilton's principle)**.** *The dynamically possible worldlines in the configuration space* $\mathcal{Q}(\mathcal{ST})$ *are the kinematically possible curves* γ *in* $\Gamma^{\mathcal{Q}}$ *which are* stationary *for the scalar Lagrangian field* $\mathcal{L} : T(\mathcal{Q}(\mathcal{ST})) \to \Re$*. That is, the set* $\Gamma^{\mathcal{L}}$ *of dynamical possibilities is:*

$$\Gamma^{\mathcal{L}} =_{df} \{\gamma \in \Gamma^{\mathcal{Q}} \mid \forall(\gamma, \gamma') \in T_{\gamma}\Gamma^{\mathcal{Q}} : dA_{\mathcal{L}}(\gamma, \gamma') = 0\} \tag{27}$$

It follows from Def. 5 that the integral curves $\gamma \in \Gamma^{\mathcal{L}}$ of the Lagrangian vector field $X^{\mathcal{L}}$ are stationary with respect to the scalar Lagrangian field $\mathcal{L}$ on $T(\mathcal{Q}(\mathcal{ST}))$.

In summary, in the Lagrangian framework, to determine the space of dynamically, i.e., physically, possible worldlines of a physical system with configuration space $\mathcal{Q}(\mathcal{ST})$ is to identify kinematically possible curves that are stationary with respect to the Lagrangian field $\mathcal{L}$ on $T(\mathcal{Q}(\mathcal{ST}))$.[26] Every dynamic, i.e., physical, possibility in this space of possibilities is a worldline $\gamma \in \Gamma^{\mathcal{L}}$ along which a system with m particles can evolve. To actualize one of these possibilities in spacetime is what it means *to be* a physical system with m particles.

[26]More precise definitions of the action and the notion of stationarity in conjunction with necessary and sufficient conditions for kinematically possible curves $\gamma \in \Gamma^{\mathcal{Q}}(\mathcal{ST})$ to be stationary for the Lagrangian field $\mathcal{L}$ see overviews by Arnold (1997); Butterfield (2007).