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Is there a quantum geography?

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Abstract In this paper I argue that a quantum theory may be the appropriate tool for describing phenomena with indeterminate boundaries in the context of the classification and delineation of geographic regions. A motivation for this claim stems from the observation that fundamental aspects of information about the physical world that follow from the success of quantum mechanics also apply to information about certain classes of geographic phenomena. Those aspects include (Rovelli, 1996; Rovelli and Vidotto, 2015): (i) information about the physical [geographic] world is fundamentally relational; (ii) information of the physical [geographic] world is fundamentally granular; and (iii) information about the physical [geographic] reflects the fundamentally indeterminate nature of certain aspects of the world at the respective scales. (The words in the brackets were added by this author.) More rigorous support for the above claim comes from recent work in theoretical physics. This work has identified three information-theoretic conditions that, when satisfied for a class of phenomena, call for a quantum theory as the appropriate theoretical framework for that class. In this paper I show that there are geographic phenomena which satisfy two of the three conditions that call for a quantum theory. I then argue that the third criterium can be validated or refuted in the geographic context by developing a quantum theory for geographic phenomena with indeterminate boundaries with classification and delineation operations as means to obtain information about those phenomena. Such a theory then can produce predictions that will either be verified by observations on the ground and thereby confirm the need for a quantum theory, or rule it out as a viable option.

0.1 Introduction

Is there a quantum geography? This seems to be a question with an obvious answer: No! Things in the quantum world are too strange to inhabit the geographic world. We do not see geographic objects in different places at the same time. Moreover, geographic phenomena do not seem to have contradicting properties at the same time. It is the aim of this paper to suggest that things are not such obvious and that there may be a quantum geography after all.

First of all, the question 'Is there a quantum geography?' is ill-posed (but catchy). Quantum mechanics is a formalism that has been used successfully to describe the behavior of entities at the sub-atomic scale. Therefore, a better way of expressing the above question may be 'Are there geographic phenomena that can be described best by a quantum theory?'. This question seems to be answerable by addressing the following (sub)-questions: (a) What is a quantum theory? (b) Are there necessary and sufficient conditions that call for a quantum theory as an adequate description? (c) If there are such conditions, are there geographic phenomena for which such conditions are satisfied.

In what follows I will address question (a) at an abstract level without going into the details of the quantum formalism. The aim of the paper is not to actually develop a quantum theory of (certain) geographic phenomena but to argue for the need to develop one. The main focus of the paper is on questions (b) and (c) – necessary and sufficient conditions that, when fulfilled, rule out classical logic and therefore classical geography which, like classical physics, is based on classical logic. Those conditions have been identified in the realm of theoretical physics in an attempt to provide an interpretation of the formalism of quantum mechanics. I will review the arguments for why theories of phenomena that satisfy the following three postulates are necessarily non-classical but quantum mechanical (Rovelli, 1996):

Postulate 1 (Limited information) *There is a maximum amount of* relevant *information that can be extracted from a system.*

Postulate 2 (Unlimited information) *It is always possible to acquire new information about a system.*

Jointly, Postulates 1 and 2 entail (as will be discussed below) that information about the systems that satisfy those postulates is subject to indeterminacy. A quantum theory then assumes:

Postulate 3 (Indeterminacy as probability) *The indeterminacy that arises from Postulates 1 and 2 manifests itself in the probabilistic nature of the processes in which information can be obtained by measurement/observation.*

After discussing Postulates 1 – 3, I am going to argue that geographic phenomena with indeterminate boundaries (Burrough and Frank, 1995), such as ecoregions (Bailey, 1983; Omernik and Griffith, 2014) and regions that are characterized by certain types of land cover (Andereson et al, 1976), fall in the class of phenomena that are subject to those criteria. Conceptually, the information-theoretic Postulates 1 and 2 are the most interesting and relevant in the context of this paper. In particular I will discuss both in the context of the classification and delineation of geographic regions, an area of geography in which boundary indeterminacy has been studied extensively (Bailey, 1983; Omernik and Griffith, 2014). Whether or not the indeterminacy that arises from Postulates 1 and 2 manifests itself probabilistically is an empirical question and can at least in principle be tested by experiment. I will sketch a toy theory that is consistent with Postulates 1 and 2 and that, when developed fully, will make predictions which are probabilistic nature and that can be tested empirically.

There is one formalism of (non-relativistic) quantum mechanics which was developed by Dirac (1930) and John von Neumann (1932). This formalism produces predictions that have been verified over and over since that time. By contrast, there are many interpretations of this formalism (Omnès, 1994;

Wikipedia, 2012). Interpretations are attempts to describe the world which brings about the phenomena that have the properties that are predicted by the formalism. According to many interpretations the world described by the formalism of quantum mechanics is essentially *non-local.* That is, according to many interpretations of the formalism, the phenomena described by it can interact instantaneously across arbitrary distances (Einstein et al, 1935; Maudlin, 2002; Musser, 2015; Redhead, 1997; Romero, 2012). Physicists have found ways to make this consistent with the theory of Special Relativity (Einstein, 1905; Kennedy, 2003) which postulates that information cannot travel faster than the speed of light. Nevertheless, the non-locality entailed by many interpretations of quantum mechanics contradicts Tobler's First Law of Geography which postulates that in geographic space "everything is related to everything else, but near things are more related than distant things." (Tobler, 1970) One of the few interpretations of quantum mechanics that preserves locality is the relational interpretation of quantum mechanics (RQM) (Rovelli, 1996). The fact that, on the relational interpretation, the formalism of quantum mechanics is consistent with Tobler's law in conjunction with the strong information-theoretic focus of this interpretation are the reasons for adopting it as the foundation of this paper.

The reminder of this paper is structured as follows: Firstly the basic ideas of the relational interpretation of quantum mechanics are discussed. In the context of this framework Postulates 1 and 2 arise which entail that the underlying logic is non-classical. I briefly discuss the commitments and intuitions that underly the probabilistic understanding of the indeterminacy that arises from Postulates 1 and 2. For closure, I also briefly introduce some aspects of the formalism of QM itself. The second part of the paper argues that phenomena with indeterminate boundaries particularly in the context of the classification and delineation are subject to Postulates 1 and 2. Those who believe that Postulates 1 and 2 are true for the considered class of phenomena are then committed to believe that, if one can verify Postulate 3 experimentally, that there is a quantum geography in the qualified way described above. I will close by sketching a toy example that illustrates how a quantum theory that produces such probabilistic predictions that can be either confirmed or refuted by observations on the ground in principle looks like.

0.2 Relational quantum mechanics

According to quantum mechanics (QM), any measurement/observation is fundamentally a physical interaction between the system S being measured and some observing system O. In relational quantum mechanics (RQM) such a physical interaction is the establishment of a *correlation* between the observed system and the observing system. This form of correlation corresponds to the notion of *information* in Shannon's information theory (Shannon, 1948). Therefore, at the core of the relational interpretation of quantum mechanics (RQM) is the recognition, that measurement/observation are bidirectional, information-theoretic processes (Yang, 2018).

The *amount* of information that one system has about another system can be quantified as the number of the elements of a set of alternatives out of which a configuration is chosen (Shannon, 1948). In this context the set of alternatives are possible ways in which the observed and observing systems can be correlated. Information is a *discrete* quantity: there is a minimum amount of information exchangeable: a single bit, or the information that distinguishes between just two alternatives. Therefore, the process of acquisition of information (a measurement/ an observation) can be framed as a "question" that an observing system asks an observed system (Wheeler, 1989). Since information is discrete, any process of acquisition of information can be decomposed into acquisitions of elementary bits of information. An elementary question that collects a single bit of information is a "yes/no question". In what follows, such yes/no questions are labeled as $Q_1, Q_2, \ldots$

In RQM any system S, viewed as an observed system, is characterized by the family of yes/no questions that can be asked to it. Following Rovelli (1996) the set of yes/no questions is written as $W(S) \equiv \{Q_i \mid i \in I\}$, for some index set I. The result of a sequence of questions $(Q_1, Q_2, Q_3, \ldots)$ to S, from an observer system O, can be represented by a binary string $(e_1, e_2, e_3, \ldots)$, where each e_i is either 0 or 1 (no or yes) and represents the response of the system S to the question Q_i.

0.2.1 The first postulate of RQM

The first postulate of RQM, Postulate 1, can be spelled out more precisely in Wheeler's (1989) information-theoretic framework (Rovelli, 1996): For all $Q_i \in W(S)$: if Q_i can be inferred from (is determined by) an infinite string of answers $(e_1, e_2, e_3, \ldots)$, then Q_i can also be inferred from (is determined by) a finite string $[e_1, ..., e_N]$ of answers. Any system S has a maximal "information capacity" N, where N is an amount of information that is expressed in bits. N bits of information exhaust everything one can say about S.

Combinatorially there are 2^N different binary strings of length N (left of Fig.0.1). Since 2^N possible answers $s^{(1)}, s^{(2)}, ..., s^{(2^N)}$ to the N yes/no questions are (by construction) mutually exclusive, one can identify 2^N questions $Q^{(1)}, \ldots, Q^{(2^N)}$ such yes answers to the question $Q_c^{(i)}$ correspond to the string of answers $s^{(i)}$. This is illustrated in the left part of Fig. 0.1 for the specific case of two yes/no questions Q_1 and Q_2 which give rise to the set $Q_c = \{Q_c^{(i)} \mid 1 \leq i \leq 2\}$ of $2^2 = 4$ combinatorially possible *complete* questions (Beltrametti et al, 1984; Hughes, 1981; Rovelli, 1996).

The set-theoretic unions of sets of complete questions $Q_c^{(i)}$ (of the same family c), give rise to a Boolean algebra (see Appendix .1) that has singleton sets of the form $\{Q_c^{(i)}\}$ as atoms (right of Fig.0.1). Intuitively, the atoms of the Boolean algebra are the 2^N different states of S that can be distinguished given N bits of information provided by answers to the yes/no questions $Q_1, \ldots, Q_N$. The non-atomic nodes of the Boolean algebra describe disjunctions of the form $Q_c^i \vee Q_c^j$ in which there is less than N bits of information is available. The maximal element of the Boolean algebra has minimal information. By contrast, the atoms have maximal information. In the right of Fig. 0.1 the Boolean algebra that arises from the set Q_c of $2^2 = 4$ combinatorially possible complete questions.

	$s_c^{(i)}$		
i	Q_1	Q_2	$Q_c^{(i)}$
0	0	0	$\neg Q_1 \wedge \neg Q_2$
1	0	1	$\neg Q_1 \wedge Q_2$
2	1	0	$Q_1 \wedge \neg Q_2$
3	1	1	$Q_1 \wedge Q_2$

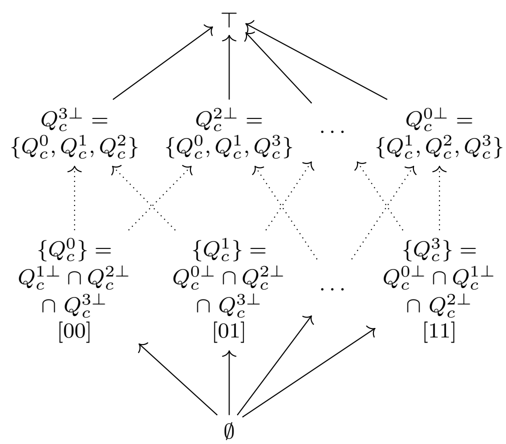


Fig. 0.1 Set $Q_c = \{Q_c^{(i)} \mid 0 \leq i < 4\}$ of 2^4 combinatorially possible complete questions $Q_c^{(i)}$ formed by two yes/no questions Q_1, Q_2 (left) and the distributive orthomodular lattice (a Boolean algebra) formed by the complete questions $Q_c^{(i)}$ (right). See also (Hughes, 1981) for details and illustrations.

The fact that there are 2^2 distinct pattern of answers to two yes/no questions logically/combinatorially possible does not guaranty that all the logical possibilities are also physically possible. For example, of the set Q_c of 2^4 logically possible complete questions only Q_S^1 and Q_S^2 are assumed to be physically possible and Q_S is the set of physically possible complete questions, i.e., $Q_S = \{Q_S^1, Q_S^2\}$. This results in the table in the left of Fig. 0.1. The corresponding Boolean algebra is depicted in the right of the Figure.

	$s_S^{(i)}$		
i	Q_1	Q_2	$Q_S^{(i)}$
1	0	1	$\neg Q_1 \wedge Q_2$
2	1	0	$Q_1 \wedge \neg Q_2$

$$\begin{tikzcd}
 & \{Q_S^1, Q_S^2\} & \\
\{Q_S^1\} = \{Q_S^2\}^{\perp} \arrow[ur] & & \{Q_S^2\} = \{Q_S^1\}^{\perp} \arrow[ul] \\
 & \emptyset \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Fig. 0.2 (left) Of the set Q_c of 2^4 logically possible complete questions only Q_S^1 and Q_S^2 are physically possible; (right) the distributive orthomodular lattice (a Boolean algebra) formed by the physically possible complete questions Q_S^1 and Q_S^2.

0.2.2 The second postulate

In the previous section, a single family c of complete questions $Q_c^{(i)}$ was considered by an observer O to gather N bits of information about the observed system S. Alternatively, O could use a different family b of N complete questions $Q_b^{(i)}$ to gather N bits of information about S. The answers to $Q_b^{(i)}$ will still have a maximal amount of information about S formed by a N-bit string. Again, unions of sets of complete questions $Q_b^{(i)}$ (of the same family b), give rise to a Boolean algebra that has the logically/combinatorially possible $\{Q_b^{(i)}\}$ as atoms. In the context of the example illustrated in Fig.0.1 and Fig.0.2 this means that there may be a second set of two yes/no questions $\{R_1, R_2\}$ which give rise to a set $Q_R = \{Q_R^{(i)} \mid 1 \leq i \leq 2\}$ of physically possible complete questions $Q_R^{(i)}$. The logical and algebraic structures of the questions in Q_R mirror those of the questions in Q_S as they are displayed in Fig.0.2.

Postulate 2 of RQM captures what happens if, after having asked the N questions such that the maximal information about S has been gathered, the system O asks a further question Q_{N+1}. According to RQM there are two extreme possibilities: Firstly: the answer to the question Q_{N+1} is fully determined by the answers $[e_1, ..., e_N]$ to the previous questions and no new information is gained. The second possibility is captured in the Postulate 2 demanding that it is always possible to obtain new information.

Jointly, Postulates 1 and 2 can be understood as follows (Rovelli, 1996): Since the amount of information that O can have about S is limited by postulate 1, it follows that, if O has a maximal amount of information about S, then, when new information about S is acquired by O, O must loose information. In particular, if a new question Q_{N+1} (not determined by the previous information gathered), is asked, then O looses (at least) one bit of the previous information. So that, after asking the question Q_{N+1}, new information is available, but the total amount of information about the system does not exceed N bits. For more details on the bidirectional nature of measurement/observation see (Yang, 2018).

At the logic/algebraic level this is captured by the fact that postulates 1 and 2 imply that the set $W(S)$ as a whole – with the families of complete questions $Q_b^{(i)}$, $Q_c^{(i)}$, ..., which classically form Boolean algebras – has the structure of an orthomodular lattice (Grinbaum, 2005). The non-classical nature of systems that adhere to postulates 1 and 2 manifests itself algebraically in the fact that, unlike Boolean algebras, orthomodular lattices may lack the property of distributivity.

	$s_S^{(i)}$			$s_R^{(i)}$			
$Q_S^{(i)}$	Q_1	Q_2	$\bigwedge_i Q_i$	R_1	R_2	$\bigwedge_i R_i$	$Q_R^{(i)}$
Q_S^1	1	0	$Q_1 \wedge \neg Q_2$	1	0	$R_1 \wedge \neg R_2$	Q_R^1
Q_S^1	1	0	$Q_1 \wedge \neg Q_2$	0	1	$\neg R_1 \wedge R_2$	Q_R^2
Q_S^2	0	1	$\neg Q_1 \wedge Q_2$	1	0	$R_1 \wedge \neg R_2$	Q_R^1
Q_S^2	0	1	$\neg Q_1 \wedge Q_2$	0	1	$\neg R_1 \wedge R_2$	Q_R^2

Table 0.1 Two sets of complete questions Q_S and Q_R for 2 bits of information (adapted from (Calude et al, 2014)).

Consider the families $Q_S = \{Q_S^1, Q_S^2\}$ and $Q_R = \{Q_R^1, Q_R^2\}$ of complete questions and the associated two bits of information as displayed in Tab. 0.1. An orthomodular lattice that satisfies postulates 1 and 2 is displayed in Fig. 0.4. To see how this structure comes about, consider the diagram in Fig. 0.3. The diagram displays the lattices that arise from the ordering of the subsets of Q_S (left) and Q_R (right) as discussed above (Fig. 0.2). The bottom element of both lattices is the empty set of questions and represents the possibility of 'no answer'. The nodes of the intermediate level represent the yes answers to exactly one question which each yield two bits of information (maximal amount of information). The top elements of the lattices represent the situation of a positive answer to at least one question in the respective set of questions. Logically, this corresponds to a yes answer to the question $Q_S^1 \vee Q_S^2$ from which neither a yes answer to the question Q_S^1 nor a yes answer to the question Q_S^2 can be inferred. As discussed above, the top node of the Boolean algebra associated with minimal information, i.e., zero bits of information are gained from a yes answer to $Q_S^1 \vee Q_S^2$. Similarly for the top element of the lattice formed by the complete questions in Q_R and the information gained from a yes answer to the question $Q_R^1 \vee Q_R^2$.

The orthomodular lattice which is consistent with postulates 1 and 2 can be constructed from the two lattices in Fig. 0.3 as follows: (i) the bottom nodes in both lattices which do not yield an answer are identified and form the bottom element of the combined lattice; (ii) a new atomic node, $\{\top_S^R\}$ is introduced in the combined lattice, which identifies the top nodes $\top_S$ and $\top_R$ of the lattices in Fig. 0.3. Both, $\top_S$ and $\top_R$, stand for a yes answer to a disjunctive question in which no information is gained. The intuition is that

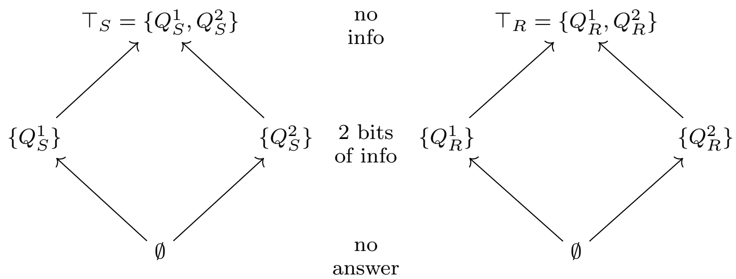


Fig. 0.3 Information content associated with sets of complete questions.

in the same sense in which there is only one empty set which represents the absence of an answer to any question, there is only one atomic node that represents the lack of information; (iii) the other two atoms of the combined lattice are $\{Q_S^1\}$ and $\{Q_R^1\}$, each of which is associated with two bits of information; (iv) the nodes $\{Q_S^2\}$ and $\{Q_R^2\}$ respectively arise as complements of $\{Q_S^1\}$ and $\{Q_R^1\}$ and as such yield two bits of information each; (v) The node $\{Q_S^1, Q_R^1\}$ is identical to the complement of the node ${\top_S^R}^{\perp}$. Since the latter represents indeterminacy, the former needs to represent indeterminacy. This is consistent with (a) the disjunctive reading of $\{Q_S^1, Q_R^1\}$ and (b) with the fact that both Q_S^1 and Q_R^1 will result in identical bits of information and thus can not distinguished.[1]

The lattice in Fig. 0.4 is an algebraic realization of the fact that, due to the finite amount of possible information, distinct sets of complete questions (Q_S and Q_R in this case) are *incompatible* in the sense that asking a question of the form $Q_S^i \vee Q_R^i$ will fail to yield determinate information as discussed above. Algebraically, this non-classical nature is reflected by the fact that the distributive law does not hold in this structure:

$$(Q_R^1 \wedge Q_S^1) \vee (Q_R^1 \wedge \top_S^R) = \emptyset \neq Q_R^1 \wedge (Q_S^1 \vee \top_S^R) = Q_R^1$$

0.2.3 Probabilities

As discussed above, Postulates 1 and 2 imply that the information that can be obtained in a setting that satisfies both postulates cannot be fully deterministic. The formalism of quantum mechanics models this indeterminacy probabilistically. That is, the formalism of quantum mechanics provides means to quantify indeterminacy by predicting the probability for sequences of responses that can be obtained from observing a system. This specific under-

[1] Usually, the lattice in Fig. 0.4 is constructed starting from the standard two-dimensional Hilbert space (e.g., (Calude et al, 2014)).

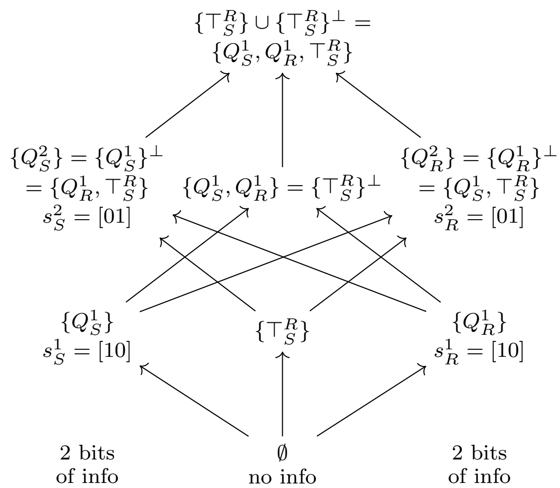


Fig. 0.4 An orthomodular lattice of the two sets of complete questions Q_S and Q_R where the arrows indicate subset relations between subsets of Q_S and Q_R and unions thereof. (Calude et al, 2014)

standing of indeterminacy is independent of the Postulates 1 and 2 and needs to be captured by additional postulates (Rovelli, 1996; Trassinelli, 2018).

In first approximation the reading of indeterminacy as probability can be captured in constraints on a family of functions of the form $p : Q_b \times Q_c \to \Re$. Those functions take the members of two sets of complete questions, Q_b and Q_c, to real numbers in a way that gives rise to a $N \times N$ matrix p^{ij} via the assignment $p^{ij} = p(Q_b^{(i)}, Q_c^{(j)})$. The aim is to constrain the functions of the form $p(Q_b^{(i)}, Q_c^{(j)})$ in such a way that their outcome can be interpreted as follows (Rovelli, 1996):

> $p^{ij} = p(Q_b^{(i)}, Q_c^{(j)})$ is the probability that a yes answer to the question $Q_b^{(i)}$ of the b-family of complete questions will follow the string $s^{(j)}$ of information that results from a yes answer to the question $Q_c^{(j)}$ of the c-family of complete questions. (0.1)

For the outcome of $p^{ij} = p(Q_b^{(i)}, Q_c^{(j)})$ to be interpretable as a probability in this sense, functions of this form need to satisfy some basic properties of probability functions (Rovelli, 1996): (i) $0 \leq p^{ij} \leq 1$; (ii) $\sum_i p^{ij} = 1$ and (iii) $\sum_j p^{ij} = 1$. That is, the function p gives rise to a $N \times N$ matrix of functions that yield real numbers between 0 and 1. All columns and rows of this matrix sum up to 1 for any families of complete questions.

The probability functions for the sets Q_S and Q_R of complete questions are displayed in Tab. 0.2. The tables are interpreted as follows. Questions that belong to the same family are compatible and yield determinate predictions.

This is displayed in the left and right subtables of Tab. 0.2. The probability that a yes answer to the question $Q_S^{(1)}$ of the S-family of complete questions will follow the string $s_S^{(1)}$ of information that results from a yes answer to the question $Q_S^{(1)}$ of the S-family of complete questions is one. That is, a sequence of identical questions always yields the same information. By contrast, the probability that a yes answer to the question $Q_S^{(2)}$ of the S-family of complete questions will follow the string $s_S^{(1)}$ of information that results from a yes answer to the question $Q_S^{(1)}$ of the S-family of complete questions is zero. That is, questions in the same family of complete questions are mutually exclusive and asking a sequence of questions of the same family does not introduce indeterminacy. In both case the information that comes from the answer to the second question is already contained in the information that is provided by a yes answer to the first question.

The table in the middle of Tab. 0.2 illustrates that the situation is very different if questions from distinct families of complete questions are asked. Both questions yield a maximum amount of information and thus new information obtained from the observed system S must overwrite existing information in the observer O. This introduces indeterminacy which probabilistically expresses itself as follows: The probability that a yes answer to the question $Q_S^{(i)}$ of the S-family of complete questions will follow the string $s_R^{(j)}$ of information that results from a yes answer to the question $Q_R^{(j)}$ of the R-family of complete questions is completely random, i.e., all possibilities are equally likely.

p^{ij}	Q_S^1	Q_S^2
Q_S^1	1	0
Q_S^2	0	1

p^{ij}	Q_R^1	Q_R^2
Q_S^1	0.5	0.5
Q_S^2	0.5	0.5

p^{ij}	Q_R^1	Q_R^2
Q_R^1	1	0
Q_R^2	0	1

Table 0.2 The outcome of $p^{ij} = p(Q_S^{(i)}, Q_S^{(j)})$, $p^{ij} = p(Q_S^{(i)}, Q_R^{(j)})$, and $p^{ij} = p(Q_R^{(i)}, Q_R^{(j)})$ for $1 \le i \le 2$.

As pointed out above, the set $W(S)$ consists not only of sets of complete questions $Q_b^{(i)}$, $Q_c^{(i)}$, $\ldots$, but for each set of complete questions Q_c, $W(S)$ also contains the questions that correspond to non-singleton subsets of Q_c. Thus, for each family Q_c of complete questions one needs to consider all the questions in the Lattice generated by the subsets of Q_c as illustrated in Figs. 0.1, 0.3, and 0.4. As above in Sec. 0.2.2, the notation $Q_c^{(j)} \vee Q_c^{(k)}$ is used to represent the question corresponding to the set $\{Q_c^{(j)}\} \cup \{Q_c^{(k)}\}$ in the respective lattices. Again, it is important to note that there is the answer yes to the question $Q_c^{(j)} \vee Q_c^{(k)}$ if and only if either there is a yes answer to the question $Q_c^{(j)}$ or there is a yes answer to the question $Q_c^{(k)}$. The amount of information that is associated with a yes answer to the question $Q_c^{(j)} \vee Q_c^{(k)}$

is less than the maximal amount of N bits of information that is associated with a answer yes to the questions $Q_c^{(j)}$ and $Q_c^{(k)}$ when asked separately.

The questions $Q_c^{(j)}$ and $Q_c^{(k)}$ are complete questions. By contrast if $j \neq k$ then $Q_c^{(j)} \vee Q_c^{(k)}$ is not a complete question. Since functions of the form $p : Q_b \times Q_c \to \Re$ are restricted to complete questions, expressions such as $p(Q_b^{(j)} \vee Q_b^{(k)}, Q_c^{(i)})$ are not defined. What is definable are *conditional* probability functions of the form $\overline{p} : \mathcal{P}(Q_b) \times Q_c \to \Re$, where $\overline{p}(\{Q_b^{(j)}, Q_b^{(k)}\}, Q_c^{(i)}) \equiv \overline{p}(Q_b^{(j)} \vee Q_b^{(k)}, Q_c^{(i)})$ is interpreted as the probability that a yes answer to $Q_b^{(j)} \vee Q_b^{(k)}$ which is associated with less than N bits of information will follow a yes answer to the question $Q_c^{(i)}$ which is associated with N bits of information. Postulates for $\overline{p}$ are:

Postulate 4 (Conditional probability (Trassinelli, 2018))

$$
\begin{aligned}
&(a)\ \overline{p}(\{Q_b^{(j)}\}, Q_c^{(i)}) \geq 0 \\
&(b)\ \overline{p}(\{Q_b^{(j)} \mid 1 \leq j \leq N\}, Q_c^{(i)}) \equiv \overline{p}(\textstyle\bigvee_{j=1}^{N} Q_b^{(j)}, Q_c^{(i)}) = 1 \\
&(c)\ \overline{p}(Q_b^{(j)} \vee Q_b^{(k)}, Q_c^{(i)}) = \overline{p}(\{Q_b^{(j)}\}, Q_c^{(i)}) + \overline{p}(\{Q_b^{(k)}\}, Q_c^{(i)})
\end{aligned}
$$

These properties of $\overline{p}$ imply as special cases the properties of $p^{ij} = p(Q_b^{(i)}, Q_c^{(j)})$ as stated above (Trassinelli, 2018). Postulate 4 is a more precise statement of Postulate 3 and thereby supersedes it. In what follows I will refer to Postulate 4 in place of Postulate 3.

0.3 The formalism of QM

Trassinelli (2018) and others have shown that Postulates 1, 2, and 4 are sufficient to derive the formalism of quantum mechanics within the framework of complex vector spaces (See Appendix .2). For the purpose of this paper it will be sufficient to briefly sketch some relevant aspects of it. The point here is to illustrate representational (and non-dynamic) aspects the formalism that focus on the formalism's consistency with Postulates 1 and 2 about the nature of information. An understanding of the formalism at this level will facilitate understanding of the probabilistic reading of indeterminacy in Postulate 4 as well as its viability in the geographic context.

0.3.1 Algebraic structure

The formalism of quantum mechanics which actually ‘implements’ structures and functions with the properties postulated in Sec. 0.2 now arises as follows.[2] Assume that there are systems O and S both of which have a maximal information capacity of N bits with respect to one another. That is there are families of complete questions that O can ask S and vice versa.

In Tab. 0.1 and Fig. 0.3 two complete sets of questions $Q_S = \{Q_S^1, Q_S^2\}$ and $Q_R = \{Q_R^1, Q_R^2\}$ for acquiring 2 bits of information were presented. In the standard Hilbert space formulation of QM (Dirac, 1930; John von Neumann, 1932) (see Appendix .2) sets of complete questions such as Q_S and Q_R give rise to bases of a 2 dimensional complex vector space $\mathcal{H}$ (a Hilbert space). The two complete questions in Q_S correspond to a system of base vectors of $\mathcal{H}$, in the sense that the question Q_S^1 corresponds to the base vector $|Q_S^1\rangle$ and Q_S^2 corresponds to the base vector $|Q_S^2\rangle$. The base of $\mathcal{H}$ that is formed by the vectors corresponding to the members of Q_S is called the Q_S-base of $\mathcal{H}$. Similarly, the two complete questions in Q_R correspond to a different system of base vectors of $\mathcal{H}$ – the Q_R-base – in the sense that the question Q_R^1 corresponds to the base vector $|Q_R^1\rangle$ and so on. This is illustrated in Fig. 0.5.

Every vector $|Q_a^i\rangle \in \mathcal{H}$ ($a \in \{S, R\}$ and $i \in \{1, 2\}$) gives rise to the ‘line’ segment $Q_a^i \approx \{\alpha\,|Q_a^i\rangle \mid \alpha \in \mathcal{C}\}$ that emerges when multiplying the vector $|Q_a^i\rangle$ by a complex number $\alpha \in \mathcal{C}$. The ‘line’ segment Q_a^i is a one-dimensional subspace of $\mathcal{H}$ induced by the span of the vector $|Q_a^i\rangle$. Jointly, the vectors $|Q_a^1\rangle$ and $|Q_a^2\rangle$ form the basis of a two-dimensional subspace of $\mathcal{H}$ by spanning a plane by means of vector addition and scalar multiplication. The plane spanned by $|Q_a^1\rangle$ and $|Q_a^2\rangle$ is designated by $Q_a^1 \vee Q_a^2$ and specified as $Q_a^1 \vee Q_a^2 \approx \{\alpha\,|Q_a^1\rangle + \beta\,|Q_a^2\rangle \mid \alpha, \beta \in \mathcal{C}\}$. One is justified to use the notation $Q_a^1 \vee Q_a^2$ to designate a two-dimensional subspace of $\mathcal{H}$ because: (i) the subspaces associated with complete questions of the form Q_a^i and $Q_a^1 \vee Q_a^2$ in conjunction with the the subset relation $\subseteq$ between the subspaces of $\mathcal{H}$ form a lattice $\mathcal{L}_a$ (Fig. 0.5 (middle)); (ii) in this lattice the least upper bound of the subspaces associated with the questions Q_a^1 and Q_a^2 is the plane associated with the question $Q_a^1 \vee Q_S^a$ via the span of the vectors $|Q_a^1\rangle$ and $|Q_a^2\rangle$. Similarly, the question $Q_a^1 \wedge Q_a^2$ is associated with the greatest lower bound of the associated subspace with respect to the subset relation of the underlying lattice $\mathcal{L}_a$.

The base vectors $|Q_R^1\rangle$ and $|Q_R^2\rangle$ share the origin with $|Q_S^1\rangle$ and $|Q_S^2\rangle$ but are rotated by 45 degree (Fig. 0.5 (left)). Jointly, $|Q_R^1\rangle$ and $|Q_R^2\rangle$ span the same subspace as $|Q_S^1\rangle$ and $|Q_S^2\rangle$ – the Hilbert space $\mathcal{H}$ as a whole. Since both systems of base vectors have the same origin and span the same space, $\mathcal{H}$, they form a joint lattice structure $\mathcal{L}_{S\oplus R}$ in Fig. 0.5 (right). With the join and meet operations $\vee$ and $\wedge$ defined as above, it is easy to verify that the

[2] Historically, the formalism was developed long before the insights into its interpretation.

lattice $\mathcal{L}$ is non-distributive:

$$(Q_R^1 \wedge Q_S^1) \vee (Q_R^1 \wedge Q_R^2) = \emptyset \neq Q_R^1 \wedge (Q_S^1 \vee Q_R^2) = Q_R^1$$

In addition one can verify that the lattice $\mathcal{L}_{S \oplus R}$ is also orthomodular. As pointed out above, orthomodular lattices are structures in which postulates 1 and 2 are satisfied.

For a more intuitive argument of the why Postulates 1 and 2 are satisfied are satisfied in a two dimensional Hilbert space with the bases induced by the sets of complete questions Q_S and Q_R, consider Fig 0.5. By construction, the amount of information provided by yes answers to any of the questions $Q_R^1, Q_R^2, Q_S^1, Q_S^2$ is maximal. Understanding the two sets of complete questions Q_S and Q_R as forming two bases of the same two dimensional vector space expresses in a mathematical language that both sets of questions yield the same amount information in virtue of describing *the same* vector space in different bases. The amount of information that can be obtained is constrained by the dimension of the Hilbert space. New information can be gained by closing a different base, i.e., by switching to a different set of complete questions and thus focusing on different aspects of the described object.

The relation between the lattice in the right of Fig 0.5 and the lattice in Fig. 0.4 can be established via embeddings that are described for example in (Calude et al, 2014). In general, one can prove that although both lattices are orthomodular, there does not exist an embedding that preserves all the properties associated with the lattice arising from QM into lattices that arise from classical logic and classical information theory (Kochen and Specker, 1967; Calude et al, 2014). Intuitively, in QM one can *quantify* indeterminacy. In a (semi-)classical framework one can *qualitatively* distinguish determinate from indeterminate situations as they arise from measurement/observation interactions.

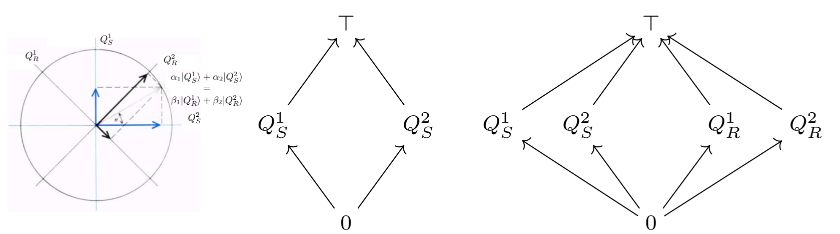


Fig. 0.5 Two complete sets of questions Q_S and Q_R for 2 bits of information in a (projection of a) 2 dimensional Hilbert space (left) and the lattices $\mathcal{L}_S$ (middle) and $\mathcal{L}_{S \oplus R}$ (right) (adapted from (Calude et al, 2014) and (Hughes, 1981)).

0.3.2 Indeterminacy as probability

Every vector in a vector space can be represented as a superposition of a system of base vectors. That is, if $|\phi\rangle$ is a vector of $\mathcal{H}$ that is described in the Q_S-base, then there are complex numbers $\alpha, \beta \in \mathcal{C}$ such that $|\phi\rangle = \alpha\,|Q_S^1\rangle + \beta\,|Q_S^2\rangle$. The formalism of QM requires that the square modulus $|\,\langle\phi|\phi\rangle\,|$ of the inner product $\langle\phi|\phi\rangle$ of the vector $|\phi\rangle$ is equal to one (see Appendix .2). For given system of base vectors, this requirement allows for the distinction of two kinds of vectors in a Hilbert space: (a) vectors that, when expressed in that base, are such that one of the coefficients α, β is equal to one and the other coefficients are equal to zero; (b) vectors $|\phi\rangle$ that, when expressed in that base, are such that none of the coefficients is equal to one, but jointly the square of the modulus of $\langle\phi|\phi\rangle$ is equal to one.

The case (a) covers all the situations in which an observing system can obtain two bits of information about the observed system in the form of a yes answer to the question associates with the base vector of the non-zero coefficient. That is, case (a) covers all the situations where there is determinate information. By contrast, case (b) covers all the situations in which an observing system cannot obtain determinate information about the observed system. Since the coefficients range over complex numbers, there is a huge number of indeterminate situations that can be distinguished. This is very different from the classical framework. Consider the orthomodular lattice of Fig. 0.4. This lattice represents a classical understanding of the indeterminacy that arises in systems with two bits of information that satisfy Postulates 1 and 2. On a classical view of the sort that is presented in Fig. 0.4, there are exactly three cases of indeterminacy that can be distinguished as the nodes $\{\top_R^S\}$, $\{\top_R^S\}^\perp$, and $\top$ in the depicted lattice. This illustrates that QM as a formalism for indeterminacy is capable of *quantifying* indeterminacy rather than only identifying indeterminate situations qualitatively.

In the formalism of QM indeterminacy is quantified probabilistically in terms of the likelihood that a yes answer to a specific complete question is obtained. In the base formed by vectors corresponding to the members of Q_S (the Q_S-base), a vector $|\phi\rangle = \alpha\,|Q_S^1\rangle + \beta\,|Q_S^2\rangle$ represents a state of an observed system S with respect to the observing system O. If S is in the state $|\phi\rangle$ with respect to O, then the probability that O receives a yes answer to the question Q_S^i is $|\,\langle\phi|Q_S^{(i)}\rangle\,|$. Here the expression $|\,\langle\phi|Q_c^{(i)}\rangle\,|$ encodes in the object language of QM the conditional probability $\overline{p}$ of Postulate 4.

0.4 An information-theoretic view of geographic information

The systematic investigation of the nature of geographic information and geographic information processing from an information-theoretic perspective was pioneered by Sinton (1978). According to Sinton, geographic information has three components that are logically interrelated but need to be treated independently: information of geographic qualities; information about the spatial location and the temporal location of the geographic phenomena that have those qualities.[3] In addition, Sinton postulates that information about the three components cannot be measured/observed at once. One component has to be fixed, one component has to be controlled, and one component can be measured/observed. In the language of Wheeler's (1989) information-theoretic view of measurement/observation processes, Sinton's conception of the nature of geographic information can be expressed as follows:

Postulate 5 (Sinton and Wheeler) *(S1) For a string of bits of information to count as geographic information it must be constituted by bits that result from answers to yes/no questions that fall in three broad classes: (i) yes/no questions about measurable/observable (geographic) qualities; (ii) yes/no questions about spatial location in geographic space; and (iii) yes/no question about temporal location.*

(S2) For a string of bits of information in the sense of (S1) to count as geographic information: Firstly, one type of yes/no questions which answers determine the bit string needs to be fixed, that is, limited to the yes answer to one question of this type. Secondly, one type of yes/no questions needs to be controlled, that is, limited to yes answers to a fixed number of questions – control questions – that yield information about some domain that is subject to a fiat subdivision[4]. Thirdly, one type of yes/no questions needs to be measured, that is, every yes answer to a control question is complemented by a yes answer to a question from the class of questions that is neither fixed nor controlled – a yes/no question in Wheeler's standard understanding.

That is, in Wheeler's (1989) information-theoretic language, Sinton's paradigm requires that: (i) there is one yes answer to k yes/no question, $Q_f^1, \ldots, Q_f^k$, pertaining to fixed information; (ii) there are n yes answers to n yes/no questions $Q_c^1 \ldots Q_c^n$ pertaining to controlled information; and (iii) there is one yes answer to yes/no questions $(Q_m)_1^1 \ldots (Q_m)_n^h$ for every bit of control information pertaining to information obtained by measurement/observation, where h is the number of yes/no questions from the class of questions that represent possible measurement/observation outcomes. With those $k + n + (n * h)$

[3] Since geographic phenomena are strictly non-relativistic, it is consistent with RQM to treat spatial location and temporal location as independent.

[4] A subdivision which boundaries are not aligned with physical discontinuities of the domain that is subdivided (Smith, 2001; Smith and Varzi, 2000; Smith, 1995).

yes/no questions there is associated an amount of $k + n + (n * h)$ bits of information and there are $2^{k+n+(n*h)}$ combinatorially possible bit strings of the form sketched in Eq. 0.2.

$$
\begin{array}{c||c|c|c|c|c|c|c|c|c|c|c|c|c|c}
s_S & Q_f^1 & \ldots & Q_f^k & Q_c^1 & \ldots & Q_c^n & (Q_m)_1^1 & \ldots & (Q_m)_1^h & (Q_m)_2^1 & \ldots & (Q_m)_2^h & \ldots & (Q_m)_n^h \\
\hline
s_S^1 & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 1 \\
s_S^2 & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 & \ldots & 0 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
s_S^L & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & \ldots & 0
\end{array}
\tag{0.2}
$$

with $L = k + n + (n * h)$

As indicated in Eq. 0.2, there is a set, s_S, of bit strings of length $k + n + (n * h)$ which has $2^{k+n+(n*h)}$ members. The paradigm of fix/control/measure reduces these combinatorial possibilities to the set S_S of possibilities that are consistent with the paradigm (Eq. 0.3).

$$
\mathsf{S}_\mathsf{S} = \left\{ s_S^i \in s_S \;\middle|\; \begin{array}{ll}
\Sigma_{j=1}^{k} s_S^i[j] = 1 \ \& & \begin{array}{c}\text{(one yes-bit of fixed}\\ \text{information for each } s_S^i)\end{array} \\
(\Sigma_{j=k+1}^{k+n+1}\ s_S^i[j]) = n \ \& & \begin{array}{c}(n \text{ yes-bits of control}\\ \text{information for each } s_S^i)\end{array} \\
(\bigwedge_{j=0}^{n-1}\ (\Sigma_{l=l_1}^{l_m}\ s_S^i[l] = 1)) = 1 & \begin{array}{c}\text{(one yes-bit for measurement}\\ \text{for each yes-control-bit}\\ \text{for each } s_S^i)\end{array} \\
l_1 = k + n + (j * h) + 1; & \\
l_m = k + n + (j * h) + h &
\end{array} \right\}
\tag{0.3}
$$

Example 1. Consider Fig. 0.6 and suppose that (a) the information that is obtained by measurement/observation is information about about the quality of elevation; (b) that the information about spatial location is controlled by projecting a fiat (Smith, 2001; Smith and Varzi, 2000; Smith, 1995) raster-shaped partition onto the ground as indicated in the image; (c) information about temporal location is fixed by allowing for a single time stamp. That is, a yes answer to one of the $Q_t^1, \ldots, Q_t^{10}$ picks out a particular time stamp. Yes answers to the yes/no questions $Q_c^1, \ldots, Q_c^{36}$ pick out particular cells in the grid structure. For every control region picked out by a control question Q_c^i there is a yes answer to one of the yes/no questions $Q_m^1, \ldots, Q_m^{110}$. By imposing those constraints the paradigm of fixing time, controlling spatial location, and measuring/observing qualities of control regions, reduces these combinatorial possibilities to the set S_S of possibilities that are consistent with the paradigm (Eq. 0.3). The yes/no questions, the yes answers to which give rise to the bit strings of information in S_S are the members of the set of complete questions Q_S. Consider the yes/no question $Q_S^{img} \in Q_S$ as depicted in Fig. 0.6. A yes answer to Q_S^{img} yields L bits of information. This information is encoded in the string $s_S^{img} \in \mathsf{S}_\mathsf{S}$. The information encoded in s_S^{img} corresponds to the information encoded in the image in the top left of Fig. 0.6. □

645	650	654	658	653	648
664	666	670	672	668	659
678	682	684	693	689	680
703	708	714	721	719	716
728	732	738	744	745	732
730	739	744	749	748	735

time	Q_t^1	Time is 11/20/2018?
	$\ldots$	$\ldots$
	Q_t^{10}	Time is 11/29/2018?
location	Q_l^1	Location is cell 1?
	$\ldots$	$\ldots$
	Q_l^{36}	Location is cell 36?
quality	Q_q^1	Quality measure is 640?
	$\ldots$	$\ldots$
	Q_q^{110}	Quality measure is 750?

$s_S \backslash Q_S$	Q_t^1	$\ldots$	Q_t^{10}	Q_l^1	$\ldots$	Q_l^{36}	$(Q_q)_1^1$	$\ldots$	$(Q_q)_1^{110}$	$(Q_q)_2^1$	$\ldots$	$(Q_q)_2^{110}$	$\ldots$	$(Q_q)_{36}^{110}$
s_S^1	1	$\ldots$	1	1	$\ldots$	1	1	$\ldots$	1	1	$\ldots$	1	$\ldots$	1
s_S^2	1	$\ldots$	1	1	$\ldots$	1	1	$\ldots$	1	1	$\ldots$	1	$\ldots$	0
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
s_S^L	0	$\ldots$	0	0	$\ldots$	0	0	$\ldots$	0	0	$\ldots$	0	$\ldots$	0

where $L = 1 + 36 + (36 * 110)$

$$\mathsf{S_S} = \left\{ s_S^i \in s_S \;\middle|\; \begin{array}{ll} (\Sigma_{j=1}^{10} s_S^i[j]) = 1 = s_S^i[1] \ \& & \text{(fix time)} \\ (\Sigma_{j=11}^{46+1}\ s_S^i[j]) = 36 \ \& & \text{(36 yes-bits of control information)} \\ (\bigwedge_{j=0}^{36-1}\ (\Sigma_{k=k_1}^{k_m}\ s_S^i[k] = 1)) = 1 & \text{(one measurement bit is 1 for each control location)} \\ k_1 = 10 + 36 + (j * 110) + 1 & \\ k_m = 10 + 36 + (j * 110) + 110 & \end{array} \right\}$$

$$Q_S^{img} = Q_t^1 \wedge \neg Q_t^2 \wedge \ldots \wedge \neg Q_t^{10} \wedge Q_c^1 \wedge \ldots \wedge Q_c^{36} \wedge \neg (Q_q)_1^1 \wedge \ldots \wedge (Q_q)_1^6 \wedge \ldots \wedge \neg (Q_q)_1^{110} \wedge \neg (Q_q)_2^1 \wedge \ldots \wedge (Q_q)_2^{11} \wedge \ldots \wedge \neg (Q_q)_{36}^{110}.$$

Fig. 0.6 Sinton's paradigm of geographic information where temporal location is fixed (11/20/2018), spatial location is controlled by fiat, and a geographic quality is measured. Q_S^{img} is the complete yes/no question the yes answer to which yields L bits of information encoded in the image in the top left. (The image in the top left is from (Bolstad, 2005).)

0.5 The nature of geographic information

The success of quantum mechanics in physics reveals three important aspects of information about the physical world (Rovelli, 1996; Rovelli and Vidotto, 2015): (i) information about the physical world is fundamentally relational (according to RQM); (ii) information of the physical world is fundamentally granular; and (iii) information about the physical world reflects the fundamentally indeterminate nature of certain aspects of the world. These properties of information manifest themselves logically in the Postulates 1, 2, and 4 as discussed above. If there is a quantum theory that captures at least certain classes of geographic phenomena, then, in analogy to (i), (ii), and (iii), information of those phenomena is fundamentally relational, granular, and affected by indeterminacy and, logically, subject to Postulates 1, 2, and 4. In what follows the relational and granular nature of geographic information and the way geographic information is affected by indeterminacy is discussed within Sinton/Wheeler framework of geographic information processing.

0.5.1 The relational nature of geographic information

The inherently relational nature of Sinton's paradigm is revealed in the explicit focus on the aspect of control that is asserted by the observing system and targeted towards the observed system in form of a fiat subdivision (Smith, 2001) of some aspect of the observed system (Postulate 5 (S2)). The assertion of control on how certain bits of information are obtained in Sinton's framework corresponds to the idea of a granular partition (Smith and Brogaard, 2002; Bittner and Smith, 2003; Bittner and Stell, 2003). The theory of Granular partitions (TGP) (Smith and Brogaard, 2002) emphasizes the bidirectional nature of the interrelationship between observing and observed system in geographic contexts such as the one sketched in Fig. 0.6. That is, control asserted by the observing system cannot be arbitrary. It has to adhere to certain features of the observed system. According to TGP, features include structural aspects such as mereology (Leonard and Goodman, 1940; Simons, 1987) as well as aspects of granularity and scale. In the context of a quantum theory, aspects of granularity and scale are of particular importance.[5] The ways in which the theory of granular partitions extends Sinton's paradigm can be seen as an investigation in the nature of the information transfer via correlations between observed and observing systems.

0.5.2 The granular nature of geographic information

In Example 1 (pg. 17) there does not seem to be a limit to the amount of information about elevation that can be had by an observer. More information can be obtained by refining cells and asking yes/no questions about the elevation in these refined cells.[6] Similarly, more information can be obtained by allowing for more precise elevation measurements.[7] It follows:

[5] The theory of granular partitions was originally linked to Griffiths' (1984) consistent history interpretation of quantum mechanics (Smith and Brogaard, 2002). However, none of the assumptions in the formalism of TGP restrict it to the consistent history interpretation.

[6] Of course the notion of elevation ceases to be meaningful if the refinement of cells reaches the atomic scale. However this is so far outside of the realm of geography that it can be ignored here.

[7] According to RQM there *are* minimal units of length (Rovelli and Vidotto, 2015). This can be ignored here.

Proposition 1. *When reading the qualities in Example 1 and Fig. 0.6 as elevation, then Example 1 constitutes a counter example to Postulate 1 and a supporting example for Postulate 2.*

Consider, again, Fig. 0.6, but now suppose that the measured quality is classificatority in nature such as the quality of land cover and land use types (Andereson et al, 1976). In virtue of the classificatory nature, there is a maximal number of land cover types that can be distinguished in a given classification scheme. In addition, there *is* a limit to the degree to which the raster cells can be subdivided while still being meaningfully associated with land cover and/or land use (or other classificatory) qualities. This is because there cannot be a land cover of type forrest in a region that is too small to contain a sufficient number of trees. This is captured in Postulate 6.

Postulate 6 (Granularity) *If the Sinton/Wheeler scheme is applied in contexts where classificatory qualities $Q_m^1 \dots Q_m^h$ of geographic regions are measured/observed, time is fixed and space is controlled via control cells referenced by yes answers to a control questions of the form $Q_l^1 \dots Q_l^n$, then there is a minimal size of control cells – a finest level of resolution/granularity – for which yes/no questions of the form:*

Q_S^k = *'Does the cell referenced by a yes answer to the question Q_l^i have the quality Q_m^j?' ($1 \leq i \leq n$, $1 \leq j \leq h$)*

still have an answer. For cells of less than minimal size – below the finest level of resolution/granularity – there is no answer to a question such as Q_S^k.

Example 2. Suppose that in this example the questions $Q_q^1, \dots, Q_q^{110}$ of Fig. 0.6 are designed to obtain information about land cover and land use qualities. In particular suppose that the symbol 645 designates the land cover type 'forrest', the symbol 670 designates 'industrial area', and so on. In the context of Sinton's scheme the yes/no questions

$$Q_t^1, \dots, Q_t^{10}, Q_l^1, \dots Q_l^{36}, (Q_q)_1^1, \dots, (Q_q)_{36}^{110}$$

play the same roles as specified in Example 1. Again, the answers to those yes/no questions give rise to the set S_{S} of bit strings that emerge from yes answers to complete questions such as $Q_S^{img} \in Q_S$. □

Proposition 2. *On the classificatorial interpretation of Example 2 the complete questions in Q_S satisfy Postulate 1, only if the control questions $Q_l^1 \dots Q_l^{36}$ acquire information about cells of* maximal *resolution.*

Proof. Every question $Q_S^i \in Q_S$ is by construction *complete* (in the sense of Sec. 0.2.1) and adheres to Sinton's scheme. Therefore, a yes answer to any of the complete questions in $Q_S^i \in Q_S$ yields the same amount of L bits of information about land use and coverage about the target area that is picked out by the control questions $Q_l^1 \dots Q_l^{36}$. More information about land use and land cover in the target area can be had only by further subdividing control cells, but this would render the question Q_S^i meaningless because it cannot be answered for cells that are smaller than cells of maximal of resolution. Thus, the amount of information of complete questions Q_S^i associated with control questions $Q_l^1 \dots Q_l^{36}$ that acquire information about cells of *maximal* resolution is maximal. Hence Postulate 1 is satisfied. □

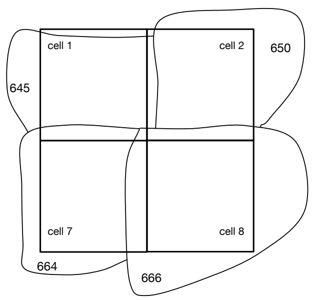
0.5.3 Unlimited amounts of limited information

On the classificatority interpretation of Fig. 0.6 in Example 2 the question arises, if there are other sets of complete questions (such as Q_R in Tab. 0.1) that can serve in support of Postulate 1 and Postulate 2. To address this issue, consider the yes/no questions $Q_t^1, \dots, Q_t^k, Q_l^1, \dots, Q_l^n$ where the fixed yes question that picks out the time stamp is Q_t^i with $1 \leq i \leq k$ and, as above, the control questions $Q_l^1, \dots Q_l^n$ acquire information about cells of maximal resolution. Now suppose that there are yes/no questions $(Q_{i/b})_j^1$ and $(Q_{i/b})_j^2$:

- $(Q_{i/b})_j^1$: "Is the cell associated with a yes answer to the control question Q_l^j an interior part of a region of a homogeneous land cover type?"
- $(Q_{i/b})_j^2$: "Does the cell associated with a yes answer to the control question Q_l^j contain a boundary between distinct land cover types?"

Suppose that the observing system has a maximal amount of information about the observed system in form of the bit string s_S^i which stems from a yes answer to the complete question Q_S^i associated with control cells at finest level of resolution as in Example 2. By assumption, yes answers to the control questions $Q_l^1, \dots Q_l^n$ pick out cells which have (mostly) *fiat* boundaries, i.e., boundaries that that do not correspond to discontinuities in the geographic world (Smith, 2001; Smith and Varzi, 2000; Smith, 1995). This means that the boundaries between control cells in general do not coincide with boundaries between regions with distinct types of land uses. This is illustrated in Example 3 and Fig. 0.7.

Example 3. Consider the image of Fig. 0.7 which displays the cells 1, 2, 7, and 8 of Fig. 0.6. On the classificatority interpretation the numbers of the image in Fig. 0.6 are interpreted as land cover types and the respective cells had the classificatory values $645, 650, 664$ and 666. As indicated in the image of Fig. 0.7 the classification of the cells in Fig. 0.6 is consistent with the land cover types of the actual regions on the ground. However, the actual boundaries that demarcate the regions on the ground lie skew to the boundaries of the raster cells that are picked out by the control questions Q_l^1, Q_l^2, Q_l^7 and Q_l^8. The tables in Fig. 0.7 illustrate, in analogy to the tables in Fig. 0.6, how the set $Q_{i/b}$ of complete questions and the strings of information $\mathsf{S}_{i/b}$ that emerge from yes answers to questions in $Q_{i/b}$ arise in a way that is consistent with Sinton's paradigm. $Q^{img}_{i/b}$ is the complete question corresponding to the image in the top of Fig. 0.7. □



time	Q_t^1	Time is 11/20/2018?
	...	...
	Q_t^{10}	Time is 11/29/2018?
location	Q_l^1	Location is cell 1?
	...	...
	Q_l^{36}	Location is cell 36?
quality	Q_q^1	Cell is land-cover homogeneous ?
	Q_q^2	Cell is land-cover inhomogeneous?

$s_{i/b}$	Q_t^1	...	Q_t^k	Q_c^1	...	Q_c^{36}	$(Q_{i/b})_1^1$	$(Q_{i/b})_1^2$	$(Q_{i/b})_2^1$	$(Q_{i/b})_2^2$	...	$(Q_{i/b})_{36}^2$
$s^1_{i/b}$	1	...	1	1	...	1	1	1	1	1	...	1
$s^2_{i/b}$	1	...	1	1	...	1	1	1	1	1	...	0
...	...	...	...	...	...	...	...	...	...	...	...	...
$s^{L_{i/b}}_{i/b}$	0	...	0	0	...	0	0	0	0	0	...	0

where $L_{i/b} = 10 + 36 + (36 * 2)$

$$\mathsf{S}_{i/b} = \left\{ s^i_{i/b} \in s_{i/b} \;\middle|\; \begin{array}{ll} (\Sigma_{j=1}^{10} s^i_{i/b}[j]) = 1 = s^i_{i/b}[1] \ \& & \text{(fix time)} \\ (\Sigma_{j=2}^{36+1}\ s^i_{i/b}[j]) = 36 \ \& & \text{(36 yes bits of control information)} \\ (\Sigma_{j=0}^{36-1}\ (\Sigma_k^{k_m}\ s^i_{i/b}[k] = 1) = 36) & \text{(one measurement bit is 1 for each control location)} \\ k = 1 + 36 + (j * 2) + 1 & \\ k_m = 1 + 36 + (j * 2) + 2 & \end{array} \right\}$$

$$Q^{img}_{i/b} = Q_t^1 \wedge \neg Q_t^2 \wedge \ldots \wedge \neg Q_t^{10} \wedge Q_c^1 \wedge \neg Q_c^2 \wedge \neg Q_c^7 \wedge \neg Q_c^8 \wedge \neg (Q_q)_1^1 \wedge (Q_q)_1^2 \wedge \neg (Q_q)_2^1 \wedge (Q_q)_2^2 \wedge \neg (Q_q)_7^1 \wedge (Q_q)_7^2 \wedge (Q_q)_8^1 \wedge \neg (Q_q)_8^2.$$

Fig. 0.7 Sinton's paradigm of geographic information where temporal location is fixed (11/20/2018), spatial location is controlled by fiat, and the quality of (in)homogeneity of land coverage is measured/observed. (The image in the top left corresponds to the four top left cells (cells 1, 2, 7, 8) in the image of Fig. 0.6.) $Q^{img}_{i/b}$ is the complete question corresponding to the image in the top left.

Proposition 3. *The complete questions in $Q_{i/b}$ of Example 3 and Fig. 0.7 satisfy Postulate 1, only if the control questions $Q_l^1 \ldots Q_l^{36}$ acquire information about cells of* maximal *resolution.*

Proof. Consider complete questions of the form $Q^i_S \in Q_S$ of Fig. 0.6 and complete questions of the form $Q^i_{i/b} \in Q_{i/b}$ of Fig. 0.7. Since both, Q^i_S and

$Q^i_{i/b}$ contain the same control questions, it follows that if Q^i_S yields a maximal amount of (classificatory) information, which is determined by the maximal resolution of the raster cells picked out by the control questions. Therefore, the question $Q^i_{i/b}$ must be yield the maximum amount of information about (in)homogeneity associated with the classification underlying Q_S and vice versa. Thus, Proposition 3 is true, if Proposition 2 is true. □

It now remains to investigate whether, jointly, the complete questions analyzed in Examples 2 and 3, satisfy Postulate 2:

Proposition 4. *Jointly, the sets of complete questions Q_S and $Q_{i/b}$ of Examples 2 and 3 satisfy Postulate 2, only if the control questions $Q^1_c \ldots Q^n_c$ acquire information about cells of* maximal *resolution.*

Proof. Consider the complete questions Q^i_S and $Q^j_{i/b}$. Question Q^i_S yields $10+36+(36*110)$ bits of information and question $Q^j_{i/b}$ yields $10+36+(36*2)$ bits of information. Thus, if Q^i_S and $Q^j_{i/b}$ are complete questions at extract information on the finest level of granularity, then the maximal amount of information that an observer can have about the observed phenomenon is $10 + 36 + (36 * 110)$ bits. Suppose that the image in Fig. 0.6 is the graphical representation of a yes answer to question Q^i_S. A yes answer to Q^i_S yields $10+36+(36*110)$ bits and thereby exhausts the amount of information that can be had. Now suppose that the observer asks question $Q^j_{i/b}$. A yes answer to this question yields $(36*2)$ bits of *new* information. This information is new because, by assumption, the boundaries between the cells picked out by the control questions are created by fiat and therefore may or may not coincide with discontinuities in the observed phenomenas. Thus, a yes answer to Q^i_S does not contain information discontinuities at the finest level of granularity. By contrast, a yes answer to $Q^j_{i/b}$ does yield information about homogeneity and inhomogeneity and thus *new* information about discontinuities in the observed phenomena at the finest level of granularity.

Since a yes answer to Q^i_S yields the maximal amount of information an observer can have about the observed phenomenon, the new information obtained by a yes answer to $Q^j_{i/b}$ must *overwrite* old information, which therefore is lost. Asking Q^i_S again and receiving a yes answer will yield genuinely new information, i.e., information that was erased by the information that was obtained by a yes answer to $Q^j_{i/b}$. The questions Q^i_S and $Q^j_{i/b}$ are geographic examples of what in quantum mechanics are called *complimentary* questions or complimentary qualities. As in the example of Q^i_S and $Q^j_{i/b}$, every question in a sequence of complementary questions, when answered with yes, will yield new information. □

Corollary 1. *The complete questions in Q_S and $Q_{i/b}$ satisfy both, Postulate 1 and Postulate 2 only if the control questions $Q^1_c \ldots Q^n_c$ acquire information about cells of maximal resolution.*

Of course, the examples discussed in the past two sections are specific instances of the famous cluster of problems that arise in the realm of the classification and delineation of geographic regions (Bailey, 1983; Omernik and Griffith, 2014). The question Q_S^i is an example of the formulation of a classification problem in the language of Wheeler's (1989) information-theoretic view of measurement/observation processes. By contrast, the question $Q_{i/b}^j$ is an example of the formulation of a delineation problem in Wheeler's (1989) language. Thus, this is an information-theoretic argument in support of the view that the classification and delineation of geographic regions are complementary measurement/observation processes at the geographic scale. Similar points were made from non-information-theoretic perspectives in (Bittner, 2011, 2017).

0.6 Information density

The arguments of the previous section about the complementary nature of complete questions, Q_c^i, about classification and complete question, $Q_{i/b}^j$, about delineation, depended critically on the assumption that the control questions (that are part of $Q_c^i \in Q_c$ as well as in $Q_{i/b}^j \in Q_{i/b}$) refer to cells at the finest level of granularity. Linking a maximal amount of information to a minimal unit of space, as it is evident in Postulate 6 and Propositions 2 and 3, makes explicit that in the context of the processing of information about the classification and delineation of geographic regions there is a maximal *information density*. The notion of maximal information density then opens the possibility that larger amounts of information can be had at coarser levels of granularity.

0.6.1 Maximal information density

Consider Fig. 0.8 and suppose that in the context of the classification and delineation of geographic phenomena the employment of Sinton's paradigm has led to fixed time, controlled space and the measurement/observation of land cover types and land cover (in)homogeneity. Suppose further that (a) the control results in a raster structure that has two cells and that each cell is of the minimal size (i.e., of maximal resolution) associated with land cover types (in the sense of Postulate 6); (b) the measurement/observation of land cover types distinguishes two types: type A and type B; and (c) the measurement/observation of land cover homogeneity distinguishes two types: homogeneous and inhomogeneous. The information-theoretic representation of what can be measured/observed within this example is encoded in eight yes/no questions. The questions are labeled $Q_c^{1,1}$, $Q_c^{1,2}$, $Q_c^{2,1}$, $Q_c^{2,2}$, $Q_{i/b}^{1,1}$, $Q_{i/b}^{1,2}$,

$Q^{2,1}_{i/b}$, $Q^{2,2}_{i/b}$ and specified as conjunctions of yes/no questions that extract fixed, controlled, and measured information as outlined in the middle of Fig. 0.8.

Due to the complementary nature of the classification qualities and delineation qualities of minimal control cells, there are two sets of complete questions:

$$Q_c = \{Q^1_c, Q^2_c, Q^3_c, Q^4_c\} \qquad Q_{i/b} = \{Q^1_{i/b}, Q^2_{i/b}, Q^3_{i/b}, Q^4_{i/b}\}$$

The amount of information that can be obtained by answering each of the four questions in the two groups is four bits.[8] That is, the maximum information density is *four bits per cell of minimal size.* The 2^4 combinatorially possible pattern of yes/no answers to these questions are listed in Tab. 0.8. Within the constraints of Sinton's paradigm only four pattern of yes/no answers are possible.

Example 4. Consider Fig. 0.9 and suppose that the image in the middle represents a specific situation on the ground – the system S that is constituted by two regions, one of land cover type A and one of land cover type B. Suppose further, that the cells marked 'cell 1' and 'cell 2' are fiat subdivisions imposed by the observer O on S which are of minimal size with respect to the measured/observed quality (land cover types and land cover (in)homogeneity). If O can have only four bits of information at the finest level of granularity, then the image in the middle of Fig. 0.9 is either characterized by a yes answer to the question $Q^{(2)}_c$ which yields the information $s^{(2)}_c = [1001]$, or by a yes answer to the question $Q^{(3)}_{i/b}$ which yields the information $s^{(2)}_{i/b} = [0110]$. □

Interpreted in the context of the limited amount of information that observer O can have about S at finest level of granularity, the image in the middle of Fig. 0.9 is an illusion. The information that can be had by O at any given time at the finest level of granularity is either a string of bits $s^{(2)}_c$ obtained as a yes answer to the question $Q^{(2)}_c$ or a string of bits $s^{(2)}_{i/b}$ obtained as a yes answer to the question $Q^{(2)}_{i/b}$, but not both.

The orthomodular lattice that arises from the two sets of complete questions Q_c and $Q_{i/b}$ at the finest level of granularity is displayed in Fig. 0.10. In analogy to Fig. 0.4 the lattice in Fig. 0.10 is an algebraic expression of the fact that, due to the limited amount of possible information, the distinct sets of complete questions Q_c and $Q_{i/b}$ are *incompatible* in and asking questions of the form $Q^i_c \vee Q^i_{i/b}$ will fail to yield determinate information.

---

[8] In the context of this example it is ignored that there are more bits of information 'hidden' in the $Q^{(i)}_c$ and $Q^{(j)}_{i/b}$.

time	Q_t	Time is 11/20/2018?
location	Q_l^1	Location is cell 1?
	Q_l^2	Location is cell 2?
quality	$Q_{i/b}^1$	Cell is land-cover homogeneous ?
$Q_{i/b}$	$Q_{i/b}^2$	Cell is land-cover inhomogeneous?
quality	Q_q^1	Cell is land-cover-type A?
Q_q	Q_q^2	Cell is land-cover-type B?

$Q_c^{1,1} \equiv Q_t \wedge Q_l^1 \wedge Q_q^1$, $Q_c^{1,2} \equiv Q_t \wedge Q_l^1 \wedge Q_q^2$, $Q_c^{2,1} \equiv Q_t \wedge Q_l^2 \wedge Q_q^1$, and $Q_c^{2,2} \equiv Q_t \wedge Q_l^2 \wedge Q_q^2$;
$Q_{i/b}^{1,1} \equiv Q_t \wedge Q_l^1 \wedge Q_{i/b}^1$, $Q_{i/b}^{1,2} \equiv Q_t \wedge Q_l^1 \wedge Q_{i/b}^2$, $Q_{i/b}^{2,1} \equiv Q_t \wedge Q_l^2 \wedge Q_{i/b}^1$, and $Q_{i/b}^{2,2} \equiv Q_t \wedge Q_l^2 \wedge Q_{i/b}^2$.

			$s_c^{(i)}$				
$\mathcal{H}_c$	$Q_c^{(i)}$	$Q_c^{1,1}$	$Q_c^{1,2}$	$Q_c^{2,1}$	$Q_c^{2,2}$		
	–	1	1	1	1	–	
	–	1	1	1	0	–	
	–	1	1	0	1	–	
	–	1	1	0	0	–	
	–	1	0	1	1	–	
$\|Q_c^{(1)}\rangle$	$Q_c^{(1)}$	1	0	1	0	$Q_{i/b}^{(1)}$	$\|Q_{i/b}^{(1)}\rangle$
$\|Q_c^{(2)}\rangle$	$Q_c^{(2)}$	1	0	0	1	$Q_{i/b}^{(2)}$	$\|Q_{i/b}^{(3)}\rangle$
	–	1	0	0	0	–	
	–	0	1	1	1	–	
$\|Q_c^{(3)}\rangle$	$Q_c^{(3)}$	0	1	1	0	$Q_{i/b}^{(3)}$	$\|Q_{i/b}^{(2)}\rangle$
$\|Q_c^{(4)}\rangle$	$Q_c^{(4)}$	0	1	0	1	$Q_{i/b}^{(4)}$	$\|Q_{i/b}^{(4)}\rangle$
	–	0	1	0	0	–	
	–	0	0	1	1	–	
	–	0	0	1	0	–	
	–	0	0	0	1	–	
	–	0	0	0	0	–	
		$Q_{i/b}^{1,1}$	$Q_{i/b}^{1,2}$	$Q_{i/b}^{2,1}$	$Q_{i/b}^{2,2}$	$Q_{i/b}^{(i)}$	$\mathcal{H}_{i/b}$
			$s_{i/b}^{(i)}$				

Fig. 0.8 Constructing a two sets, Q_c and $Q_{i/b}$ of complete questions that each capture four bits of information about the land cover type or four bits of information about the land cover homogeneity of two *minimal* raster cells.

0.6.2 Information at coarse levels of granularity

To illustrate that the notion of maximal information density is consistent with larger amounts of information at coarser levels of granularity consider Example 5:

Example 5. Consider, again, Fig. 0.9 but now suppose that the cells labeled 'cell 1' and 'cell 2' are of a size that is much larger than the minimal size (say 10 times larger or so) with respect to the measured/observed quality. That is, on this interpretation the figure represents a specific configuration S' at a

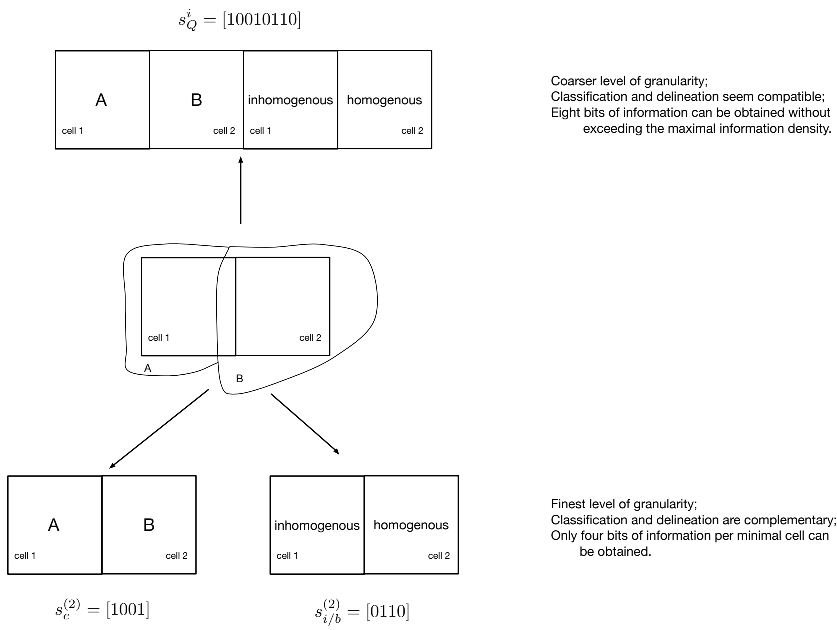


Fig. 0.9 Amount of information that can be obtained at non-finest level of granularity is 8 bits (top) and the amount of information that can be obtained at finest level of granularity is 4 bits (bottom).

coarse level of granularity. At this coarser level of granularity the observing system O can have an amount of (at least) eight bits of information about the observed system S' *without* exceeding the maximal information density of four bits per cell at the finest level of granularity. The eight bits of information specify: Cell 1 is inhomogeneous *and* of type A and cell 2 is homogeneous *and* of type B. The string of eight bits of information that corresponds to the image in the middle of Fig. 0.9 described at a coarser level of granularity is $s^i_Q = [10010110]$ as indicated in the top of the figure. □

In contrast to the non-distributive orthomodular lattice that arises from the two sets of complete questions Q_c and $Q_{i/b}$ at the finest level of granularity (Fig. 0.10), at coarser levels of granularity at which (at least) eight bits of information can be obtained per cell, the indeterminate nodes $\{\top^{i/b}_c\}$ and $\{\top^{i/b}_c\}^\perp$ of Fig 0.10 disappear and the lattice becomes a boolean algebra of the form sketched in the right of Fig. 0.1. This represents at the algebraic level that at the finest level of granularity the theory that describes the information that an observer can have about the observed system is very non-classical, i.e., logical conjunction and disjunction are non-distributive. At coarser levels of granularity descriptions become more classical, i.e., logical conjunction and disjunction are distributive.

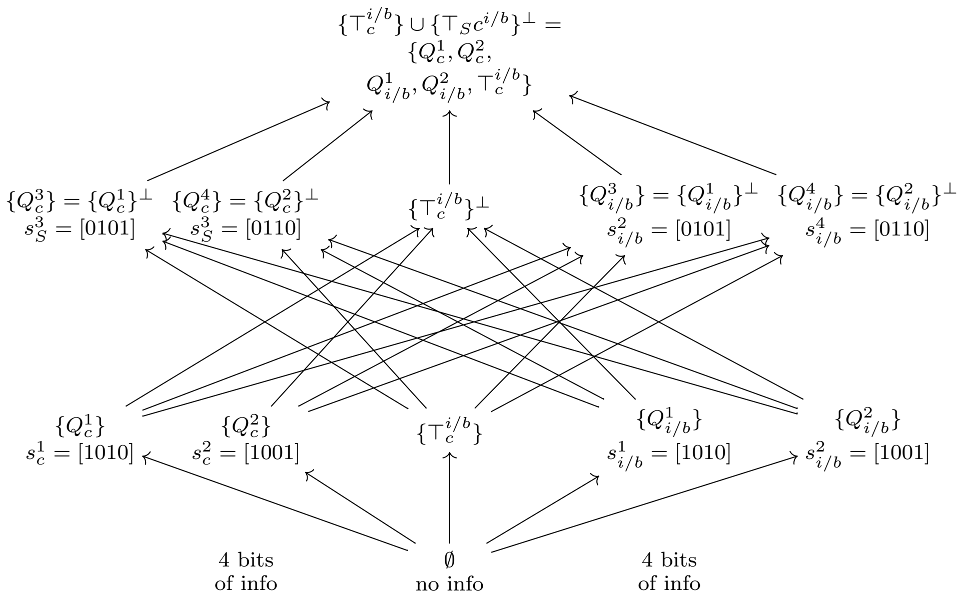


Fig. 0.10 An orthomodular lattice of the two sets of complete questions Q_c and $Q_{i/b}$ where the arrows indicate subset relations between subsets of Q_c and $Q_{i/b}$ and unions thereof. (Calude et al, 2014)

0.7 Sketch of a quantum theory

The discussion of the previous sections supports the hypothesis that a quantum theory may be the proper theory of the information that arises in the context of the classification and delineation of geographic phenomena at finest level of granularity. This is because the analysis of the classification and delineation of geographic phenomena from an information-theoretic perspective provides examples that satisfy both, Postulate 1 and Postulate 2, which, according to the relational interpretation of QM, in conjunction with Postulate 4, demand a quantum theory. At this point, however, it is not clear whether in the the context of the classification and delineation of geographic phenomena the indeterminacy that is entailed by Postulates 1 and 2, manifests itself probabilistically. To test whether or not this is indeed the case one needs to build a quantum theory and test its probabilistic predictions. It goes beyond the scope of this paper to actually develop such a theory. Nevertheless it will be useful to sketch a toy theory that illustrates how such a theory could look like. The examples developed in Sec. 0.6 will continue to serve as an illustration.

Analogous to the discussion in Sec. 0.3, the Hilbert space of the standard formalism of QM can be constructed from the sets Q_c and $Q_{i/b}$ of complete questions. The four complete questions in Q_c correspond to a system of base vectors – the Q_c-base – of a four-dimensional Hilbert space, $\mathcal{H}$. That is, the question Q_c^1 corresponds to the base vector $|Q_c^{(1)}\rangle$, etc. Similarly, the four complete questions in $Q_{i/b}$ correspond to a different system of base vectors of $\mathcal{H}$ – the $Q_{i/b}$-base. This is illustrated in Tab. 0.8.

As in the example in the left of Fig. 0.5, the systems of base vectors corresponding to the sets of complete questions Q_c and $Q_{i/b}$ are rotated with respect to one another in the four-dimensional Hilbert space $\mathcal{H}$. The orthomodular lattice that is formed by the subspaces generated by the vectors in this four dimensional Hilbert space is displayed in Fig. 0.11. In analogy to the lattice in the right of Fig. 0.5 there are two complementary sets of atoms in the lattice of Fig. 0.11.

Every vector in a vector space can be represented as a superposition of a system of base vectors. That is, if $|\phi\rangle$ is a vector of $\mathcal{H}$ that is described in the Q_c-base, then there are complex numbers $\alpha, \beta, \gamma, \delta \in \mathcal{C}$ such that $|\phi\rangle = \alpha\,|Q_c^{(1)}\rangle + \beta\,|Q_c^{(2)}\rangle + \gamma\,|Q_c^{(3)}\rangle + \delta\,|Q_c^{(4)}\rangle$. If the vector $|\phi\rangle$ represents the state of an observed system S with respect to an observing system O, then the probability that O receives a yes answer to the question $Q_c^{(i)}$ is $|\,\langle\phi|Q_c^{(i)}\rangle\,|$.

Base vectors are just vectors. Thus, one can express the base vectors of the base associated with the questions in $Q_{i/b}$ (the $Q_{i/b}$-base) in terms of the base vectors associated with the questions in Q_c (the Q_c-base). The fact that the complete questions in Q_c are incompatible with or complementary to the complete questions in $Q_{i/b}$ is represented in the formalism of QM as follows: If a vector $|\phi\rangle$ is determinate when expressed in the Q_c-base (case (a) in Sec. 0.3.2), then the vector $|\phi\rangle$ is maximally indeterminate when expressed in the $Q_{i/b}$-base (case (b) in Sec. 0.3.2). In particular, if $|\,\langle\phi|Q_c^{(i)}\rangle\,| = 1$ for some $i \in 1\ldots4$, then $|\,\langle\phi|Q_{i/b}^{(i)}\rangle\,| = \frac{1}{4}$ for every $i \in 1\ldots4$. That is, determinacy expressed probabilistically as certainty and maximal indeterminacy expressed probabilistically as complete randomness. Similarly, if $|\,\langle\phi|Q_{i/b}^{(i)}\rangle\,| = 1$ for some $i \in 1\ldots4$, then $|\,\langle\phi|Q_c^{(i)}\rangle\,| = \frac{1}{4}$ for every $i \in 1\ldots4$. Some examples of intermediate degrees of indeterminacy and their probabilistic representations are displayed in Tab. 0.12.

0.8 Conclusion

So, is there a quantum geography? Or better: Are there geographic phenomena that can be described best by a quantum theory? The answer to this question is definitively not 'No'. In fact, there seem to be good reasons to believe that the answer is 'Yes'. In support of this answer stand regional geographic phenomena with indeterminate boundaries that typically are identified by classification and delineation processes. This was illustrated above in the context of the classification and delineation of geographic regions that are characterized by their land use and land coverage.

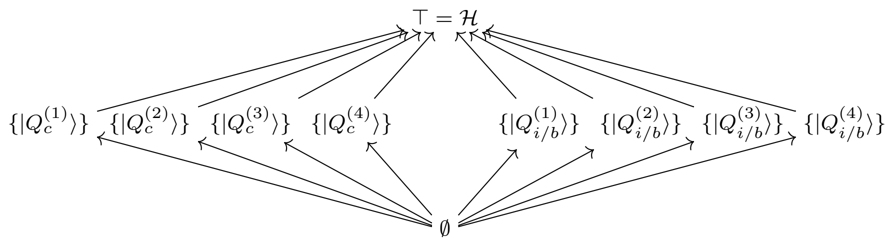


Fig. 0.11 The orthomodular structure of a four dimensional Hilbert space $\mathcal{H}$ with bases Q_c and $Q_{i/b}$ where $\{|Q_c^{(1)}\rangle\}$ stands for the subspace $\{\alpha_c\,|Q_c^{(1)}\rangle \mid (|\alpha_c| \in [0,1])\}$ of $\mathcal{H}$ spanned by the base vector $|Q_c^{(1)}\rangle$ and $\top = \{|\phi\rangle = \alpha_c\,|Q_c^{(1)}\rangle + \beta_c\,|Q_c^{(2)}\rangle + \gamma_c\,|Q_c^{(3)}\rangle + \delta_c\,|Q_c^{(4)}\rangle \mid (|\langle\phi|\phi\rangle| = 1)\} = \{|\psi\rangle = \alpha_{i/b}\,|Q_{i/b}^{(1)}\rangle + \beta_{i/b}\,|Q_{i/b}^{(2)}\rangle + \gamma_{i/b}\,|Q_{i/b}^{(3)}\rangle + \delta_{i/b}\,|Q_{i/b}^{(4)}\rangle \mid (|\langle\psi|\psi\rangle| = 1)\}$

	α	β	γ	δ	$\|\phi\rangle = \alpha\,\|Q_c^{(1)}\rangle + \beta\,\|Q_c^{(2)}\rangle + \gamma\,\|Q_c^{(3)}\rangle + \delta\,\|Q_c^{(4)}\rangle$
1	1	0	0	0	probability of $\|\langle\phi\|Q_c^{(1)}\rangle\| = 1$ to obtain a yes answer to $Q_c^{(1)}$
2	0	1	0	0	probability of $\|\langle\phi\|Q_c^{(2)}\rangle\| = 1$ to obtain a yes answer to $Q_c^{(2)}$
3	0	0	1	0	probability of $\|\langle\phi\|Q_c^{(3)}\rangle\| = 1$ to obtain a yes answer to $Q_c^{(3)}$
4	0	0	0	1	probability of $\|\langle\phi\|Q_c^{(4)}\rangle\| = 1$ to obtain a yes answer to $Q_c^{(4)}$
5	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	0	equal probability of $\|\langle\phi\|Q_c^{(1)}\rangle\| = \|\langle\phi\|Q_c^{(2)}\rangle\| = \frac{1}{2}$ to obtain a yes answer to $Q_c^{(1)}$ or $Q_c^{(2)}$ and no probability to obtain a yes answer to $Q_c^{(3)}$ or $Q_c^{(4)}$ $(\|\langle\phi\|Q_c^{(3)}\rangle\| = \|\langle\phi\|Q_c^{(4)}\rangle\| = 0$.
6	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	0	equal probability of $\|\langle\phi\|Q_c^{(1)}\rangle\| = \|\langle\phi\|Q_c^{(3)}\rangle\| = \frac{1}{2}$ to obtain a yes answer to $Q_c^{(1)}$ or $Q_c^{(3)}$ and no probability to obtain a yes answer to $Q_c^{(2)}$ or $Q_c^{(4)}$.
...	...	...	...	...	...
m	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}$	0	equal probability of $\|\langle\phi\|Q_c^{(1)}\rangle\| = \|\langle\phi\|Q_c^{(1)}\rangle\| = \|\langle\phi\|Q_c^{(3)}\rangle\| = \frac{1}{3}$ to obtain a yes answer to $Q_c^{(1)}$ or $Q_c^{(2)}$ or $Q_c^{(3)}$ and no probability to obtain a yes answer to $Q_c^{(4)}$.
...	...	...	...	...	...
n	$\frac{1}{\sqrt{4}}$	$\frac{1}{\sqrt{4}}$	$\frac{1}{\sqrt{4}}$	$\frac{1}{\sqrt{4}}$	equal probability of $\frac{1}{4}$ to obtain a yes answer to $Q_c^{(1)}$ or $Q_c^{(2)}$ or $Q_c^{(3)}$ or $Q_c^{(4)}$.

Fig. 0.12 Examples of vectors in the base corresponding to the complete questions in Q_c and their probabilistic interpretation.

The argument of why a quantum theory may be a good tool to describe those phenomena has three major premisses: First: There are three necessary and sufficient conditions that call for a quantum theory as an adequate description (Postulates 1 – 3); Second: The above class of phenomena satisfy two of those conditions (Postulates 1 and 2). Third: A quantum theory can be developed along the lines sketched above. This theory would produce predictions that, if verified empirically, indicate that Postulate 3 is satisfied. The focus of this paper was on first two premisses. The truth of the third premise is yet to be determined.

References

Andereson JR, Hardy EE, Roach JT, Witmer RE (1976) A Land Use and Land Cover Classification System for Use with Remote Sensor Data. Tech. rep., Geological Survey Professional Paper 964

Bailey RG (1983) Delineation of ecosystem regions. Environmental Management 7:365–373

Beltrametti EG, Cassinelli G, Carruthers PA (1984) The logic of quantum mechanics. Encyclopedia of Mathematics and its Applications, vol 15, Cambridge University Press, DOI 10.1017/CBO9781107340725

Birkhoff G (1948) Lattice Theory. American Mathematical Society, New York

Bittner T (2011) Vagueness and the tradeoff between the classification and delineation of geographic regions. Int J Geographical Information Systems (IJGIS) 25(5):825–850

Bittner T (2017) Towards a Quantum Theory of Geographic Fields. In: Clementini E, Donnelly M, Yuan M, Kray C, Fogliaroni P, Ballatore A (eds) 13th International Conference on Spatial Information Theory (COSIT 2017), Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Dagstuhl, Germany, Leibniz International Proceedings in Informatics (LIPIcs), vol 86, pp 5:1–5:14, DOI 10.4230/LIPIcs.COSIT.2017.5, URL `http://drops.dagstuhl.de/opus/volltexte/2017/7766`

Bittner T, Smith B (2003) A Theory of Granular Partitions. In: Duckham M, Goodchild MF, Worboys MF (eds) Foundations of Geographic Information Science, London: Taylor & Francis, pp 117–151

Bittner T, Stell J (2003) Stratified rough sets and vagueness. In: Kuhn W, Worboys M, Timpf S (eds) Spatial Information Theory. Cognitive and Computational Foundations of Geographic Information Science. International Conference COSIT'03, Springer, pp 286–303

Bolstad P (2005) GIS Fundamentals. Eider Press, Minnesota

Burrough P, Frank AU (eds) (1995) Geographic Objects with Indeterminate Boundaries. GISDATA Series II, Taylor and Francis, London

Calude CS, Hertling PH, Svozil K (2014) Embedding Quantum Universes into Classical Ones. ArXiv e-prints `1402.5199`

Dirac P (1930) Principles of Quantum Mechanics. Oxford : Clarendon Press

Einstein A (1905) On the Electrodynamics of Moving Bodies. Ann Phys 17(891)

Einstein A, Podolsky B, Rosen N (1935) Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? Physical Review 47:777–780, DOI 10.1103/PhysRev.47.777

Griffiths RB (1984) Consistent histories and the interpretation of quantum mechanics. Journal of Statistical Physics 36(1):219–272, DOI 10.1007/BF01015734, URL https://doi.org/10.1007/BF01015734

Grinbaum A (2005) Information-theoretic princple entails orthomodularity of a lattice. Foundations of Physics Letters 18(6):563–572, DOI 10.1007/s10702-005-1129-0, URL https://doi.org/10.1007/s10702-005-1129-0

Hughes RIG (1981) Quantum logic. Scientific American 245(4):202–213, URL http://www.jstor.org/stable/24964585

John von Neumann (1932) Mathematical Foundations of Quantum Mechanics. Springer Verlag

Kennedy J (2003) Space, Time, and Einstein: An Introduction. McGill-Queen's University Press, URL http://books.google.com/books?id=laalPR8p6fgC

Kochen S, Specker EP (1967) The problem of hidden variables in quantum mechanics. Journal of Mathematics and Mechanics 17(1):59–87, URL http://www.jstor.org/stable/24902153

Leonard HS, Goodman N (1940) The calculus of individuals and its uses. The Journal of Symbolic Logic 5(2):45–55, URL http://www.jstor.org/stable/2266169

Maudlin T (2002) Quantum Non-Locality and Relativity: Metaphysical Intimations of Modern Physics. Blackwell Publishers

Musser G (2015) Where is here?. Scientific American 313(5):70, URL http://search.ebscohost.com.gate.lib.buffalo.edu/login.aspx?direct=true&db=f5h&AN=110284477&site=ehost-live&scope=site

Omernik JM, Griffith GE (2014) Ecoregions of the Conterminous United States: Evolution of a Hierarchical Spatial Framework. Environmental Management DOI 10.1007/s00267-014-0364-1

Omnès R (1994) The Interpretation of Quantum Mechanics. Princeton: Princeton University Press

Redhead M (1997) Quantum Non-Locality and Relativity: Metaphysical Intimations of Modern Physics by Tim Maudlin. The Philosophical Quarterly 47(186):118–120, DOI 10.2307/2956486, URL http://www.jstor.org/stable/2956486

Romero GE (2012) Philosophical Problems of Space-Time Theories. In: Gravitation and Cosmology

Rovelli C (1996) Relational quantum mechanics. Int J Theor Phys 35:1637–1678, DOI 10.1007/BF02302261, quant-ph/9609002

Rovelli C, Vidotto F (2015) Covariant Loop Quantum Gravity. Cambridge University Press

Shannon CE (1948) A mathematical theory of communication. The Bell System Technical Journal 27(3):379–423, DOI 10.1002/j.1538-7305.1948.tb01338.x

Simons P (1987) Parts, A Study in Ontology. Clarendon Press, Oxford

Sinton DF (1978) The inherent structure of information as a constraint to analysis: Mapped thematic data as a case study. In: Dutton G (ed) Harvard Papers on Geographic Information Systems, Addison Wesley, pp 1–17

Smith B (1995) On Drawing Lines on a Map. In: Frank A, Kuhn W (eds) Conference on Spatial Information Theory, COSIT, vol 988, Springer-Verlag, Semmering, Austria, pp 475–484

Smith B (2001) Fiat Objects. Topoi 20(2):131–48

Smith B, Brogaard B (2002) Quantum Mereotopology. Annals of Mathematics and Artificial Intelligence 35(1–2)

Smith B, Varzi A (2000) Fiat and Bona Fide Boundaries. Philosophy and Phenomenological Research 60(2):401–420

Tobler W (1970) A Computer Movie Simulating Urban Growth in the Detroit Region. Economic Geography 46(2):234–240

Trassinelli M (2018) Relational quantum mechanics and probability. Foundations of Physics 48(9):1092–1111, DOI 10.1007/s10701-018-0207-7, URL https://doi.org/10.1007/s10701-018-0207-7

Wheeler JA (1989) Information, physics, quantum: The search for links. In: Proceedings III International Symposium on Foundations of Quantum Mechanics, pp 354–358

Wikipedia (2012) Interpretations of quantum mechanics — Wikipedia, The Free Encyclopedia. URL \url{http://en.wikipedia.org/w/index.php?title=Interpretations_of_quantum_mechanics&oldid=506521010}, [Online; accessed 26-August-2012]

Wikipedia contributors (2016) Complemented lattice — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Complemented_lattice&oldid=745481682, [Online; accessed 3-November-2018]

Wikipedia contributors (2018) Lattice (order) — Wikipedia, the free encyclopedia. https://en.wikipedia.org/w/index.php?title=Lattice_(order)&oldid=866284134, [Online; accessed 4-November-2018]

Yang JM (2018) A relational formulation of quantum mechanics. Scientific Reports 8(1):13,305, DOI 10.1038/s41598-018-31481-8, URL https://doi.org/10.1038/s41598-018-31481-8

.1 Boolean algebra and orthomodular lattices

A Boolean algebra is a specific form of an orthomodular lattice, which in turn is a special lattice – a partially ordered set $(X, \leq)$, with join and meet operations $(\vee, \wedge)$, with unique maximal $(\top)$ and minimal elements $(\emptyset)$. The join and meet operations $\vee$ and $\wedge$ are defined in the standard way such that $\vee$ yields the least upper bound of its arguments in the underlying partially ordered set and $\wedge$ yields the greatest lower bound (Birkhoff, 1948; Grinbaum, 2005; Wikipedia contributors, 2018). In addition, a Boolean algebra has an *ortho-complementation*, a function that maps each element a to an ortho-complement $a^{\perp}$ such that (Beltrametti et al, 1984; Wikipedia contributors, 2016): (i) $a^{\perp} \vee a = 1$ and $a^{\perp} \wedge a = 0$; (ii) $a^{\perp\perp} = a$; (iii) if $a \leq b$ then $b^{\perp} \leq a^{\perp}$. A Boolean algebra is a ortho-complemented lattice that is distributive, i.e., $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, etc. is always true. By contrast, a orthomodular lattice is a ortho-complemented lattice in which the weaker condition if $a \leq c$, then $a \vee (a^{\perp} \wedge c) = c$ is always true. Orthomodular lattices also describe the mathematical structure of Hilbert spaces that are exploited in quantum mechanics (Beltrametti et al, 1984; Hughes, 1981; Rovelli, 1996).

.2 Hilbert space

A Hilbert space $\mathcal{H}$ is a *complex* vector space with an *inner product*. In what follows Dirac's notation for vectors in Hilbert spaces (Dirac, 1930) is used. The members of a Hilbert space $\mathcal{H}$ are written as ket vectors of the form $|\phi\rangle$ where ϕ is a name/label. As vector spaces Hilbert spaces are closed under vector addition and scalar multiplication. That is if $|\phi\rangle, |\psi\rangle \in \mathcal{H}$ then $\alpha|\phi\rangle + \beta|\psi\rangle \in \mathcal{H}$, where α and β are complex numbers that modify the length of a vector via scalar multiplication and $+$ is the vector addition. The inner product $\langle\psi|\phi\rangle$ of the vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ (defined below) is a complex number.

A base $\mathcal{B} = |Q_1\rangle, \ldots, |Q_n\rangle$ of a n-dimensional Hilbert space $\mathcal{H}$ is a system of vectors such that every member of $\mathcal{H}$ can be expressed as a vector sum of the base vectors. A base is orthonormal if the inner product of distinct base vectors is zero and all base vectors are of unit length, i.e., $\langle Q_i|Q_j\rangle = 1$ if $i = j$ and $\langle Q_i|Q_j\rangle = 0$ otherwise. If the vector $|\phi\rangle = \alpha_1|Q_1\rangle + \ldots + \alpha_n|Q_n\rangle$ then there exists a dual vector $\langle\phi| = \overline{\alpha}_1|Q_1\rangle + \ldots + \overline{\alpha}_n|Q_n\rangle$ where $\overline{\alpha}_i$ is the complex conjugate of α_i. If $|\phi\rangle = \alpha_1|Q_1\rangle + \ldots + \alpha_n|Q_n\rangle$ and $|\psi\rangle = \beta_1|Q_1\rangle + \ldots + \beta_n|Q_n\rangle$ then the inner product of $|\phi\rangle$ and $|\psi\rangle$ designated by $\langle\psi|\phi\rangle$ is the sum of the products of the components of $\langle\psi|$ and $|\phi\rangle$ computed as $\sum_i \overline{\beta_i}\alpha_i$. In what follows $|\langle\phi|\psi\rangle|$ stands for the squared modulus of the scalar product $\langle\phi|\psi\rangle$. The value of $|\langle\phi|\psi\rangle|$ is a real number between zero and one and is interpreted in Wheeler's (1989) information theoretic framework as the probability that a yes answer to the question encoded in $|\phi\rangle$ is followed by a yes answer to the question encoded in $|\psi\rangle$. Details can be found in any text book on quantum mechanics. The classic reference is (Dirac, 1930).