# Number Theory in Geometry

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*Myhill Lectures, October 18-19-20, 2016, at SUNY, Buffalo*

It is a great honor to deliver Myhill lectures. I thank Mohan Ramachandran, David Hemmer and other colleagues here for the invitation to deliver these lectures this year.

I had spent a month in spring of 1973 here to attend courses on Algebraic Geometry and Algebraic Groups given by Alexander Grothendieck. I have very fond memory of that stay and of my interaction with Grothendieck.

The topic of my talk is number theory in geometry. I will say a few words about history of mathematics to show how geometry has played an important role in the development of mathematics. I will also mention some contributions of Indian Mathematicians of antiquity since these are not widely known. Humans, and apparently some animals too, can and need to count. We count discrete objects, such as siblings, animals, friends, trees, . . . . Therefore, integers have existed in our consciousness for a very very long time.

Babylonians had discovered the place-value representation of integers over 3 millennia BCE; they used base 60[1], so they needed 60 distinct symbols to write the numbers. Historians of mathematics believe that Indians independently came up with place-value representation to base 10 during the Vedic period (c. 1500 BCE). It is not clear if Indus-valley civilization had numerals in their writing; some experts have opined that they indeed had, and that they used 8 as the base. In any case, base 10 is much simpler than base 60 – imagine a child memorizing the multiplication table up to $59\times 59$. Using decimal notation, elementary arithmetical operations like addition and multiplication became quite easy. Division led to rational numbers–the ratio of two integers. Subtraction was problematic until negative numbers (and zero) were introduced. Paninin's (c. 5th century BCE) Sanskrit grammar uses the word "shunya" which means "void" or "nothing"; this word may have inspired the numeral "0". Via its Arabic translation "sifr" or "sifar" we got the word "zero". The Chinese and Indians arrived at the concept of negative numbers very early on, but in Europe until the mid 19th century negative numbers were looked-down with suspicion[2]. Complex numbers were introduced by the Italian mathematician G. Cardono around 1545 in his quest for solutions of

[1]In measurement of time and angles multiples and submultiples of 60 appear. This testifies to the wide influence of the Babylonians on astronomy.

[2]See David Mumford's article "*What's so baffling about negative numbers?*"

equations of degree 3. But it was only with the vivid geometric interpretation of complex numbers and their multiplication law in terms of points of a plane by Gauss and Argand that such numbers became widely accepted in 19th century, and later found use across numerous areas of science such as electrical engineering, signal processing, and quantum mechanics.

With the introduction of negative numbers, Chinese could come up with solutions of systems of linear equations in several unknowns. The classics of Chinese mathematics "Nine Chapters in Mathematical Art" which began in the first millennium BCE, with new commentaries added by Lieu Hui in 263 CE, contains the method of Gaussian elimination. Indians were interested in solutions of quadratic equations in one variable. The standard solution by completing squares was given by Sridhar in the 9th century. The Sanskrit name for "algebra" is "*beeja-ganit*" that literally means the mathematics of roots (or solutions) of polynomial equations, so it is quite an appropriate name. The word "algebra" comes from the title "*Al-jabr w'al muqabala*" of a book of Al-Khwarizimi (a Persian mathematician born in the 8th century) which presented solutions of linear and quadratic equations. Al-Khwarizimi was familiar with Brahmagupta's book "Brahma-sphuta-siddhanta" on the topic written two centuries earlier. The word "*algorithm*", used so frequently these days, is derived from the title "*Algoritmi de numero Indorum*" of the Latin translation of his book " *Calculation with Hindu numerals*".

Indians were also greatly interested in Diophantine problems early on–that is, finding integral solutions of polynomial equations with integral coefficients in two or more unknowns. For example, the "Chakravala" method to solve Pell's equation $x^2 - ny^2 = 1$, where $n$ is a positive integer which is not a square, was discovered by Brahmagupta in the 7th century. He also knew that two solutions can be "multiplied" to obtain a third solution[3]. The solutions of a given Pell's equation thus form a group; however the concept of a group was introduced only in the 18th century. Bhaskara II in 1150 extended the Chakravala method and obtained a general solution of Pell's equation. For example, he found the smallest solution for $n = 61$ to be $x = 1766319049$ and $y = 226153980$, quite a feat! Apparently a reason for interest in Pell's equation was to find rational approximations of $\sqrt{n}$. If $x = a$ and $y = b$ is a solution of the equation, then $a/b$ is a good approximation of $\sqrt{n}$ for large $b$.

It is amusing to note that in 1657, Fermat posed Pell's equation as a challenge to the mathematicians of Europe and England. A method to find solutions of the equation in terms of the theory of continued fractions was found by the Swiss

[3]Given two solutions $x = a_1$, $y = b_1$ and $x = a_2$, $y = b_2$ of $x^2 - ny^2 = 1$, then as $(a_1 + \sqrt{n}b_1)(a_2 + \sqrt{n}b_2) = (a_1a_2 + nb_1b_2) + \sqrt{n}(a_1b_2 + a_2b_1)$, $x = a_1a_2 + nb_1b_2$, $y = a_1b_2 + a_2b_1$ is again a solution. This is the third solution provided by Brahmagupta.

mathematician Euler and presented in a polished form by Lagrange in 1766. Euler was responsible for mistakenly naming the equation after Pell, one millennium after Brahmagupta considered these equations

Rituals, geometry, and astronomy were prime motivators for the development of early mathematics. Sulvasutra, from the late Vedic period (8th century BCE), developed plane geometry in detail to construct altars for the fire rituals (yajnas). Sulvasutra contains the Pythagoras theorem (and uses its converse to assert that if $c^2 = a^2 + b^2$, then the triangle is right-angled). It follows that a square of side 1 has diagonal of length $\sqrt{2}$, which is not an integer, in fact not even a rational number, as unique factorization of integers implies. Thus arose nonrational numbers from geometry. Note that for any positive integer $n$, $\sqrt{n}$ is a solution of the equation $x^2 - n = 0$ with integral coefficients. More numbers arise as solutions of such equations. All these numbers are called algebraic numbers.

Due to their symmetry and their occurrence in nature, circles and spheres held special fascination. The circumference, the area, and the volume of circle, sphere, and ball were known in various cultures. Archimedes was very proud of his computation of the area of the surface of a sphere and volume of a ball. Such computations later gave rise to integration (or integral calculus). These computations involve a new number $\pi$. It was known to Aryabhata in the 6th century (and most certainly to Nilakantha who in the 15th century wrote a commentary on Aryabhata's work) that $\pi$ is not a rational number. Only much later in the 19th century it was proved by Lindemann that $\pi$ is a transcendental number (i.e., not an algebraic number). This settled the problem, already considered in Sulvasutra, and by early Greeks, whether a circle can be squared. The answer is "no". From early on, there were attempts to find good rational approximations to the value of $\pi$ and in the 14th century Madhava found the value correct to 11 decimal places.

Mathematics is the only science which rigorously treats actual and potential infinity. Since the time of Georg Cantor (in the 19th century) we know that there are different kinds of infinities. The smallest one is called the countable infinity. It is the cardinality of the set of integers. As there are only countably many equations with integral coefficients, there are only countably many algebraic numbers. On the other hand, it was shown by Cantor that the cardinality of the set of real numbers is a larger infinity. So "most" real numbers are transcendental. However, given a real number, it is usually very hard to decide whether it is transcendental.

Early interest in astronomy and spherical geometry gave rise to trigonometric functions: sine, cosine, tan, arctan etc. The rate of change and the second-order differences of the trigonometric functions and the notion of instant velocity of planets (as opposed to the mean velocity) were studied by Indian mathematicians in the 1st millennium CE. Inspired by these, the notion of derivative as a limit of rate of change was introduced by Madhava's school. This school flourished in southern Indian state of Kerala for about 200 years beginning in the 14th century and it came up with the Taylor expansion of sine, cosine, and arctan. In particular, they discovered the series $(1 - \frac{1}{3} + \frac{1}{5} - ...)$ for $\pi/4$ which is called the Gregory series these days. All this happened almost three hundred years before Newton and Leibniz. The mathematicians of Madhava's school were aware that the Gregory series converges very slowly, so they, and much later Ramanujan, found several rapidly converging series.

Gauss used geometry in his investigations of binary quadratic forms–that is, homogeneous quadratic polynomials in two variables. Dirichlet and Minkowski used geometry to study the structure of number fields and the ring of integers and group of units in such fields. Thus, a new branch of mathematics known as "Geometry of Numbers" was born. Many spectacular advances in number theory during the last 30 years, such as Wiles' solution of Fermat's Last Theorem, would have been impossible without advances in at the interface of geometry and algebra due to such great mathematicians as Grothendieck and Langlands. Manjul Bhargava's work for which he has been awarded a Fields' Medal two years ago, uses geometry extensively to obtain results in number theory.

Though mathematics is largely a creation of human mind, it has been used to accurately model natural phenomena and to formulate laws governing them. This led the physicist Eugene Wigner to talk about "unreasonable effectiveness" of mathematics in physical sciences. Now math has found many practical applications in our day-to-day life. From security and privacy of financial transactions using prime factorization of integers, encryption, weather prediction and analysis of the stock market, in digitalization of music and movies on CDs, in medical imaging and in many more things. It is also being used in biology and genetics. But nature may have been using very sophisticated mathematics much longer than us. My daughter, who is a neuroscientist, has explained to me that to keep the visual image of the world fixed during body and head motion, the part of the brain that controls eye position performs calculus, integrating motion and counter-rotating the eyes. Also, insects and mammals can keep a continuous tally of their geographical position as they move around; this also involves integrating motion into an updated position estimate so that on their way back they are able to take the shortest path. Just like in CDs, where scratches do not destroy the ability of the music to be read out, the brain also appears to use sophisticated codes related to number theory and algebra, to correct errors during these updates.

Due to the increasing use of mathematics in different areas, the need to learn it well is growing. So it is not surprising that the mathematics departments offer the largest number of service courses and the largest number of undergraduates are taught by mathematicians.

Now after these historical remarks in which I described how geometry played an important role in the development of mathematics in general and number theory in particular, I will describe briefly two interesting geometric problems in which number theory has been used in my recent work.

Mathematicians like to classify things: to make a complete list in a suitable form or find ways to distinguish objects in terms of numbers and invariants attached to them. Let us see a concrete example of difficulties in giving a usable classification. Let us say that we want to classify people. We can try to classify by the following easily determined "invariants": gender, name, nationality, age, height, profession, .... But none of these by themselves are adequate: There are too many Prasad's, in fact many exactly with the same name as mine, too many 41 year olds... We may try to classify people by a combination of the above "invariants", but as is obvious, unless we use a long list of invariants we will not be able to determine a person uniquely from the list of invariants under consideration. Of course, one hopes that a finer invariant like finger-print, or DNA-map does identify a person.

In topology, which studies properties of geometric objects (called "spaces") which do not change under bending and stretching, an invariant defined by Euler (in the 18th century), and refined by Poincaré, is the *Euler-Poincaré characteristic* (to be abbreviated Euler characteristic in the following). For a simplicial complex, it is the alternating sum of the number of $n$-dimensional simplices. It is clear that for simplicial complexes, the Euler characteristic is an easily computable integer. For other geometric objects, Euler characteristic may not be so easily computable. But still among all the known topological invariants, the Euler characteristic appears to be the simplest to compute. Algebraic geometry is an area of mathematics which studies geometric objects described by polynomial equations, for example circles, ellipses, parabola, hyperbolas encountered in a high school geometry course, and their higher dimensional generalizations called *algebraic varieties*. By algebraic variety I will mean its points whose coordinates are complex numbers, not just points whose coordinates are real numbers. In algebraic geometry we have a good classification of algebraic curves (that is, 1-dimensional objects). For example, the Euler characteristic completely classifies the topological type of an algebraic curve (an algebraic curve is a Riemann surface!). It is a remarkable fact, for which Gerd Faltings got a Fields Medal in 1986, that the properties of the set of points of an algebraic curve whose coordinates are rational numbers are controlled by its Euler characteristic.

The curves with Euler characteristic zero are called elliptic curves. They are topologically a torus, and are defined by an equation of the form $y^2 = x^3 - ax - b$.

Elliptic curves carry an additional structure, that of a commutative group. In Manjul Bhargava's Fields Medal lecture, he focused on his recent work concerning number-theoretic problems about elliptic curves, done in collaboration with several others, including my former colleague – and U of M graduate – Chris Skinner.

Algebraic varieties of next dimension, that is of dimension 2, are called *algebraic surfaces*. The great Italian mathematicians of the 19th and early 20th century gave a short list of invariants that provided a crude but useful classification of algebraic surfaces. The widest collection in their classification are called surfaces "of general type", and for those surfaces it is known that the Euler characteristic has to be at least 3. An example of a surface which achieves this minimum is the so-called complex projective plane, an analogue of the real projective plane which was discovered by Renaissance artists who developed the first principles of perspective drawing. Only in 1979 was another such example found, by the great algebraic geometer David Mumford. Mumford called his example a "fake projective plane" (*fpp*) because it shares certain additional topological invariants with the usual complex projective plane (it has the same Betti numbers, namely, $1, 0, 1, 0, 1$).

It remained a challenging problem for almost three decades to determine all algebraic surfaces with Euler characteristic equal to 3. It was proved with much effort that any such surface is covered by the "complex unit 2-ball" $\mathbf{B}^2_{\mathbf{C}} = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 < 1\}$, the analogue for complex numbers of the familiar 2-dimensional open unit disc in the real plane, which in turn is just the "complex unit 1-ball" in the complex plane of Argand-Gauss. What does one mean by such a covering? The complex 2-ball has evident symmetries in the form of rotations around the origin, but from the perspective of hyperbolic geometry it has an abundant supply of additional symmetries which form a continuous group denoted $\mathrm{PU}(2, 1)$. Number theoretic structures on this group identify distinguished classes of collections of symmetries, called *arithmetic lattices*. There are infinitely many such lattices. When we choose such a lattice of symmetries and create a new space by identifying points which can be moved from one to the other by applying a symmetry in the chosen lattice, this new space is called the "quotient" of the complex 2-ball by the chosen lattice of symmetries and the complex 2-ball is a "cover" of this quotient space. It was proved independently by Bruno Klingler and Sai-Kee Yeung that any fake projective plane (i.e., a smooth complex algebraic surface with Betti numbers $1, 0, 1, 0, 1$) arises as the quotient of complex 2-ball $\mathbf{B}^2_{\mathbf{C}}$ by a torsion-free arithmetic lattice. Sai-Kee Yeung later proved that in fact any smooth complex algebraic surface with Euler characteristic 3 is such a quotient. But this begs the question: how many such lattices actually give rise to a surface with Euler characteristic 3? If one picks a lattice of symmetries of the 2-ball at random then it generally does not yield a surface with this Euler characteristic. Some years ago in a seminar talk at the University of Michigan, the speaker Sai-Kee Yeung presented his result, based on geometry alone (by consideration of Hilbert schemes), that the number of such possible lattices is at most $10^{5.5\,\text{million}}$. This is quite a large number, considering that the universe is believed to have only around $10^{80}$ electrons. This seminar talk got me interested in the question.

A major difficulty in determining all the smooth complex projective algebraic surfaces with Euler characteristic 3 is that we do not know how to concretely visualize and triangulate every surface which is obtained as a quotient of the complex 2-ball by an arithmetic lattice $\Gamma$. These surfaces do not imbed in three dimensions! But my volume formula for arithmetic quotients allows one to compute the Euler characteristic without an explicit knowledge of the surface and its triangulation. The formula involves several number-theoretic and group-theoretic invariants. The Euler characteristic is an integral multiple of

$$\frac{D_\ell^{5/2}\zeta_k(2)\mathrm{L}_{\ell|k}(3)}{3^x(16\pi^5)^d D_k h_{\ell,3}}\prod e'(P_v),$$

where $k$ is a totally real number field of degree $d$, $\ell$ is a totally complex quadratic extension of $k$, $D_k$ and $D_\ell$ are the absolute discriminants of $k$ and $\ell$ respectively, $h_{\ell,3}$ is the 3-component of the class number of $\ell$, $x$ is a nonnegative integer, the group $G$ is a $k$-form of $\mathrm{SU}(2,1)$, the $P_v$'s are parahoric subgroups of $G(k_v)$ and $e'(P_v)$ are positive integers determined by using the Bruhat-Tits theory. The data $k$, $\ell$, $G$ and the $P_v$ are uniquely determined by $\Gamma$, and conversely they determine the "class" of $\Gamma$; each class consists of fundamental groups of finitely many surfaces under consideration.

By using deep results in number theory and Bruhat-Tits theory, Sai-Kee Yeung and I showed that there are 28 classes of *fpp*'s, and one more class with $(k,\ell) = \mathscr{C}_{11} = (\mathbf{Q}(\sqrt{3}), \mathbf{Q}(e^{\pi i/6}))$, which may give a surface with Euler characteristic equal to 3. We exhibited at least one *fpp* in each of the 28 classes and outlined a procedure to determine each class completely by computation. The desired computations were carried out jointly by Donald Cartwright (in Australia) and Tim Steger (in Italy), at times using over 50 computers. They found that the 28 classes altogether contain 100 *fpp*'s, and as they are now explicitly given, we know many geometric properties of these surfaces. The class attached to the pair $\mathscr{C}_{11}$ mentioned earlier gave a completely unexpected smooth projective algebraic surface with Euler characteristic equal to 3, and the first Betti number (which is 2) distinguishes it from all the others. As this is a unique surface of its kind, it is attracting considerable attention. An interesting consequence of our work is that the complex projective plane is determined by the topological invariant known as integral homology. Now we also know that for every positive integral multiple $3n$ of 3, there is a smooth projective complex algebraic surface with Euler characteristic $3n$. Using inputs from number theory, we have also found fake $\mathbf{P}^4_{\mathbf{C}}$ and fake Grassmannian $\mathrm{Gr}_{2,3}$.

The geometric problem which I want to describe next, and in whose solution arithmetic has played a crucial role, is the problem formulated by Marc Kac in the following interesting way: "Can one hear the shape of a drum?" To explain the geometric meaning of the problem, let us assume that we are given a nice geometric object, say a compact Riemannian manifold–Einstein worked with these in his theory of general relativity which is now being used in GPS technology. The surface of an idealized drum is such a geometric object. There is a partial differential equation called the wave equation which encodes the modes of vibrations of the manifold. The partial differential equation is given in terms of an operator called the Laplace-Beltrami operator which is determined by the geometry (in fact, just the metric) of the Riemannian manifold and the frequencies (or the wave-lengths) of the pure-tones are determined by the eigenvalues of the Laplace-Beltrami operator.

So a precise mathematical formulation of Mark Kac's question is whether the shape of a drum is determined by the eigenvalues of the Laplace-Beltrami operator. Now the same Laplace-Beltrami operator comes-up in many situations: in diffusion equation for heat and fluid flow, studies of waves, and in quantum mechanics. Therefore, it is interesting to ask Kac's question in a more general setting, namely whether a compact Riemannian manifold can be recovered by knowing its spectrum, that is by the set of eigenvalues of the Laplace-Beltrami operator together with their multiplicities. Two manifolds are called *isospectral* if they have the same spectra. It was shown by the venerable mathematician Hermann Weyl that two manifolds with same spectrum have equal dimension, equal volume, and equal scalar curvature. However, later examples provided by John Milnor (in dimension 16) and Marie-France Vigneras (in dimension 2, of compact Riemann surfaces) showed that the answer is in general "no". A Japanese mathematician, T. Sunada, gave a general construction of isospectral manifolds $M_1$ and $M_2$. His construction had the special property that the manifolds $M_1$ and $M_2$ he produced could be obtained from a single manifold $M$ by forming quotients by two finite groups of isometries. Two manifolds obtained this way are called *commensurable*. So a more reasonable formulation of Kac's question is whether commensurability is a consequence of isospectrality, at least for some "nice" class of manifolds.

Andrei Rapinchuk and I have investigated this question for the most symmetric of Riemannian manifolds, so-called Riemanniam locally symmetric spaces. We impose the condition that a topological invariant known as the fundamental group is "arithmetic" (this condition holds automatically for spaces of rank $> 1$ by Margulis' arithmeticity theorem!). This condition opens the door to applying number-theoretic tools and obtain results which are out of reach by purely geometric methods.

In fact, we focused on a more general question than that posed by Kac: it is known that when spectra of two compact locally symmetric spaces are equal then the sets of lengths of closed geodesics must coincide. We say that two such manifolds are *isolength* when they share the same sets of lengths of closed geodesics. In the special case of the locally symmetric spaces studied in my work with Rapinchuk, the closed geodesics correspond to "semi-simple" elements in the fundamental group, and given such an element $\gamma$ there is an explicit formula for the length $\ell(\gamma)$ of the corresponding geodesic:

$$\ell(\gamma)^2 = \sum_{\alpha \in \Phi} (\log |\alpha(\gamma)|)^2$$

where $\Phi$ is a finite set encoding the nature of the continuous symmetries of the space and the numbers $\alpha(\gamma)$ are algebraic (i.e., not transcendental) in view of the arithmeticity condition imposed on the fundamental group. So the logarithms of the $|\alpha(\gamma)|$'s must be transcendental, by the theorem of Gelfond and Schneider on Hilbert's 7th problem, which Hilbert thought would be one of the most difficult in his famous list of 23 problems posed at the 1900 International Congress of Mathematicians (but which was among the first to be solved!).

Using transcendental number theory (the solution of Hilbert seventh problem by Gel'fond and Schneider in the rank 1 case and Schanuel's[4] conjecture in higher rank), Rapinchuk and I showed that the isolength hypothesis on a pair of locally symmetric spaces implies a new algebraic property relating their fundamental groups, a property we call "weak commensurability" (it is a-priori weaker than the geometric condition of commensurability discussed earlier). By using deep results in number theory, we showed that this weak commensurability property has very strong consequences for isolength locally symmetric spaces. For example, in the case of hyperbolic spaces whose dimension is even or one less than a multiple of 4 we showed that weak commensurability implies commensurability! In particular, for such spaces the modified form of Kac's question in terms of commensurability has a positive answer. This was very surprising, and prior to our work using input from number theory there were no techniques to prove such results beyond dimension 3.

It is crucial that in our work we considered lengths of closed geodesics rather than eigenvalues of the Laplace-Beltrami operator, since the former is given by a

[4]Stephen Schanuel was a professor here at SUNY, Buffalo.

concrete formula (shown above) with direct connection to number theory whereas eigenvalues of the Laplace-Beltrami operator are notoriously difficult to compute, even for 2-dimensional spaces. So it was very fortunate that we could get far by focusing on the isolength property. In our work, there was even pay-back to number theory in the form of new local-global principles for algebraic structures of interest to number theorists.

Thank you!