Vague distance predicates

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Abstract

A formal theory of vague distance predicates is presented which combines a crisp region-based geometry with a theory of vague size predicates in a supervaluation-based formal framework. In the object language of the axiomatic theory, logical and semantic properties of vague distance predicates that are context- and domain-independent are formalized. Context and domain-dependent aspects are addressed in the meta-language of the theory by incorporating context- and domain-specific restrictions on the canonical interpretations. This allows to relate the ontological and qualitative analysis in the object language to numeric values as they are commonly used in scientific discourses.

Vagueness, distance predicates, mereo-geometry, formal ontology, applied ontology

1 Introduction

Vague geometric predicates such as close-to, near-to, far-away, etc. are commonly used not only in natural language discourses [44, 29], but also in spatial sciences such as geography [46, 22] and in other scientific disciplines including biology and medicine [43, 7].

In philosophy and linguistics logical theories have been developed to explicate logical and semantic properties of vague predicates, e.g., [23, 35, 31, 49]. However, such logics are rarely used to develop axiomatic theories that specify the logic and the semantics of specific vague predicates. In particular, there is a lack of axiomatic theories which specify the semantics and the logic of vague geometric predicates such as roughly-the-same-distance, close-to, near-to, far-away, etc, as they are used in geography, biology, and the medical sciences to refer to relations between spatially extended phenomenas.

By contrast, in artificial intelligence several formal tools have been developed that facilitate reasoning about relations referred to by predicates such as roughly-the-same-size-as, close-to, near-to, far-away, high, low, etc. [28, 16, 20]. Many of those proposals are based on *order of magnitude reasoning* (OMR) [36, 33, 18, 19]. Unfortunately, many attempts to formalize geometric relations in this way have two major shortcomings: Firstly, many are based on the assumption that the spatial extension of regions can be ignored and that distance relations between two regions can be modeled in terms of distance relations between two points, e.g., [28, 16, 20]. Secondly, existing proposals do not appropriately account for the feature of vagueness that affects the predicates that are used to refer to such relations.

Ignoring the extension of spatial phenomena may be appropriate in certain areas of physics, but it is clearly inappropriate in geography, biology, and medicine where size and shape are extremely important. While mathematics provides a range of sophisticated tools for representing spatial regions as point sets, such tools may be inappropriate for representing the geometry of geographic and biological phenomena (a) for ontological reasons [21, 32, 15, 41, 14, 13], as well as (b) for reasons of cognitive adequacy [42, 22]. Alternative approaches to topology and geometry have been proposed that are based on regions as representational primitives, e.g., [21, 32, 15, 37, 45, 26, 9, 39, 47, 48]. Unfortunately, with the exception of [47, 48], none of these approaches can account sufficiently for the vagueness of geometric predicates such as close-to, near-to, and far-away, etc. In contrast to the supervaluation based [23, 35] approach to formalizing vagueness used here (Section 2), the author of [47, 48] proposes an approach to fuzzifying geometric relations by extending work on fuzzy logic by [34].

The axiomatic theory that is presented in this paper attempts to overcome such shortcomings by combining a version of region-based geometry [45, 9, 4] with work from order of magnitude reasoning, especially [19], and work on semantic theories of vagueness, especially [23, 35]. In the resulting mereo-geometry one is able to formalize logical properties of vague predicates such as roughly-the-same-size-as, negligible-with-respect-to, roughly-sphere-like-shaped, close-to, near-to and far-away-from. In addition one is able to formalize context-independent aspects of the vagueness affecting such predicates.

Context and domain-specific aspects of vagueness are addressed in the meta-language of the axiomatic theory by choosing certain parametrized constraints in the formal models. Several examples are given, which demonstrate that within the proposed formalism, vague geometric predicates can be linked quite naturally to numerical interpretations that are used frequently in scientific discourses related to geography, biology, and medicine.

The remainder of the paper is structured as follows: In Section 2 the syntax and the semantics of a first-order modal logic with identity which provides the logical foundation for the axiomatic description of crisp and vague size and distance predicates. In Sections 3 and 4 the axiomatization of a mereology with crisp and vague size predicates of [8] is reviewed. A region-based geometry is sketched in Section 5 in conjunction with a discussion of the ways in which it is linked to the underlying mereology with size predicates. Based on this foundation the logic and the semantics of vague distance predicates is studied in Sections 6 – 8.

2 Logical preliminaries

Vagueness here is understood as a semantic phenomenon and is modeled within a framework that is based on supervaluation [23, 35]. To specify a semantics for vague predicates in the object language of the mereo-geometry (VMG), a first-order modal logic with identity is used. The syntax and semantics of the language of VMG are defined in the standard ways based on [30].

The language of VMG includes variables and constant symbols (*Var* and *Const*) as well as the primitive predicate symbols P, $\sim$, $\approx$, $Sp \in Pred$. In addition, the modalities U and its dual S are included.[1] Intuitively, $\mathsf{U}\alpha$ is interpreted as α is 'unequivocally' true, i.e., true under all precisifications. $\mathsf{S}\alpha$ is interpreted as α is true under some precisification.

The axiomatic theory is formed by axioms constraining the interpretations of these symbols. Wherever possible, axioms and theorems are stated as non-modal first order sentences. Leading universal quantifiers are generally omitted. All theorems are computer-verified and their computational representation can be accessed at `http://www.buffalo.edu/~bittner3/Theories/VagueSizeDistance/`. In the computational representation modal sentences are translated into first order sentences in the standard way [25, p516].

Canonical models of VMG, are Ω-structures, of the form

$$\langle \Omega, \mathcal{D}, \mathsf{V}, \sqsubseteq, \|\|, \mathsf{dist} \rangle. \tag{1}$$

The set Ω is a convex set of possible precisifications and a sub-interval of the real numbers between 0 and 1. The members of $\mathcal{D}$ are the non-empty regular closed subsets of $\Re^n$ [1] that have a finite Lebesgue measure [12] larger than zero. The relation $\sqsubseteq$ is the subset relation restricted to members of $\mathcal{D}$. $\|\|$ is a function from members of $\mathcal{D}$ to the positive real numbers such that $\|\mathsf{d}\|$ yields the Lebesgue measure of the set d.[2] The function dist yields the distance between members of $\mathcal{D}$, i.e., $\mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2)$ is the greatest lower bound of the distance between any member of d_1 and any member of d_2. The distance between members of d_1 and d_2 is the distance function *dist* of the metric space $(\Re^n, \mathit{dist})$.

V is the interpretation function. It maps the members of *Const* to members of $\mathcal{D}$. The interpretation of constants is the same at all precisification points. If $F \in \mathit{Pred}$ is a n-ary predicate, then $\mathsf{V}(F)$ is a set of $n+1$-tuples of the form $(\mathsf{d}_1, \ldots, \mathsf{d}_n, \omega)$ with $\mathsf{d}_1, \ldots, \mathsf{d}_n \in \mathcal{D}$ and $\omega \in \Omega$. V_μ is a function that maps terms (variables and constants) to members of $\mathcal{D}$ and formulas to truth values in the

[1] The notations U and S are adopted from [2].

[2] The Lebesgue measure is a formalization of the intuitive notion of the length of d if d is a regular subset of $\Re^1$. $\|\mathsf{d}\|$ is the area of d if d a regular subset of $\Re^2$ and $\|\mathsf{d}\|$ is the volume of d if d a regular subset of $\Re^3$.

standard ways. For example,

$$\begin{aligned}&\mathsf{V}_\mu(F\ t_1 \ldots t_n, \omega) = 1 \text{ if } \langle \mathsf{V}_\mu(t_1), \ldots, \mathsf{V}_\mu(t_n), \omega\rangle \in \mathsf{V}(F)\\ &\quad\text{and 0 otherwise;}\\ &\mathsf{V}_\mu(\mathsf{U}\alpha, \omega) = 1 \text{ if } \mathsf{V}_\mu(\alpha, q) = 1 \text{ for all } q \in \Omega \text{ and 0 otherwise.}\end{aligned} \tag{2}$$

A well-formed formula α is true in a given Ω-structure, if and only if $\mathsf{V}_\mu(\alpha, \omega) = 1$ for all $\omega \in \Omega$ and all variable assignments. (Variables range over all members of $\mathcal{D}$ at all precisification points $\omega \in \Omega$.) The formula α is VMG-valid if α is true in all Ω structures.

VMG includes the usual rules and axioms of first order logic with identity, a rule of necessitation for U, and the S5-axiom schemata K, T, and 5 [30]. S is defined in the usual way as the dual of U.

3 Mereology with size predicates

As the mereological basis the primitive binary predicate P is introduced in the formal theory. Intuitively, '$P\ xy$' is interpreted as "the region x is part of region y". Formally, the parthood predicate P is at all precisification points, $\omega \in \Omega$, interpreted as the relation $\sqsubseteq$ among the members of $\mathcal{D}$:

$$\mathsf{V}(P) \quad =_{df} \quad \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \mathsf{d}_1 \sqsubseteq \mathsf{d}_2\} \tag{3}$$

In terms of P the binary predicates of proper parthood (D_{PP}) and overlap (D_O) as well as the ternary predicates of summation (D_+) and difference (D_-) are introduced in the usual ways:

D_O	$O\ xy \equiv (\exists z)(P\ zx \wedge P\ zy)$
D_{PP}	$PP\ xy \equiv P\ xy \wedge \neg P\ yx$
D_+	$+xyz \equiv (w)(Owz \leftrightarrow (O\ wx \vee O\ wy))$
D_-	$-\ xyz \equiv (w)(O\ wz \leftrightarrow (\exists w_1)(P\ w_1x \wedge \neg O\ w_1y \wedge O\ w_1w))$

The standard axioms of an extensional mereology for regions in which sums of finitely many sums exist [41, 13] are added:

$A1$	$P\ xx$		
$A2$	$P\ xy \wedge P\ yx \rightarrow x = y$	$A4$	$\neg P\ xy \rightarrow (\exists z)(-\ xyz)$
$A3$	$P\ xy \wedge P\ yz \rightarrow P\ xz$	$A5$	$(\exists z)(+xyz)$

The distinction between crisp and vague predicates is made explicit by axioms that postulate or deny the existence of boundary cases. The n-ary predicate $F \in \mathit{Pred}$ is crisp if and only if for all $x_1 \ldots x_n$ either unequivocally $\neg F\ x_1 \ldots x_n$ or unequivocally $F\ x_1 \ldots x_n$. By contrast, the n-ary predicate F is vague if and only if there are $x_1 \ldots x_n$ such that on some precisification $F\ x_1 \ldots x_n$ and on some precisification $\neg F\ x_1 \ldots x_n$. As abbreviations the sentence operators D and I are introduced ($D_{\mathsf{D}}, D_{\mathsf{I}}$).[3]

[3]The operator I corresponds to Fine's indefiniteness operator [23] and the operator D corresponds to Pinkal's definiteness operator [35].

D_{D}	$\mathsf{D}\alpha \equiv \mathsf{U}\neg\alpha \vee \mathsf{U}\alpha$	$A6$	$\mathsf{D}(P\ xy)$
D_{I}	$\mathsf{I}\alpha \equiv \mathsf{S}\alpha \wedge \mathsf{S}\neg\alpha$		

Thus, Axiom (A6) requires that the predicate P is crisp.

In the formal theory the *same-size* predicate $\sim$ is introduced [8]. At all precisification points the predicate $\sim$ is interpreted as the relation of having the same Lebesgue measures:

$$\mathsf{V}(\sim) \quad =_{df} \quad \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \|\mathsf{d}_1\| = \|\mathsf{d}_2\|\} \tag{4}$$

In terms of $\sim$ one can define: the size of x is *less than or equal* to the size of y if and only if there is a region z that is a part of y and has the same size as x ($D_{\leq}$); the size of x is *less than* the size of y if and only if the size of x is less than or equal to the size of y and the size of y is not less than or equal to the size of x ($D_{<}$).

$D_{\leq}$	$x \leq y \equiv (\exists z)(z \sim x \wedge P\ zy)$	$D_{<}$	$x < y \equiv x \leq y \wedge \neg(y \leq x)$

The axioms (A7-A13) ensure that for every fixed precisification ω, $\mathsf{V}(\leq)$ is a pre-order such that $\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathsf{V}(\sim)$ iff $\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle, \langle \mathsf{d}_2, \mathsf{d}_1, \omega\rangle \in \mathsf{V}(\leq)$. Since for fixed precisifications $\mathsf{V}(\sim)$ is an equivalence relation (A7-A9), $\mathsf{V}(\leq)$ induces a partial order on the equivalence classes induced by $\mathsf{V}(\sim)$. (A11) then ensures that this partial order is a total order. (A10) links $\sim$ to the underlying mereology, i.e., if x is part of y and x and y have the same size then y is part of x. Axiom (A13) requires that the predicate $\sim$ is crisp.

$A7$	$x \sim x$		
$A8$	$x \sim y \rightarrow y \sim x$	$A11$	$x \leq y \vee y \leq x$
$A9$	$x \sim y \wedge y \sim z \rightarrow x \sim z$	$A12$	$x \leq y \wedge y \leq x \rightarrow x \sim y$
$A10$	$P\ xy \wedge x \sim y \rightarrow P\ yx$	$A13$	$\mathsf{D}(x \sim y)$

The theory formed by the axioms and definitions discussed in this section is called QSizeR.

4 Vague size predicates

The crisp mereology with size predicates, QSizeR, is extended by the relational predicates $\approx$, $\preceq$, $\prec$, and $\ll$ [8]. The resulting theory is called VSizeR. In terms of the primitive $\approx$ (roughly-the-same-size) one can define: Region x is *negligible in size with respect to* region y if and only if there are regions z_1 and z_2 such that (1) x and z_1 have the same size, (2) z_1 is a part of y, (3) z_2 is the difference of z_1 in y and (4) z_2 and y have roughly the same size ($D_{\ll}$). The vague ordering predicates $\preceq$ and $\prec$ are defined in the obvious ways.

$D_{\ll}$	$x \ll y \equiv (\exists z_1)(\exists z_2)(z_1 \sim x \wedge P\ z_1 y \wedge -y z_1 z_2 \wedge z_2 \approx y)$
$D_{\preceq}$	$x \preceq y \equiv x \leq y \vee x \approx y$
$D_{\prec}$	$x \prec y \equiv x \preceq y \wedge \neg(y \preceq x)$

In Ω-structures the predicates $\approx$, $\ll$, $\preceq$ and $\prec$ are interpreted as follows:

$$\begin{array}{ll} \mathsf{V}(\approx) & = \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega \rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid 1/(1+\omega) \leq \|\mathsf{d}_1\|/\|\mathsf{d}_2\| \leq 1+\omega\} \\ \mathsf{V}(\ll) & = \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega \rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \|\mathsf{d}_1\|/\|\mathsf{d}_2\| < \omega/(1+\omega)\} \\ \mathsf{V}(\preceq) & = \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega \rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \|\mathsf{d}_1\|/\|\mathsf{d}_2\| \leq 1+\omega\} \\ \mathsf{V}(\prec) & = \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega \rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \|\mathsf{d}_1\|/\|\mathsf{d}_2\| < 1/(1+\omega)\} \end{array} \tag{5}$$

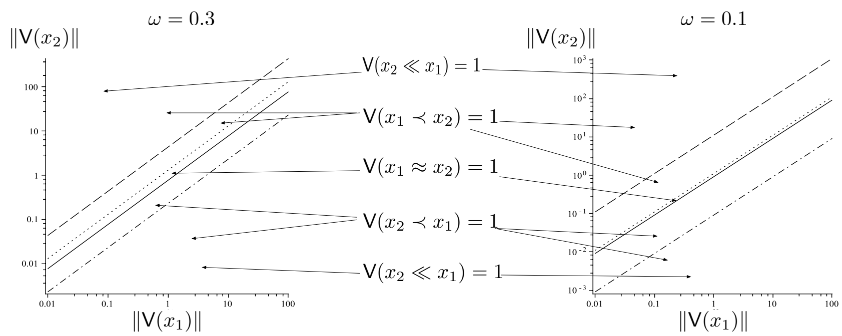

Figure 1: Interpretations of the formulas $x_1 \approx x_2$, $x_1 \preceq x_2$, $x_1 \prec x_2$, $x_1 \ll x_2$ at the precisification points $\omega = 0.3$ and $\omega = 0.1$. [8]

Unlike $\sim$, which has a crisp and unique interpretation (formally reflected by the *parameter-free equation* in (4)), the interpretations of $\approx$, $\preceq$, $\prec$, and $\ll$ are vague and admit ranges of interpretations. In the formal models this is reflected by the use of *parameterized constraints*. There are large classes of constraints that potentially can serve as interpretations for the vague predicates $\approx$, $\preceq$, $\prec$, and $\ll$. The constraints that are used here are displayed in the middle of Figure 1. Of course, other choices are possible, however these particular constraints have been proven useful in a wide range of applications [33, 18]. At the bottom of Figure 1 the interpretations of the formulas $x_1 \approx x_2$, $x_1 \preceq x_2$, $x_1 \prec x_2$, $x_1 \ll x_2$ using the chosen class of constraints are displayed for the precisification points $\omega = 0.3$ and $\omega = 0.1$.

The axioms for vague size predicates are collected in Table 1. $\approx$ is reflexive and symmetric (A19, A20). Axioms (A17, A18) link $\approx$ and $\ll$ to the crisp size predicates of QSizeR. The remaining axioms characterize the vagueness of the predicates defined in terms of $\approx$. For an extended discussion see [8]. The contributions of the various axioms of VSizeR to the constraints on the logically admissible range of the precisification parameters in Ω are collected in Table 1. Jointly, the following constraints need to be satisfied:

Theorem 1 ([8]) *The axioms (A14-A30) of VSizeR are true in Ω-structures,* $\langle \Omega, \mathcal{D}, V, \subseteq, \|\|\rangle$*, where* $\Omega \subseteq [\omega_\downarrow, \omega_\uparrow] \subset (0, -\frac{1}{2} + \frac{1}{2}\sqrt{5}] \subset \Re$ *such that* $2\omega_\downarrow + {\omega_\downarrow}^2 \leq \omega_\uparrow$ *and* $\omega_\downarrow$ *and* $\omega_\uparrow$ *are bounds on a convex set of possible precisifications.*

number	axiom	true in Ω-structures if
A14	$x \approx x$	$0 \leq \omega_\downarrow \leq \omega_\uparrow < 1$
A15	$x \approx y \rightarrow y \approx x$	$0 \leq \omega_\downarrow \leq \omega_\uparrow < 1$
A16	$x \approx y \wedge x \leq z \wedge z \leq y \rightarrow (z \approx x \wedge z \approx y)$	$0 \leq \omega_\downarrow \leq \omega_\uparrow < 1$
A17	$x \ll y \wedge y \leq z \rightarrow x \ll z$	$0 \leq \omega_\downarrow \leq \omega_\uparrow < 1$
A18	$(\exists y)(\mathsf{U}(x \ll y))$	$0 < \omega_\downarrow < \omega_\uparrow < 1$
A19	$(\exists y)(\mathsf{U}(x \approx y) \wedge \neg(x \sim y))$	$0 < \omega_\downarrow < \omega_\uparrow < 1$
A20	$(\exists y)\mathsf{I}(x \ll y)$	$0 < \omega_\downarrow < \omega_\uparrow < 1$
A21	$(\exists y)(\mathsf{I}(x \prec y)$	$0 < \omega_\downarrow < \omega_\uparrow < 1$
A22	$\mathsf{S}(x \ll y) \rightarrow \mathsf{U}(x \prec y)$	$0 < \omega_\downarrow < \omega_\uparrow < 1$
A23	$\mathsf{U}(x \approx y) \wedge \neg(x \sim y) \rightarrow$	
	$(\exists z)(\mathsf{U}(z \approx x) \wedge \neg\mathsf{U}(z \approx y))$	$0 < \omega_\downarrow < \omega_\uparrow < 1$
A24	$\mathsf{U}(x \approx y) \wedge \mathsf{U}(y \approx z) \rightarrow \mathsf{S}(x \approx z)$	$0 < 2\omega_\downarrow + {\omega_\downarrow}^2 \leq \omega_\uparrow \leq 1$
A25	$(\exists x)(\mathsf{U}(x \ll z) \rightarrow (\exists x)(\exists y)(\mathsf{U}(x \approx y) \wedge$	
	$\mathsf{U}(x \ll z) \wedge \neg\mathsf{U}(y \ll z))$	$0 < \omega_\downarrow < \omega_\uparrow < 1$
A26	$(\exists x)(\exists y)(\mathsf{U}(x \approx y) \wedge \mathsf{U}(z \ll x) \wedge \neg\mathsf{U}(z \ll y))$	$0 < \omega_\downarrow < \omega_\uparrow < 1$
A27	$\mathsf{U}(x \ll y) \wedge \mathsf{U}(y \approx z) \rightarrow \mathsf{S}(x \ll z)$	$0 < -\frac{\omega}{\omega-1} \leq \omega_\uparrow \leq 1$
A28	$\mathsf{U}(x \approx y) \wedge \mathsf{U}(y \ll z) \rightarrow \mathsf{S}(x \ll z)$	$0 < -\frac{\omega}{\omega-1} \leq \omega_\uparrow \leq 1$
A29	$x \ll y \wedge y \approx z \rightarrow x \prec z$	$0 < \omega < -\frac{1}{2} + \frac{1}{2}\sqrt{5}$
A30	$x \approx y \wedge y \ll z \rightarrow x \prec z$	$0 < \omega < -\frac{1}{2} + \frac{1}{2}\sqrt{5}$

Table 1: The axioms of VSizeR are true in Ω-structures with the listed properties ($\Omega = [\omega_\downarrow, \omega_\uparrow]$) where $\omega_\downarrow$ and $\omega_\uparrow$ are bounds on a convex set of possible precisifications [8].

5 A crisp region-based geometry

The purpose of this section is to provide a self-contained axiomatic basis of a simple mereo-geometry for the study of vague shape predicates such as roughly-sphere-like-shaped and vague distance predicates such as close-to, near-to, far-away, etc. The aim is not to fully develop a region-based geometry. For an extended discussion of how the theory fragment presented here relates to Bennett's version of Tarski's mereo-geometry [3], see [6]. For an overview of alternative region-based geometries see [10]. A simplified version of the mereo-geometry presented here was used in [5] to formalize certain mereo-geometrical aspects of biological structures.

5.1 Definitions

QSizeR is extended by the primitive predicate *Sp*. The formula '*Sp* x' is interpreted as *x is a sphere-shaped region* or sphere for short. In Ω-structures *Sp* is interpreted as:

$$\mathsf{V}(Sp) \quad =_{df} \quad \{\langle \mathsf{d}, \omega \rangle \in \mathsf{Sp} \times \Omega\} \tag{6}$$

In terms of *Sp* one can define (Fig. 2): x is *maximal with respect to y in z* if and only if (1) x, y, and z are spheres, (2) x and y are non-overlapping parts of z, and (3) every sphere u that has x as a part either is identical to x, overlaps y, or is not a part of z (D_{Mx}) (Fig. 2(a)). x is a *concentric* proper part of y

if and only if (1) x and y are spheres, (2) x is a proper part of y and (3) all spheres that are maximal with respect to x in y have the same size (D_{CoPP}) (Fig. 2(b)); Sphere y is a smallest connecting sphere for regions x and z and only if (1) x and z are disconnected, (2) y is connected to x and z, and (3) y is smaller or equal in size with respect to all spheres that are connected to x and z (D_{MCS}). (Fig. 2(c)). Two regions x and y are *connected* if and only if there is a sphere z that overlaps x and y and all spheres that are concentric proper parts of z also overlap x and y (D_C) (Fig. 2(d)). (A similar definition of C was employed in [4].)

$$
\begin{array}{ll}
D_{Mx} & Mx\ xyz \equiv Sp\ x \wedge Sp\ y \wedge Sp\ z \wedge P\ xz \wedge P\ yz \wedge \neg O\ xy \wedge \\
 & \qquad (u)(Sp\ u \wedge P\ xu \rightarrow (x = u \vee O\ uy \vee \neg P\ uz)) \\
D_{CoPP} & CoPP\ xy \equiv Sp\ x \wedge Sp\ y \wedge PP\ xy \wedge (u)(v)(Mx\ uxy \wedge Mx\ vxy \rightarrow u \sim v) \\
D_{MCS} & MCS\ xyz \equiv Sp\ y \wedge C\ xy \wedge C\ yz \wedge \neg C\ xz \wedge \\
 & \qquad (w)(Sp\ w \wedge C\ xw \wedge C\ wz \rightarrow y \leq w) \\
D_C & C\ xy \equiv (\exists z)(Sp\ z \wedge O\ zx \wedge O\ zy \wedge (u)(CoPP\ uz \rightarrow (O\ ux \wedge O\ uy))
\end{array}
$$

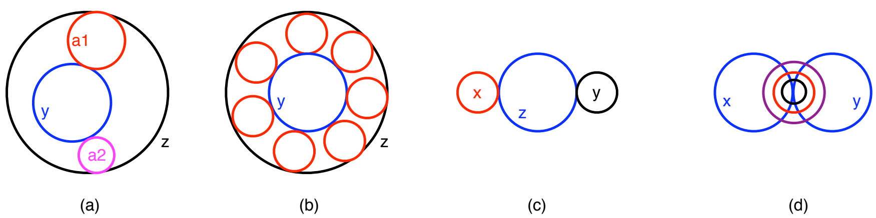

Figure 2: Top: Definitions using the sphere predicate. Bottom: Illustration of the definitions for the case of spheres in $\Re^2$: (a) D_{Mx}: $Mx\ a_1yz \wedge Mx\ a_2yz$, (b) D_{CoPP}: $CoPP\ yz$, (c) D_{MCS}: $MCS\ xzy$, and (d) D_C: $C\ xy$.

In Ω-structures the connection predicate C holds between regions d_1 and d_2 if and only if the distance between them is zero:

$$
\begin{aligned}
\mathsf{V}(C) &= \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2) = 0\} \\
&= \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \mathsf{d}_1 \cap \mathsf{d}_2 = \emptyset\}
\end{aligned} \tag{7}
$$

In the axiomatic theory an axiom is included requiring that the predicate Sp is crisp (A26). One then can prove that C is crisp (T1).

$$A26 \quad \mathsf{D}(Sp\ x) \qquad\qquad T1 \quad \mathsf{D}(C\ xy)$$

It follows that the mereo-topological base of the presented mereo-geometry is crisp.

5.2 Existential axioms

The following spheres are required to exist: Every region has a sphere as a proper part (A27). If sphere x is a proper part of sphere y then there is a sphere z that is maximal with respect to x in y (A28). If y is smaller in size to sphere x then there is a concentric proper part of x with the same size as y (A29). If the spheres x and y have the same size then there are spheres z_1 and z_2 such that z_1 is a concentric proper part of z_2 and x and y are maximal with respect to z_1 in z_2 (A30). For all disconnected regions there exists a minimal connecting sphere (A31).

$A27 \quad (\exists z)(Sp\ z \wedge PP\ zx)$
$A28 \quad Sp\ x \wedge Sp\ y \wedge PP\ xy \rightarrow (\exists z)(Mx\ zxy)$
$A29 \quad Sp\ x \wedge y < x \rightarrow (\exists z)(CoPP\ zx \wedge z \sim y)$
$A30 \quad Sp\ x \wedge Sp\ y \wedge x \sim y \rightarrow (\exists z_1)(\exists z_2)(CoPP\ z_1 z_2 \wedge Mx\ x z_1 z_2 \wedge Mx\ y z_1 z_2)$
$A31 \quad \neg C\ xy \rightarrow (\exists z)(MCS\ xzy)$

It follows from (A27) that all regions are mereological sums of spheres (T2). Axioms (A28) and (A29) ensure that the definitions D_{CoPP} and D_C work as intended. From (A27) and (A29) it follows that every sphere has a concentric proper part (T3). Thus regions are infinitely sub-divisible. (A30) ensures that on the sub-domain of spheres, the interpretation of $\sim$ is restricted to the congruence relation between spheres.

$T2 \quad O\ wx \leftrightarrow (\exists z)(Sp\ z \wedge P\ zx \wedge O\ wz)$
$T3 \quad Sp\ x \rightarrow (\exists y)(CoPP\ yx)$

One can also prove: C is reflexive (T4); C symmetric (T5); if x is part of y, and z is connected to x then y is connected to z (T6); all minimal connecting spheres for two disconnected regions have the same size (T7).

$T4 \quad C\ xx$
$T5 \quad C\ xy \rightarrow C\ yx$
$T6 \quad P\ xy \wedge C\ zx \rightarrow C\ yz$
$T7 \quad MCS\ x z_1 y \wedge MCS\ x z_2 y \rightarrow z_1 \sim z_2$

5.3 Links to the underlying mereotopology

The converse of theorem (T6) is not provable and is added as an axiom: if everything that connects to x also connects to y then x is a part of y (A32). Moreover, *CoPP* is required to be transitive (A33).

$A32 \quad (z)(Czx \rightarrow Czy) \rightarrow P\ xy$
$A33 \quad CoPP\ xy \wedge CoPP\ yz \rightarrow CoPP\ xz$
$T8 \quad P\ xy \leftrightarrow (z)(Czx \rightarrow Czy)$
$T9 \quad x = y \leftrightarrow (z)(Czx \leftrightarrow Czy)$

From axioms (A2 and A32) and theorem (T4) it follows: x is a part of y if and only if everything connected to x is also connected to y (T8); x and y are identical if and only if everything is connected to x if and only if it is connected to y (T9). Thus *CoPP* is a restricted proper parthood predicate and the connectedness predicate C has the usual properties one expects in the domain of spatial regions. (See also [37].)

The theory that extends QSizeR with the axioms and definitions presented in this section is called RBG.

6 Vague distance predicates

It is common to understand the distance between two extended regions x and y as the greatest lower bound of the distances between any point in x and any point in y. This, however, does not seem to be sufficient to capture the semantics of predicates such as close-to, far-away, etc. For example, a (large) road-sized region may be (on the scale of the road) close to a (small) pebble-sized region in an adjacent ditch, but the pebble-sized region maybe not (on the scale of the pebble) be close to the road-sized region. In this subsection vague distance predicates between regions are introduced in a way that takes into account the size of the regions. The formal basis of this section is formed jointly by RBG and VSizeR. The theory which extends VSizeR and RBG by the definitions for distance predicates is called VDistR.

6.1 Vague size-depended distance predicates

Region x is *close* to region y if and only if there is a sphere z such that z is connected to both x and y and z is negligible in size with respect to x (D_{Cl}). x is *strictly close* to y if and only if x is close to y but not connected to y (D_{SCl}). x is *near* to y if and only if there is a sphere z such that z is connected to x and y and the size of z is less than or roughly equal to the size of x (D_N). x is *strictly near* to y if and only if x is near to y but not close to y (D_{SN}). x is *away* from y if and only if x is less than and not roughly equal in size with respect to all spheres w that are connected to x and y (D_A). x is *far away* from y if and only if x is negligible in size with respect to all spheres w that are connected to x and y (D_{FA}). Axiom (A31) ensures that there always is a sphere z such that z is connected to x and y. x is *moderately away* from y if and only if x is away from y but not far away from y (D_{MA}).[4]

D_{Cl}	$Cl\ xy \equiv (\exists z)(Sp\ z \wedge C\ zx \wedge C\ zy \wedge z \ll x)$
D_{SCl}	$SCl\ xy \equiv Cl\ xy \wedge \neg C\ xy$
D_N	$N\ xy \equiv (\exists z)(Sp\ z \wedge C\ zx \wedge C\ zy \wedge z \preceq x)$
D_{SN}	$SN\ xy \equiv N\ xy \wedge \neg Cl\ xy$
D_A	$A\ xy \equiv (w)(Sp\ w \wedge C\ wx \wedge C\ wy \rightarrow x \prec w)$
D_{FA}	$FA\ xy \equiv (w)(Sp\ w \wedge C\ wx \wedge C\ wy \rightarrow x \ll w)$
D_{MA}	$MA\ xy \equiv A\ xy \wedge \neg FA\ xy$

One can prove that all distance predicates are vague:

$T10$	$(\exists x)(\exists y)\mathsf{I}(Cl\ xy)$	$T12$	$(\exists x)(\exists y)\mathsf{I}(A\ xy)$
$T11$	$(\exists x)(\exists y)\mathsf{I}(N\ xy)$	$T13$	$(\exists x)(\exists y)\mathsf{I}(FA\ xy)$

[4] Consider the distance predicates $\{C, c, M, f, F\}$ between points of [24] and [28]. Roughly, for regions of the same size, *SCl* corresponds to C, *SN* corresponds to c, *MA* corresponds to M, and *FA* corresponds to the disjunction of f and F. Note, however that the definitions given here are for regions rather than for points and take the size of the regions into account. The predicate *SCl* roughly corresponds to the predicate 'near' of [40].

Let $\|\mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2)\|$ be the Lebesgue measure of a n-ball with the diameter $\mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2)$. In Ω-structures the distance predicates *Cl*, *N*, *SCl*, *SN*, *A*, *MA*, and *FA* are interpreted as follows:

$$
\begin{array}{lcl}
\mathsf{V}(Cl) & = & \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \mathsf{d}_1 \cap \mathsf{d}_2 \neq \emptyset \text{ or} \\
 & & \quad \|\mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2)\|/\|\mathsf{d}_1\| < \omega/(1+\omega)\} \\
\mathsf{V}(SCl) & = & \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \mathsf{d}_1 \cap \mathsf{d}_2 = \emptyset \text{ and} \\
 & & \quad \|\mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2)\|/\|\mathsf{d}_1\| < \omega/(1+\omega)\} \\
\mathsf{V}(N) & = & \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \mathsf{d}_1 \cap \mathsf{d}_2 \neq \emptyset \text{ or} \\
 & & \quad \|\mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2)\|/\|\mathsf{d}_1\| \leq 1+\omega\} \\
\mathsf{V}(SN) & = & \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \\
 & & \quad \omega/(1+\omega) \leq \|\mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2)\|/\|\mathsf{d}_1\| \leq 1+\omega\} \\
\mathsf{V}(A) & = & \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \\
 & & \quad \|\mathsf{d}_1\|/\|\mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2)\| < 1/(1+\omega)\} \\
\mathsf{V}(FA) & = & \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \\
 & & \quad \|\mathsf{d}_1\|/\|\mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2)\| < \omega/(1+\omega)\} \\
\mathsf{V}(MA) & = & \{\langle \mathsf{d}_1, \mathsf{d}_2, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \Omega \mid \\
 & & \quad \omega/(1+\omega) \leq \|\mathsf{d}_1\|/\|\mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2)\| < 1/(1+\omega)\}
\end{array}
\tag{8}
$$

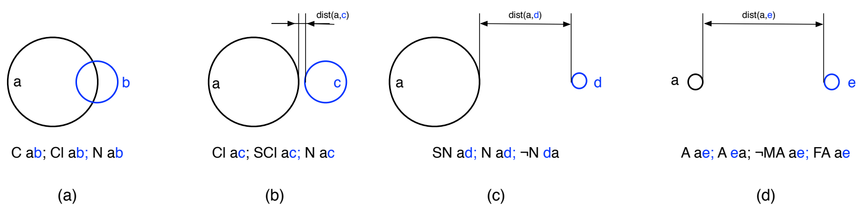

Figure 3: Examples of vague distance predicates between two-dimensional discs in the plane. Assumptions: $\|\mathsf{a}\| = 1$, $\omega = 0.2$, $\mathsf{dist}(\mathsf{a}, \mathsf{c}) < 0.461$, $0.461 \leq \mathsf{dist}(\mathsf{a}, \mathsf{d}) \leq 1.24$, $\mathsf{dist}(\mathsf{a}, \mathsf{e}) > 2.77$. (All distances are understood in units such that in the respective figure $\|\mathsf{a}\| = 1$.)

Consider Figure 3 and assume that the discs that are displayed on the left hand sides in subfigures (a)–(d) (they are all referred to as 'a') all have a Lebesgue measure of 1, i.e., $\|\mathsf{a}\| = 1$. The relation between the diameter and the Lebesgue measure for 2-balls of $\Re^2$ can be expressed as $\|d\| = \frac{\pi}{4}d^2$. Using this relationship and the constraints in (8) one can compute what counts as close, near, etc. for the a region a of unit size at specific precisification points. This is visualized in Figure 4 for precisification points that fall in the range of $0.01 \leq \omega < -\frac{1}{2} + \frac{1}{2}\sqrt{5}$. Consider the precisification point $\omega = 0.2$. In Figure 3(b) the regions a and c count as close if and only if the distance $\mathsf{dist}(\mathsf{a}, \mathsf{c})$ is smaller than 0.461. (Distance and diameter units are such that in the respective figure $\|\mathsf{a}\| = 1$.) The regions a and c count as near if and only if the distance $\mathsf{dist}(\mathsf{a}, \mathsf{c})$ is smaller than 1.24 units. In Figure 3(d) the regions a and e count as far apart if and only if the distance $\mathsf{dist}(\mathsf{a}, \mathsf{e})$ is larger than 2.77 units.

In Definitions D_{Cl} – D_{MA} the first parameter serves as a *reference* region which size determines what counts as near, close, far-away, etc. To some it may

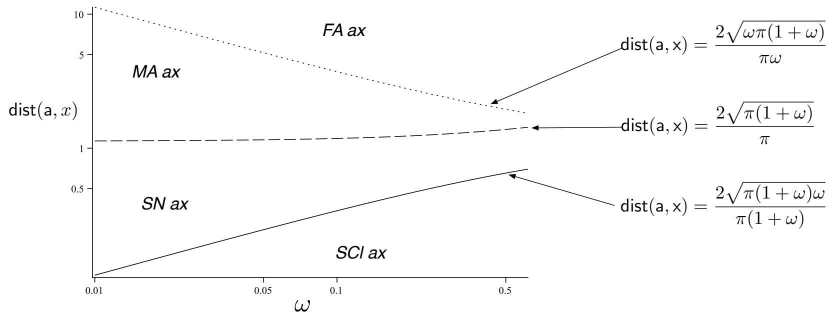

Figure 4: Interpretation of vague distance predicates for regions of normalized sizes ($\|\mathsf{a}\| = 1$) at precisification points $0.01 \leq \omega < -\frac{1}{2} + \frac{1}{2}\sqrt{5}$.

be more intuitive to use the second parameter as the reference region. To those the following definition of far-away may be more intuitive:

$$FA'\ xy \equiv (w)(Sp\ w \wedge C\ wx \wedge C\ wy \rightarrow y \ll w)$$

In any case, it should be obvious that such changes can be made easily and that they do not seriously affect the study of the logical properties of these predicates.

6.2 Logical properties of vague distance predicates

One can prove that the predicates *Cl*, *N*, *SCl*, *SN*, *A*, *MA*, and *FA* form the implication hierarchy depicted in the left part of Figure 5. In addition one can prove that there exist sets of jointly exhaustive and pair-wise disjoint (JEPD) predicates as depicted in the right part of Figure 5. In the remainder the letter Δ is used to refer (in the meta language) to the set of jointly exhaustive and pairwise disjoint predicates that form the bottom of the lattice in the right part of Figure 5. One can also prove that *Cl* and *N* are reflexive and that *SCl*, *SN*, *A*, *MA*, and *FA* are irreflexive.

6.3 Parthood and vague distances

One can prove a number of compositional theorems about the logical interrelationships between parthood and the various vague distance predicates. Those theorems show that there are some structural similarities between the vague distance predicates close-to, near, etc. and the crisp connectedness predicate C (in particular T6): One can prove that if x is a part of y and x is close to z then y is close to z (T15a). Similarly one can prove that if x is a part of y and z is close to x then z is close to y (T15b). Similar theorems for the interrelationships of parthood and nearness and apartness can be proved (T16a-T17b).

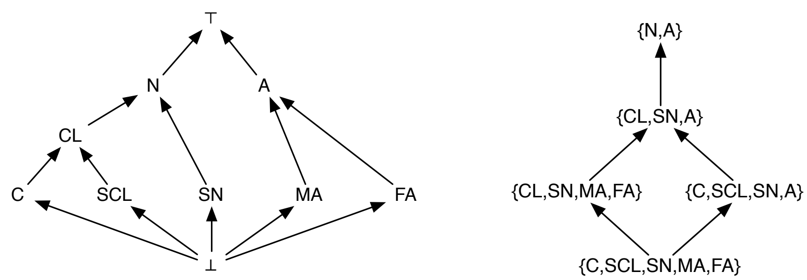

Figure 5: Logical interrelationships of the vague distance predicates. Left: Implication hierarchy of the vague distance predicates. Right: Sets of JEPD vague distance predicates.

*T*15*a*	$P\ xy \wedge Cl\ xz \rightarrow Cl\ yz$	*T*15*b*	$P\ xy \wedge Cl\ zx \rightarrow Cl\ zy$
*T*16*a*	$P\ xy \wedge N\ xz \rightarrow N\ yz$	*T*16*b*	$P\ xy \wedge N\ zx \rightarrow N\ zy$
*T*17*a*	$P\ xy \wedge A\ yz \rightarrow A\ xz$	*T*17*b*	$P\ xy \wedge A\ zy \rightarrow A\ zx$

Besides those structural similarities there are also differences when comparing the logical relationships between connectedness and parthood and the logical relationships between vague distance predicates and parthood: There should not be theorems or axioms for vague distance predicates that are structurally similar to axiom (A32). Consider the formula F1 = '$(z)(Nzx \rightarrow Nzy) \rightarrow P\ xy$'. If F1 was a theorem then the formula F2= '$(z)(Nzx \leftrightarrow Nzy) \leftrightarrow x = y$' would be an immediate consequence. That is, via F2 the crisp identity predicate would become definable in terms of a vague predicate such as N.

6.4 Symmetry

None of the defined distance predicates is symmetric. As pointed out above, a road-sized region may (on the scale of the road) be close to a pebble-sized region in an adjacent ditch, even if the pebble-sized region is not (on the scale of the pebble) close to the road-sized region.

To see how this asymmetry is represented in the underlying Ω-structures, consider Figure 3(c). Under the assumptions listed in the caption of the figure, ($\mathsf{V}(N\ ad, \omega)$=1) if and only if the distance between $\mathsf{V}(a)$ and $\mathsf{V}(d)$ is less than 1.24 units on a scale based on the size of the disc $\mathsf{V}(a)$ – the reference region. If d is chosen as the reference region such that $\|\mathsf{V}(d)\| = 1$, then $\mathsf{V}(N\ da, \omega) = 1$ only if the distance between $\mathsf{V}(d)$ and $\mathsf{V}(a)$ is smaller than 1.24 units on a scale based on the size of $\mathsf{V}(d)$. Since $\mathsf{V}(d)$ is much smaller than $\mathsf{V}(a)$, 1.24 units on a scale based on $\|\mathsf{V}(d)\|$ cover less distance than 1.24 units on a scale based on $\|\mathsf{V}(a)\|$. Therefore, if ($\mathsf{V}(N\ ad, \omega)$=1) then the distance between $\mathsf{V}(d)$ and $\mathsf{V}(a)$ is larger than 1.24 units on a scale based on $\|\mathsf{V}(d)\|$. Thus ($\mathsf{V}(N\ da, \omega) = 0$).

By contrast, consider Figure 3(d). Since a and e are discs of the same size, it follows that $\mathsf{V}(A\ ae, \omega) = 1$ if and only if $\mathsf{V}(A\ ea, \omega) = 1$. More generally, one can prove restricted forms of symmetry for the vague distance predicates. The respective theorems of VDistR are collected in Table 2. As displayed in the table, four cases are distinguished depending on which size predicate is used to impose restrictions on the size of the regions between which a given distance predicate is supposed to hold:

Firstly. For regions of *exactly the same size* one can prove that all distance predicates are symmetric. (Row 1 in Table 2.) Secondly. One can prove that if the size of x is less than or equal to the size of y then: if *Cl* xy then *Cl* yx. Similarly for *SCl*, N, A, and *FA*. The predicates *SN* and *MA* are not symmetric under those relaxed constraints. For *SN* and *MA* only results about coarser predicates can be proved. Such coarser predicates are expressed as disjunctions of predicates in Δ. (Row 2 in Table 2.)

Thirdly. If the vague size predicate $\preceq$ is used to express the constraints on the region size the one can prove only results about much coarser relations. Again, in Table 2 such coarser predicates are expressed as disjunctions of predicates in Δ. For example, if the size of x is less than or *roughly equal* to the size of y then: if x is strictly close to y then either y is strictly close to x or y is strictly near to x. (Rows 3 and 4 in Table 2.)

Fourthly. If the vague size predicate $\prec$ is used to express the constraints on the region sizes then symmetry results for *Cl*, *SCl*, N, A, *FA* can be proved. For *SN* and *MA* only results about *coarser* predicates that could be expressed as disjunctions of predicates in Δ can be proved. (Row 4 in Table 2.)

	Cl xy	*SCl* xy	N xy	*SN* xy	A yx	*MA* yx	*FA* yx
$x \sim y$	*Cl* yx	*SCl* yx	N yx	*SN* yx	A xy	*MA* xy	*FA* xy
$x \leq y$	*Cl* yx	*SCl* yx	N yx	N yx	A xy	A xy	*FA* xy
$x \preceq y$	N yx	*SCl* yx *SN* yx	N yx *MA* yx	*SCl* xy *SN* yx *MA* yx	A xy *SN* xy	*SN* xy A xy	A xy
$x \prec y$	*Cl* yx	*SCl* yx	N yx	N yx	A xy	A xy	*FA* xy

Table 2: Theorems about the restricted symmetry of vague distance predicates. Sets of predicates are interpreted as disjunctions.

7 Logical interrelations between distance and shape predicates

Additional restrictions on the shape of regions may be needed to characterize distance predicates such as near, far, etc. Consider the ring-shaped region depicted in Figure 6(d). When making a judgement about the distance relation between a ring-shaped region such as d and a solid disc with the same Lebesgue measure, the judger may associate different scales to each regions although both are of the same size. (Intuitively, a ring-shaped region such as d may appear to be 'larger' than a solid disc with the same Lebesgue measure.) Thus, it seems to be reasonable to restrict vague distance predicates to *roughly-sphere-like-shaped* regions.

Region x is *sphere-like-shaped* if and only if there is a sphere y such that (a) x and y are of the same size, (b) every region that is connected to y is close to x, and (c) every region that is connected to x is close to y (D_{Sl}). Region x is *roughly sphere-like-shaped* if and only if there is a sphere y such that (a) x and y are of the same size, (b) every region that is connected to y is near to x, and (c) every region that is connected to x is near to y (D_{RSl}). Region x is *sphere-unlike* if and only if x is not roughly sphere-like (D_{Sul}).

$$
\begin{array}{ll}
D_{Sl} & Sl\,x \equiv (\exists y)(Sp\ y \wedge y \sim x \wedge (z)(C\ yz \rightarrow Cl\ xz) \wedge (z)(C\ xz \rightarrow Cl\ yz) \\
D_{RSl} & RSl\,x \equiv (\exists y)(Sp\ y \wedge y \sim x \wedge (z)(C\ yz \rightarrow N\ xz) \wedge (z)(C\ xz \rightarrow N\ yz) \\
D_{Sul} & Sul\,x \equiv \neg RSl\,x
\end{array}
$$

Let $\|dist(p_s, p_d)\|$ be the Lebesgue measure of a n-ball with a diameter that is identical to the distance between the points p_s and p_d ($dist(p_s, p_d)$). In Ω-structures the shape predicates Sl and RSl are interpreted as follows:

$$
\begin{array}{rl}
\mathsf{V}(Sl) = & \{\langle \mathsf{d}, \omega\rangle \in \mathcal{D} \times \Omega \mid \exists \mathsf{s} \in \mathsf{Sp} : \langle \mathsf{d}, \mathsf{s}, \omega\rangle \in \mathsf{V}(\sim) \ \& \\
& \forall p_s \in \mathsf{s} : \exists p_d \in \mathsf{d} : \|dist(p_s, p_d)\| / \|\mathsf{d}\| < \omega/(1+\omega) \ \& \\
& \forall p_d \in \mathsf{d} : \exists p_s \in \mathsf{s} : \|dist(p_s, p_d)\| / \|\mathsf{d}\| < \omega/(1+\omega)\} \\
\mathsf{V}(RSl) = & \{\langle \mathsf{d}, \omega\rangle \in \mathcal{D} \times \Omega \mid \exists \mathsf{s} \in \mathsf{Sp} : \langle \mathsf{d}, \mathsf{s}, \omega\rangle \in \mathsf{V}(\sim) \ \& \\
& \forall p_s \in \mathsf{s} : \exists p_d \in \mathsf{d} : \|dist(p_s, p_d)\| / \|\mathsf{d}\| \leq (1+\omega) \ \& \\
& \forall p_d \in \mathsf{d} : \exists p_s \in \mathsf{s} : \|dist(p_s, p_d)\| / \|\mathsf{d}\| \leq (1+\omega)\}
\end{array} \tag{9}
$$

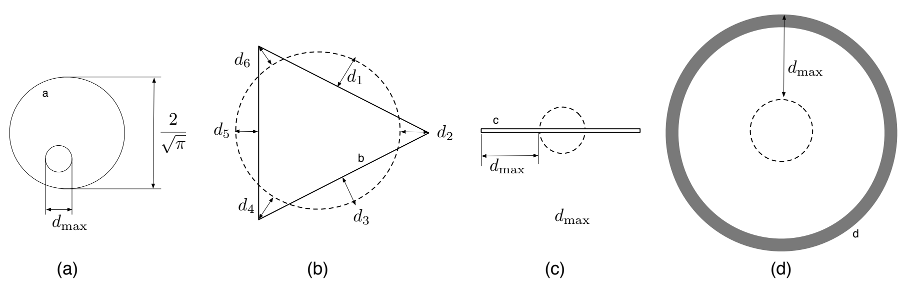

Figure 6: Examples and counter examples of sphere-like shaped, roughly-sphere-like shaped, and sphere-unlike shaped regions.

Let $\mathcal{D}$ be the regular closed subsets of $\Re^2$ and let $\mathsf{a}, \mathsf{b}, \mathsf{c}, \mathsf{d} \in \mathcal{D}$ be the regions depicted respectively in Figures 6(a)–(d). Assume that $\|\mathsf{a}\| = \|\mathsf{b}\| = \|\mathsf{c}\| = \|\mathsf{d}\| = 1$ and assume a fixed choice of the precisification parameter ω. Consider Figure 6(a). For the holed region a to count as sphere-like-shaped (respectively roughly-sphere-like-shaped) on the precisification ω, there must be a sphere s such that for every point of a there is some point of s such that the distance between them does not exceed the value at position ω in the bold (dashed) graph of Figure 7. Similarly vice versa. For example, for a to count as sphere-like-shaped (respectively roughly-sphere-like-shaped) on the precification 0.01, the diameter of the hole ($d_{\max}$) cannot be larger than $2 * 0.112$ units (respectively $2 * 1.14$ units). Clearly, the regions a and b will count as sphere-like-shaped (roughly-sphere-like-shaped) for a wider range of precifications than the regions c and d. Note also, that vague predicates such as *sphere-like-shaped* and *roughly sphere-like-shaped* on most interpretations do not distinguish a ball of wool from a solid 3-disc. That is, the rough overall shape of a whole on most interpretations will not distinguish between solid wholes and wholes of some (local) shape that are 'folded' into another (global) shape.

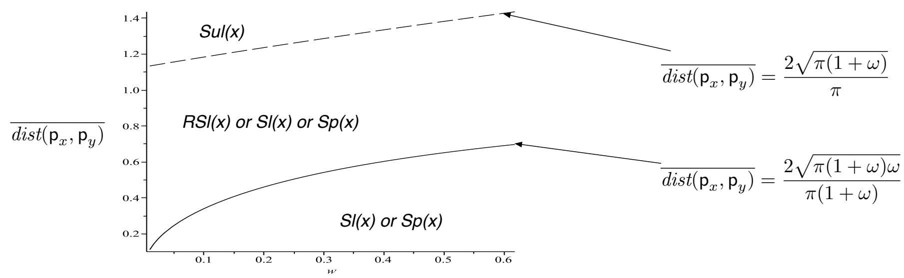

Figure 7: Interpretation of vague size predicates for regions of normalized sizes ($\|x\| = 1$) at various precisification points ($0.01 \leq \omega < -\frac{1}{2} + \frac{1}{2}\sqrt{5}$). $\overline{dist(\mathsf{p}_x, \mathsf{p}_y)}$ is the smallest distance such that for every point of x there is some point of b within this distance and vice versa.

It follows from the definitions D_{Sl}, D_{RSl}, and D_{Sul} that if x is a sphere then x is sphere-like-shaped (T65) and if x is sphere-like-shaped then x is roughly-sphere-like-shaped (T66). Moreover, the predicates *Sl*, *RSl*, and *Sul* are all vague (T67-T69).

$$T65 \quad Sp\ x \rightarrow Sl\ x$$
$$T66 \quad Sl\ x \rightarrow RSl\ x$$

$$T67 \quad (\exists x)\mathsf{I}(Sl\ x)$$
$$T68 \quad (\exists x)\mathsf{I}(RSl\ x)$$
$$T69 \quad (\exists x)\mathsf{I}(Sul\ x)$$

Corresponding to the vague distance predicates defined in Section 6 shape-restricted distance predicates are then defined in the obvious ways. For example:

$$D_{\overline{Cl}} \quad \overline{Cl}\ xy \equiv RSl\ x \wedge RSl\ y \wedge Cl\ xy$$
$$D_{\overline{N}} \quad \overline{N}\ xy \equiv RSl\ x \wedge RSl\ y \wedge N\ xy$$

8 Composition of vague distance predicates

An important aspect of the logic of relational predicates is their logical composition. The composition of binary predicates provides the basis for automated reasoning [17]. Automated reasoning of this kind can be used for data maintenance, data mining, and other tasks that are important to handle the large amounts of data that are so common in many modern sciences [50, 11].

The composition of distance predicates such as C, SCl, SN, MA, and FA in a region based geometry roughly corresponds to the addition of vectors in analytical geometry [28, 16]. Thus, in addition to size and shape, for the composition of distance predicates one has to take into account the relative arrangement of the related regions. There are various ways of qualitatively describing the relative arrangements of points (e.g., [27, 38]). Here a similar technique that is based on comparing distances between regions will be used.

The resulting formal theory is called VMG or vague region-based geometry.

8.1 Comparing distances

The distance between x and y is *smaller than or equal to the distance between x and z* if and only if: (i) x is connected to y or (ii) any minimal connecting sphere of x and y are smaller than any minimal connecting sphere of x and z ($D_{Dist_{\leq}}$). The predicates $Dist_{=}$ and $Dist_{<}$ are defined in the usual ways. Examples are depicted in Figure 8.

$$
\begin{array}{ll}
D_{Dist_{\leq}} & Dist_{\leq}\ xyz \equiv C\ xy \vee (\exists w_1)(\exists w_2)(MCS\ xw_1y \wedge MCS\ xw_2z \wedge w_1 \leq w_2) \\
D_{Dist_{=}} & Dist_{=}\ xyz \equiv Dist_{\leq}\ xyz \wedge Dist_{\leq}\ xzy \\
D_{Dist_{<}} & Dist_{<}\ xyz \equiv Dist_{\leq}\ xyz \wedge \neg Dist_{=}\ xyz
\end{array}
$$

In Ω-structures $Dist_{\leq}\ xyz$ holds if and only if the greatest lower bound of distances between points in x and points in y is less than or equal to the greatest lower bound of distances between points in x and points in z:

$$\mathsf{V}(Dist_{\leq}) = \{\langle \mathsf{d}_1, \mathsf{d}_2, \mathsf{d}_3, \omega\rangle \in \mathcal{D} \times \mathcal{D} \times \mathcal{D} \times \Omega \mid \mathsf{dist}(\mathsf{d}_1, \mathsf{d}_2) \leq \mathsf{dist}(\mathsf{d}_1, \mathsf{d}_3)\} \tag{10}$$

Unlike the distance predicates Cl, N, A, FA, the predicate $Dist_{\leq}$ is crisp (T23) and independent of the size and the shapes of the related regions. Obviously, vague versions of the predicates $Dist_{\leq}$, $Dist_{=}$, and $Dist_{<}$ can be defined analogously using the vague size predicate $\preceq$.

One can prove that predicates $Dist_{\leq}$, $Dist_{=}$, and $Dist_{<}$ have the expected logical properties: The distance of x to itself is less than or equal to the distance to any other region (T24); If the first argument is fixed, then $Dist_{\leq}$ is transitive (T29); For arbitrary triples of regions either $Dist_{\leq}\ xyz$ or $Dist_{\leq}\ xzy$ (T28); If the first argument is fixed, then $Dist_{=}$ is reflexive (T25), symmetric (T26) and transitive (T30); If the first argument is fixed, then $Dist_{<}$ is asymmetric (T27).

$$
\begin{array}{ll}
T23 & \mathsf{D}(Dist_{\leq}\ xyz) \\
T24 & Dist_{\leq}\ xxy \\
T25 & Dist_{=}\ xxx \\
T26 & Dist_{=}\ xyz \rightarrow Dist_{=}\ xzy \\
T27 & Dist_{<}\ xyz \rightarrow \neg Dist_{<}\ xzy \\
T28 & Dist_{\leq}\ xyz \vee Dist_{\leq}\ xzy \\
T29 & Dist_{\leq}\ xyz \wedge Dist_{\leq}\ xzw \rightarrow Dist_{\leq}\ xyw \\
T30 & Dist_{=}\ xyz \wedge Dist_{=}\ xzw \rightarrow Dist_{=}\ xyw
\end{array}
$$

8.2 Spatial arrangement of triples of regions

Using the predicate $Dist_{\leq}$ the spatial arrangement of triples of regions can be classified in a systematic way and fifteen jointly exhaustive and pairwise disjoint classes of spatial configurations can be identified.

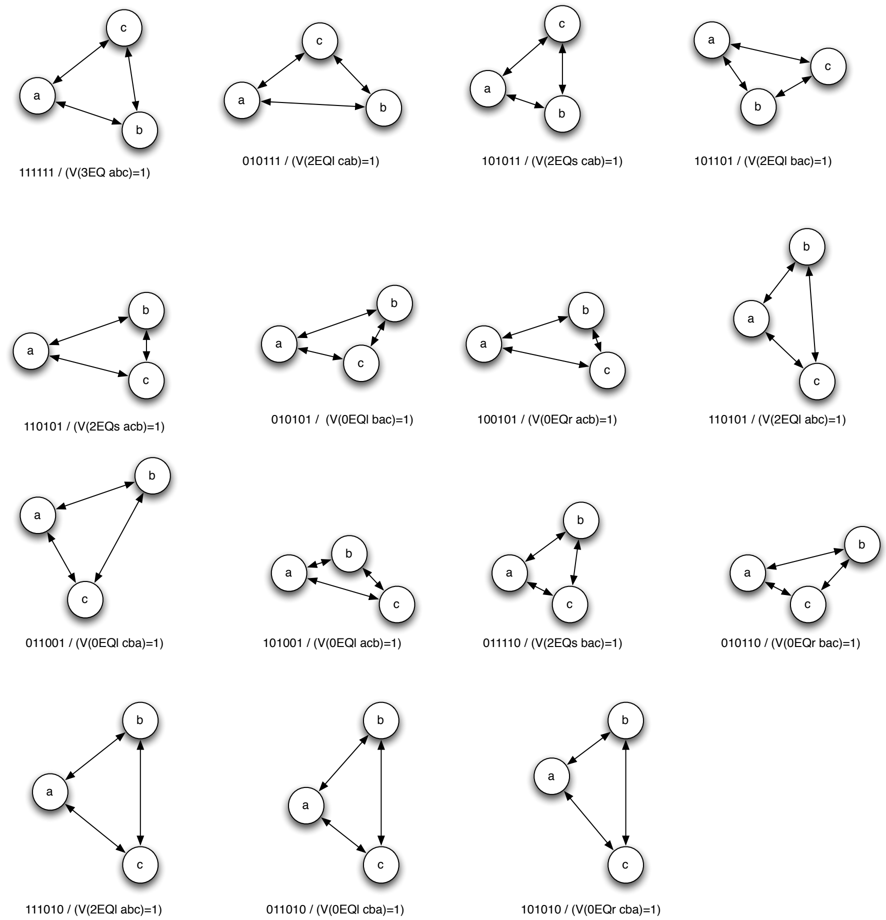

Figure 8: Fifteen realizable configuration of triples of discs in the plane with truth value pattern in the order $\mathsf{V}(Dist_{\leq}\ abc)$, $\mathsf{V}(Dist_{\leq}\ acb)$, $\mathsf{V}(Dist_{\leq}\ bac)$, $\mathsf{V}(Dist_{\leq}\ bca)$, $\mathsf{V}(Dist_{\leq}\ cab)$, $\mathsf{V}(Dist_{\leq}\ cba)$, and the corresponding classification in terms of *3EQ*, *2EQs*, *2EQl*, *0EQl*, and *0EQr*.

	$\mathsf{V}(Dist_{\leq}$ $abc) =$	$\mathsf{V}(Dist_{\leq}$ $acb) =$	$\mathsf{V}(Dist_{\leq}$ $bac) =$	$\mathsf{V}(Dist_{\leq}$ $bca) =$	$\mathsf{V}(Dist_{\leq}$ $cab) =$	$\mathsf{V}(Dist_{\leq}$ $cba) =$
1	1	1	1	1	1	1
2	0	1	1	1	1	1
3	1	0	1	1	1	1
4	0	0	1	1	1	1
5	1	1	0	1	1	1
...	...	...	...	...	...	...
37	1	1	0	1	1	0
...	...	...	...	...	...	...
39	1	0	0	1	1	0
...	...	...	...	...	...	...
64	0	0	0	0	0	0

Table 3: Characterizing spatial arrangement using $Dist_{\leq}$.

Let a, b, and c be three regions which can be of arbitrary shape and size. There are six permutations in which these three regions can occur exactly once as parameters of $Dist_{\leq}$. (See the header of Table 3). For every permutation, $\mathsf{V}(Dist_{\leq})$ is either 1 or 0. This results in $2^6 = 64$ combinatorially possible pattern of truth assignments. Some of of those pattern are depicted in the body of Table 3. Of those 64 combinatorially possible truth pattern only a few are realizable in the models of VMG. For example, the pattern of truth assignments in row (4) of Table 3 is not realizable. This is because, by theorem (T28) not both, $Dist_{\leq}\ abc$ and $Dist_{\leq}\ acb$ can be false. Similarly, $Dist_{\leq}\ bac$ and $Dist_{\leq}\ bca$ cannot be both false and neither can be $Dist_{\leq}\ cab$ and $Dist_{\leq}\ cba$. There are 27 of the 64 combinatorially possible assignments of truth pattern that satisfy Theorem (T28).

Consider the pattern of truth values in row (5) of Table 3. If $Dist_{\leq}\ abc$ and $Dist_{\leq}\ acb$ both are true, then $Dist_{=}\ abc$ is true (by $D_{Dist_{=}}$). Similarly, if $Dist_{\leq}\ cab$ and $Dist_{\leq}\ cba$ both are true, then $Dist_{=}\ cab$ is true. In VMG one can prove that if $Dist_{=}\ xyz$ and $Dist_{=}\ yzx$ are both are true, then $Dist_{=}\ zyx$ is true (T31).

$$T31 \quad Dist_{=}\ xyz \wedge Dist_{=}\ yzx \rightarrow Dist_{=}\ zxy$$

Therefore, if $Dist_{\leq}\ abc$, $Dist_{\leq}\ acb$, $Dist_{\leq}\ cab$ and $Dist_{\leq}\ cba$ are all true, then neither $Dist_{\leq}\ bac$ nor $Dist_{\leq}\ bca$ can be false. Thus, the pattern of truth assignments in row (5) of Table 3 is not realizable in models of VMG. Similarly the pattern of truth assignments in rows (2) and (3) are not realizable. In general, no row of Table 3 that has only a single '0' is realizable in models of VMG. Of the 27 assignments of truth pattern in Table 3 that satisfy Theorem (T28) only 20 satisfy Theorem (T31).

Consider the pattern truth valueS in rows (37) and (39) of Table 3. These pattern do not satisfy Theorem (T32).

$$T32 \quad Dist_{\leq}\ xyz \wedge Dist_{\leq}\ yzx \rightarrow Dist_{\leq}\ zyx$$

In the case of row (37) theorem (T32) requires that if $Dist_{=}\ abc$ and $Dist_{<}\ bca$ are both true, then $Dist_{\leq}\ cba$ cannot be false. Similarly for row (39). Of the 20 assignments of truth pattern that satisfy Theorems (T28 and T31) only 15 satisfy Theorem (T32). All these 15 pattern of truth values are realizable in models of VMG. As a set these 15 pattern describe the possible spatial arrangements of triples of regions in a jointly exhaustive and pairwise disjoint fashion. Examples for discs in the plane are depicted in Figure 8.

Each of the 15 realizable configurations for triples of regions falls into one of five coarser classes (D_{3EQ} - D_{0EQr}). Examples for discs in the plane are depicted in Figure 8.

$$
\begin{array}{ll}
D_{3EQ} & 3EQ\ xyz \equiv Dist_{=}\ xyz \wedge Dist_{=}\ yzx \\
D_{2EQs} & 2EQs\ xyz \equiv Dist_{=}\ xyz \wedge Dist_{<}\ yzx \\
D_{2EQl} & 2EQl\ xyz \equiv Dist_{=}\ xyz \wedge Dist_{<}\ yxz \\
D_{0EQl} & 0EQl\ xyz \equiv Dist_{<}\ xyz \wedge Dist_{<}\ yzx \wedge Dist_{<}\ zxy \\
D_{0EQr} & 0EQr\ xyz \equiv Dist_{<}\ xyz \wedge Dist_{<}\ yzx \wedge Dist_{<}\ zyx
\end{array}
$$

8.3 Composition tables

Let RC be a meta-variable such that $\mathsf{V}(RC\ xyz) = 1$ for exactly one of the 15 possible classes of configurations formed by triples of regions x, y, and z as depicted in Figure 8. Let SR_1 and SR_2 be meta-variables ranging over the size predicates $\sim$, $<$, and $>$. Let R and S range over predicates in $\Delta = \{C, SCl, SN, MA, FA\}$ (Figure 5) and let T be a disjunction of predicates in Δ. In a composition table for Δ the values of RC, SR_1, and SR_2 are fixed and all formulas such that

$$
\begin{array}{l}
\mathsf{V}(RC\ xyz \wedge SR_1\ xy \wedge SR_2\ yz \rightarrow (R\ xy \wedge S\ yz \rightarrow T\ xz)) = 1 \\
\mathsf{V}(RC\ xyz \wedge SR_1\ xy \wedge SR_2\ yz \wedge R\ xy \wedge S\ yz) \neq 0
\end{array}
\tag{11}
$$

are explicated for the various combinations of R and S.

Given that there are 15 possible values for RC and three possible values for SR_1 and SR_2, there is a potentially large number of composition tables. Many of them may be empty due to contradicting combinations of predicates excluded in Equation 11. It is beyond the scope of this paper to go through all of these tables. Here it will be sufficient to consider one class of examples:

Let the meta-variable RC in Equation (11) range over the predicates $3EQ$, $2EQs$, and $2EQl$. Under this assumption, the resulting composition tables are identical and independent of the choices of size predicates for SR_1 and SR_2. This is summarized in Table 4. Other composition tables can be derived using theorems such as:

$$
\begin{array}{llll}
T33 & Dist_{\leq}\ xyz \wedge FA\ xy \rightarrow FA\ xz & T37 & Dist_{\leq}\ zyx \wedge Cl\ xz \wedge x \leq y \rightarrow Cl\ yz \\
T34 & Dist_{\leq}\ xyz \wedge Cl\ xz \rightarrow Cl\ xy & T38 & Dist_{\leq}\ zyx \wedge x \leq y \wedge FA\ yz \rightarrow FA\ xz \\
T35 & Dist_{\leq}\ xyz \wedge N\ xz \rightarrow N\ xy & T39 & Dist_{\leq}\ zyx \wedge N\ xz \wedge x \leq y \rightarrow N\ yz \\
T36 & Dist_{\leq}\ xyz \wedge A\ xy \rightarrow A\ xz & T40 & Dist_{\leq}\ zyx \wedge x \leq y \wedge A\ yz \rightarrow A\ xz
\end{array}
$$

	$C\ yz$	$SCl\ yz$	$SN\ yz$	$MA\ yz$	$FA\ yz$
$C\ xy$	Cxz	–	–	–	–
$SCl\ xy$	–	$SCl\ xz$	–	–	–
$SN\ xy$	–	–	$SN\ xz$	–	–
$MA\ xy$	–	–	–	$MA\ xz$	–
$FA\ xy$	–	–	–	–	$FA\ xz$

Table 4: Composition table for the spatial arrangements such that $\mathsf{V}(\mathit{3EQ}\ xyz) = 1$, $\mathsf{V}(\mathit{2EQs}\ xyz) = 1$, and $\mathsf{V}(\mathit{2EQl}\ xyz) = 1$.

9 Conclusions

A formal theory of vague distance predicates was presented which combines a fragment of a crisp region-based geometry, with order of magnitude reasoning about size relations, and a supervaluation semantics of vague predicates. In this axiomatic theory context-independent logical and semantic properties of vague distance predicates were formalized.

Besides the study of context-independent logical and semantic properties of vague distance predicates in the object language of the formal theory, context- and domain-dependent aspects of such predicates were taken into account in the meta-language of the formal theory by choosing certain constraints and parameters that determine the canonical interpretations of the vague distance predicates. This integrated approach is an example of how to link a formal-ontological analysis to numerical interpretations that are used frequently in scientific discourses related to geography, biology, and medicine.

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