Notes on Differential Equations

University at Buffalo Edition 2

by Jiří Lebl

with additional exercises, examples, and notes on using Python
by James Javor, John Ringland, Brian Hassard, and James Faran

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See https://www.jirka.org/diffyqs/ for more information (including contact information).

The LaTeX source for the source work is available for possible modification and customization at github: https://github.com/jirilebl/diffyqs
The LaTeX source for this derivative work is available for possible modification and customization at github: https://github.com/UBmath/306

Cover photo courtesy of University at Buffalo | Chad Cooper.

Contents

Introduction

0.1 Notes about this book

Note: A section for the instructor.

0.1.1 History of the source work

This book originated from my class notes for Math 286 at University of Illinois at Urbana-Champaign (`https://www.math.uiuc.edu`) in Fall 2008 and Spring 2009. It is a first course on differential equations for engineers. Using this book, I also taught Math 285 at UIUC, Math 20D at University of California, San Diego (`https://www.math.ucsd.edu/`), and Math 4233 at Oklahoma State University (`https://math.okstate.edu`). Normally these courses are taught with Edwards and Penney, *Differential Equations and Boundary Value Problems: Computing and Modeling* [EP], or Boyce and DiPrima's *Elementary Differential Equations and Boundary Value Problems* [BD], and this book aims to be more or less a drop-in replacement. Other books I used as sources of information and inspiration are E.L. Ince's classic (and inexpensive) *Ordinary Differential Equations* [I], Stanley Farlow's *Differential Equations and Their Applications* [F], now available from Dover, Berg and McGregor's *Elementary Partial Differential Equations* [BM], and William Trench's free book *Elementary Differential Equations with Boundary Value Problems* [T]. See the Further Reading chapter at the end of the book.

The source work's website `https://www.jirka.org/diffyqs/` contains additional resources.The LaTeX source of source work is also available for possible modification and customization at github (`https://github.com/jirilebl/diffyqs`).

0.1.2 Acknowledgments by author of source work

Firstly, I would like to acknowledge Rick Laugesen. I used his handwritten class notes the first time I taught Math 286. My organization of this book through chapter 5, and the choice of material covered, is heavily influenced by his notes. Many examples and computations are taken from his notes. I am also heavily indebted to Rick for all the advice he has given me, not just on teaching Math 286. For spotting errors and other suggestions, I would also like to acknowledge (in no particular order): John P. D'Angelo, Sean Raleigh, Jessica Robinson, Michael Angelini, Leonardo Gomes, Jeff Winegar, Ian

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0.1.3 Acknowledgments by authors of the derivative work

The authors thank Asela Abeya for assistance in preparing the additional exercises, and Joseph Hundley for managing the Amazon KDP edition.

0.1.4 Contents of this derivative work

This book contains the Introduction and Chapters 1 (except 1.9 on PDE), 2, 3, 6 (except 6.5 on PDE), 7, and 8 of the source work. Chapter 4 and 5 of the source work (Fourier series and PDEs, More on eigenvalue problems), and its Appendix on Linear Algebra, are not included.

Additions are some some extra exercises (inserted starting at numbers 51 and 151) and examples, and brief instructions on using Python for some graphical, numerical, and symbolic tasks. There is an associated Python module *resources306.py* available at https://raw.githubusercontent.com/UBmath/306/master/resources306.py.

The organization of this book to some degree requires chapters be done in order. The dependence of the material covered is roughly:

$$
\begin{tikzcd}
 & \text{Introduction} \arrow[d] & \\
 & \text{Chapter 1} \arrow[d] & \\
 & \text{Chapter 2} \arrow[dl] \arrow[dr] \arrow[dddr] & \\
\text{Chapter 3} \arrow[d] & & \text{Chapter 6} \\
\text{Chapter 8} & & \\
 & & \text{Chapter 7}
\end{tikzcd}
$$

0.2 Introduction to differential equations

Note: more than 1 lecture, §1.1 in [EP], chapter 1 in [BD]

0.2.1 Differential equations

The laws of physics are generally written down as differential equations. Therefore, all of science and engineering use differential equations to some degree. Understanding differential equations is essential to understanding almost anything you will study in your science and engineering classes. You can think of mathematics as the language of science, and differential equations are one of the most important parts of this language as far as science and engineering are concerned. As an analogy, suppose all your classes from now on were given in Swahili. It would be important to first learn Swahili, or you would have a very tough time getting a good grade in your classes.

You saw many differential equations already without perhaps knowing about it. And you even solved simple differential equations when you took calculus. Let us see an example you may not have seen:

$$\frac{dx}{dt} + x = 2\cos t. \tag{1}$$

Here x is the *dependent variable* and t is the *independent variable*. Equation (1) is a basic example of a *differential equation*. It is an example of a *first order differential equation*, since it involves only the first derivative of the dependent variable. This equation arises from Newton's law of cooling where the ambient temperature oscillates with time.

0.2.2 Solutions of differential equations

Solving the differential equation means finding x in terms of t. That is, we want to find a function of t, which we call x, such that when we plug x, t, and $\frac{dx}{dt}$ into (1), the equation holds; that is, the left hand side equals the right hand side. It is the same idea as it would be for a normal (algebraic) equation of just x and t. We claim that

$$x = x(t) = \cos t + \sin t$$

is a *solution*. How do we check? We simply plug x into equation (1)! First we need to compute $\frac{dx}{dt}$. We find that $\frac{dx}{dt} = -\sin t + \cos t$. Now let us compute the left-hand side of (1).

$$\frac{dx}{dt} + x = \underbrace{(-\sin t + \cos t)}_{\frac{dx}{dt}} + \underbrace{(\cos t + \sin t)}_{x} = 2\cos t.$$

Yay! We got precisely the right-hand side. But there is more! We claim $x = \cos t + \sin t + e^{-t}$ is also a solution. Let us try,

$$\frac{dx}{dt} = -\sin t + \cos t - e^{-t}.$$

We plug into the left-hand side of (1)

$$\frac{dx}{dt} + x = \underbrace{(-\sin t + \cos t - e^{-t})}_{\frac{dx}{dt}} + \underbrace{(\cos t + \sin t + e^{-t})}_{x} = 2\cos t.$$

And it works yet again!

So there can be many different solutions. For this equation all solutions can be written in the form

$$x = \cos t + \sin t + Ce^{-t},$$

for some constant C. Different constants C will give different solutions, so there are really infinitely many possible solutions. See Figure 1 for the graph of a few of these solutions. We will see how we find these solutions a few lectures from now.

Solving differential equations can be quite hard. There is no general method that solves every differential equation. We will generally focus on how to get exact formulas for solutions of certain differential equations, but we will also spend a little bit of time on getting approximate solutions. And we will spend some time on understanding the equations without solving them.

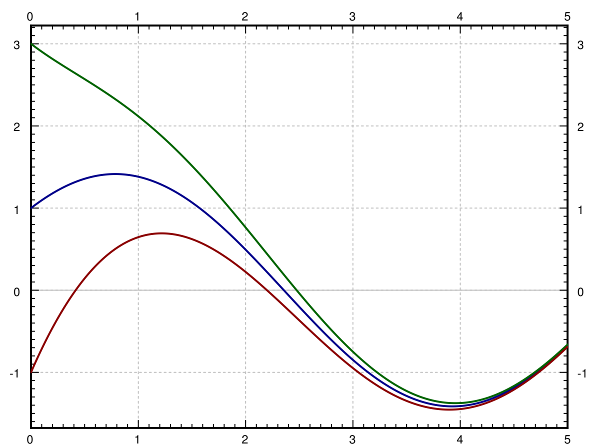


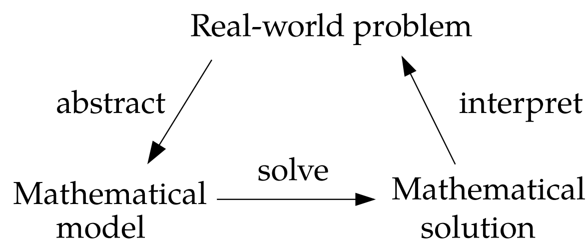
Figure 1: Few solutions of $\frac{dx}{dt} + x = 2\cos t$.

Most of this book is dedicated to *ordinary differential equations* or ODEs, that is, equations with only one independent variable, where derivatives are only with respect to this one variable. If there are several independent variables, we get *partial differential equations* or PDEs.

Even for ODEs, which are very well understood, it is not a simple question of turning a crank to get answers. When you can find exact solutions, they are usually preferable to approximate solutions. It is important to understand how such solutions are found. Although in real applications you will leave much of the actual calculations to computers, you need to understand what they are doing. It is often necessary to simplify or transform your equations into something that a computer can understand and solve. You may even need to make certain assumptions and changes in your model to achieve this.

To be a successful engineer or scientist, you will be required to solve problems in your job that you never saw before. It is important to learn problem solving techniques, so that you may apply those techniques to new problems. A common mistake is to expect to learn some prescription for solving all the problems you will encounter in your later career. This course is no exception.

0.2.3 Differential equations in practice



So how do we use differential equations in science and engineering? First, we have some *real-world problem* we wish to understand. We make some simplifying assumptions and create a *mathematical model*. That is, we translate the real-world situation into a set of differential equations. Then we apply mathematics to get some sort of a *mathematical solution*. There is still something left to do. We have to interpret the results. We have to figure out what the mathematical solution says about the real-world problem we started with.

Learning how to formulate the mathematical model and how to interpret the results is what your physics and engineering classes do. In this course, we will focus mostly on the mathematical analysis. Sometimes we will work with simple real-world examples so that we have some intuition and motivation about what we are doing.

Let us look at an example of this process. One of the most basic differential equations is the standard *exponential growth model*. Let P denote the population of some bacteria on a Petri dish. We assume that there is enough food and enough space. Then the rate of growth of bacteria is proportional to the population—a large population grows quicker. Let t denote time (say in seconds) and P the population. Our model is

$$\frac{dP}{dt} = kP,$$

for some positive constant $k > 0$.

Example 0.2.1: Suppose there are 100 bacteria at time 0 and 200 bacteria 10 seconds later. How many bacteria will there be 1 minute from time 0 (in 60 seconds)?

First we need to solve the equation. We claim that a solution is given by

$$P(t) = Ce^{kt},$$

where C is a constant. Let us try:

$$\frac{dP}{dt} = Cke^{kt} = kP.$$

And it really is a solution.

OK, now what? We do not know C, and we do not know k. But we know something. We know $P(0) = 100$, and we know $P(10) = 200$. Let us plug these conditions in and see what happens.

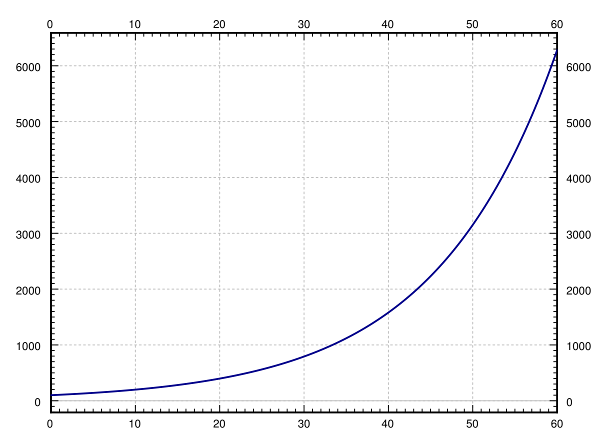


Figure 2: Bacteria growth in the first 60 seconds.

$$100 = P(0) = Ce^{k0} = C,$$
$$200 = P(10) = 100\, e^{k10}.$$

Therefore, $2 = e^{10k}$ or $\frac{\ln 2}{10} = k \approx 0.069$. So

$$P(t) = 100\, e^{(\ln 2)t/10} \approx 100\, e^{0.069t}.$$

At one minute, $t = 60$, the population is $P(60) = 6400$. See Figure 2 on the previous page.

Let us talk about the interpretation of the results. Does our solution mean that there must be exactly 6400 bacteria on the plate at 60s? No! We made assumptions that might not be true exactly, just approximately. If our assumptions are reasonable, then there will be approximately 6400 bacteria. Also, in real life P is a discrete quantity, not a real number. However, our model has no problem saying that for example at 61 seconds, $P(61) \approx 6859.35$.

Normally, the k in $P' = kP$ is known, and we want to solve the equation for different *initial conditions*. What does that mean? Take $k = 1$ for simplicity. Suppose we want to solve the equation $\frac{dP}{dt} = P$ subject to $P(0) = 1000$ (the initial condition). Then the solution turns out to be (exercise)

$$P(t) = 1000\, e^{t}.$$

We call $P(t) = Ce^{t}$ *the general solution*, as every solution of the equation can be written in this form for some constant C. We need an initial condition to find out what C is, in order to find the *particular solution* we are looking for. Generally, when we say "particular solution," we just mean some solution.

0.2.4 Four fundamental equations

A few equations appear often and it is useful to just memorize what their solutions are. Let us call them the four fundamental equations. Their solutions are reasonably easy to guess by recalling properties of exponentials, sines, and cosines. They are also simple to check, which is something that you should always do. No need to wonder if you remembered the solution correctly.

First such equation is

$$\frac{dy}{dx} = ky,$$

for some constant $k > 0$. Here y is the dependent and x the independent variable. The general solution for this equation is

$$y(x) = Ce^{kx}.$$

We saw above that this function is a solution, although we used different variable names.

Next,

$$\frac{dy}{dx} = -ky,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = Ce^{-kx}.$$

***Exercise* 0.2.1:** *Check that the y given is really a solution to the equation.*

Next, take the *second order differential equation*

$$\frac{d^2y}{dx^2} = -k^2 y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx).$$

Since the equation is a second order differential equation, we have two constants in our general solution.

***Exercise* 0.2.2:** *Check that the y given is really a solution to the equation.*

Finally, consider the second order differential equation

$$\frac{d^2y}{dx^2} = k^2 y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 e^{kx} + C_2 e^{-kx},$$

or

$$y(x) = D_1 \cosh(kx) + D_2 \sinh(kx).$$

For those that do not know, cosh and sinh are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

They are called the *hyperbolic cosine* and *hyperbolic sine*. These functions are sometimes easier to work with than exponentials. They have some nice familiar properties such as $\cosh 0 = 1$, $\sinh 0 = 0$, and $\frac{d}{dx}\cosh x = \sinh x$ (no that is not a typo) and $\frac{d}{dx}\sinh x = \cosh x$.

***Exercise* 0.2.3:** *Check that both forms of the y given are really solutions to the equation.*

Example 0.2.2: In equations of higher order, you get more constants you must solve for to get a particular solution. The equation $\frac{d^2y}{dx^2} = 0$ has the general solution $y = C_1 x + C_2$; simply integrate twice and don't forget about the constant of integration. Consider the initial conditions $y(0) = 2$ and $y'(0) = 3$. We plug in our general solution and solve for the constants:

$$2 = y(0) = C_1 \cdot 0 + C_2 = C_2, \qquad 3 = y'(0) = C_1.$$

In other words, $y = 3x + 2$ is the particular solution we seek.

An interesting note about cosh: The graph of cosh is the exact shape of a hanging chain. This shape is called a *catenary*. Contrary to popular belief this is not a parabola. If you invert the graph of cosh, it is also the ideal arch for supporting its weight. For example, the gateway arch in Saint Louis is an inverted graph of cosh—if it were just a parabola it might fall. The formula used in the design is inscribed inside the arch:

$$y = -127.7 \text{ ft} \cdot \cosh(x/127.7 \text{ ft}) + 757.7 \text{ ft}.$$

0.2.5 Checking and plotting solution formulas with Python

We can check that formulas really are solutions of a differential equations using the symbolic computation capabilities of *sympy*. This is especially useful in complicated cases where we may not totally trust our own hand-computations. In the screenshot on the next page, we recheck the results of 0.2.2. See § 0.1 for how to obtain the "resources306" module.

Solution plots like the one on the next page can also be created using *numpy* instead of *sympy*, as shown below. The idea is to generate a sequence of points along the graph and join them with straight line segments. We use several hundred points so that the curve looks smooth. In the example (a), we plot the curve $y = e^{-2t}$ for t between -1 and 2. In example (b), we plot a family of curves indexed by the parameter c which runs over the values $\{4, 5, 6, \ldots, 15\}$. There are many options for modifying and decorating such plots: to find out how, ask Google "matplotlib how to add title to plot", etc.

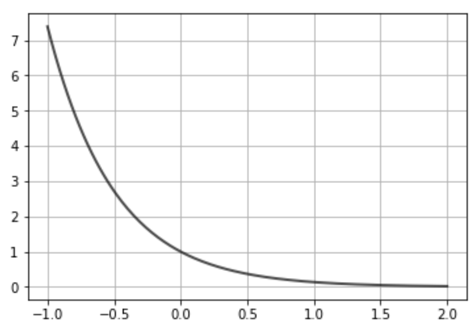
(a)

```
# the following line imports sympy as sp, numpy as np, matplotlib.pyplot as plt
from resources306 import *
t = np.linspace(-1,2,200)
plt.plot( t, np.exp(-2*t) );
plt.grid()
```

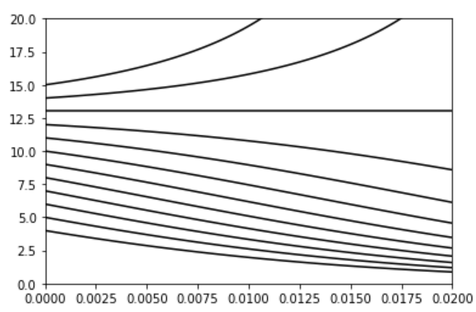
(b)

```
t = np.linspace(0,0.02,100)
for c in np.linspace(4,15,12):
        plt.plot( t, 13/(1-(1-13/c)*np.exp(91*t)) )
plt.xlim(0,.02)
plt.ylim(0,20)
```

(a)



(b)



```
from resources306 import *
```

Construct the alleged family of solutions:

```
t,C = sp.symbols('t C')
x = sp.cos(t) + sp.sin(t) + C*sp.exp(-t)
x
```

$Ce^{-t} + \sin(t) + \cos(t)$

Verify that they all satisfy the differential equation $\frac{dx}{dt} + x = 2\cos t$:

```
sp.diff(x,t) + x == 2*sp.cos(t)
```

```
True
```

Checking "LHS == RHS" tests whether the two expressions are identical: it will not catch if they are different but equivalent, like $\cos^2 t$ and $1 - \sin^2 t$. A more robust tactic is to test if "LHS – RHS" simplifies to 0:

```
sp.simplify( sp.diff(x,t) + x - 2*sp.cos(t) ) == 0
```

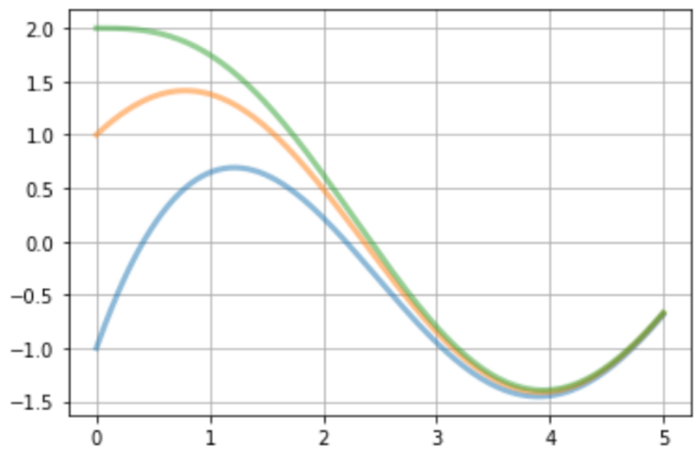
```
True
```

Plot a few of the solutions:

```
afewsolutions = [x.subs({C:c}) for c in [-2,0,1]]
afewsolutions
```

$\left[\sin(t) + \cos(t) - 2e^{-t},\ \ \sin(t) + \cos(t),\ \ \sin(t) + \cos(t) + e^{-t}\right]$

```
for solution in afewsolutions:
    expressionplot( solution, t, 0, 5 )
plt.grid()
```



0.2.6 Exercises

Note: Exercises with numbers 101 and higher have solutions in the back of the book.

***Exercise* 0.2.4:** *Show that $x = e^{4t}$ is a solution to $x''' - 12x'' + 48x' - 64x = 0$.*

***Exercise* 0.2.5:** *Show that $x = e^{t}$ is not a solution to $x''' - 12x'' + 48x' - 64x = 0$.*

***Exercise* 0.2.6:** *Is $y = \sin t$ a solution to $\left(\frac{dy}{dt}\right)^2 = 1 - y^2$? Justify.*

***Exercise* 0.2.7:** *Let $y'' + 2y' - 8y = 0$. Now try a solution of the form $y = e^{rx}$ for some (unknown) constant r. Is this a solution for some r? If so, find all such r.*

***Exercise* 0.2.8:** *Verify that $x = Ce^{-2t}$ is a solution to $x' = -2x$. Find C to solve for the initial condition $x(0) = 100$.*

***Exercise* 0.2.9:** *Verify that $x = C_1e^{-t} + C_2e^{2t}$ is a solution to $x'' - x' - 2x = 0$. Find C_1 and C_2 to solve for the initial conditions $x(0) = 10$ and $x'(0) = 0$.*

***Exercise* 0.2.10:** *Find a solution to $(x')^2 + x^2 = 4$ using your knowledge of derivatives of functions that you know from basic calculus.*

***Exercise* 0.2.11:** *Solve:*

a) $\dfrac{dA}{dt} = -10A, \quad A(0) = 5$

b) $\dfrac{dH}{dx} = 3H, \quad H(0) = 1$

c) $\dfrac{d^2y}{dx^2} = 4y, \quad y(0) = 0, \quad y'(0) = 1$

d) $\dfrac{d^2x}{dy^2} = -9x, \quad x(0) = 1, \quad x'(0) = 0$

***Exercise* 0.2.12:** *Is there a solution to $y' = y$, such that $y(0) = y(1)$?*

***Exercise* 0.2.13:** *The population of city X was 100 thousand 20 years ago, and the population of city X was 120 thousand 10 years ago. Assuming constant growth, you can use the exponential population model (like for the bacteria). What do you estimate the population is now?*

***Exercise* 0.2.14:** *Suppose that a football coach gets a salary of one million dollars now, and a raise of 10% every year (so exponential model, like population of bacteria). Let s be the salary in millions of dollars, and t is time in years.*

a) *What is $s(0)$ and $s(1)$.*

b) *Approximately how many years will it take for the salary to be 10 million.*

c) *Approximately how many years will it take for the salary to be 20 million.*

d) *Approximately how many years will it take for the salary to be 30 million.*

***Exercise* 0.2.51:** *Verify that the function(s) solve the following differential equations (DEs):*

a) $y' = -5y\ ;\ y = 3e^{-5x}$

b) $y' = \cos(3x)$; $y = \frac{1}{3}\sin(3x) + 7$

c) $y' = 2y$; $y = ce^{2x}$, *where c is any real number.*

d) $y'' + y' - 6y = 0$; $y_1 = e^{2x}$, $y_2 = e^{-3x}$

e) $y'' + 16y = 0$; $y_1 = \cos(4x)$, $y_2 = \sin(4x)$

***Exercise* 0.2.101:** *Show that* $x = e^{-2t}$ *is a solution to* $x'' + 4x' + 4x = 0$.

***Exercise* 0.2.102:** *Is* $y = x^2$ *a solution to* $x^2y'' - 2y = 0$? *Justify.*

***Exercise* 0.2.103:** *Let* $xy'' - y' = 0$. *Try a solution of the form* $y = x^r$. *Is this a solution for some* r? *If so, find all such* r.

***Exercise* 0.2.104:** *Verify that* $x = C_1e^t + C_2$ *is a solution to* $x'' - x' = 0$. *Find* C_1 *and* C_2 *so that* x *satisfies* $x(0) = 10$ *and* $x'(0) = 100$.

***Exercise* 0.2.105:** *Solve* $\frac{d\varphi}{ds} = 8\varphi$ *and* $\varphi(0) = -9$.

***Exercise* 0.2.106:** *Solve*:

a) $\dfrac{dx}{dt} = -4x, \quad x(0) = 9$

b) $\dfrac{d^2x}{dt^2} = -4x, \quad x(0) = 1, \quad x'(0) = 2$

c) $\dfrac{dp}{dq} = 3p, \quad p(0) = 4$

d) $\dfrac{d^2T}{dx^2} = 4T, \quad T(0) = 0, \quad T'(0) = 6$

***Exercise* 0.2.151:** *Substitute* $y = e^{rx}$ *into the following DEs, solve for* r, *then write the general linear combination of the solution(s) of the form* e^{rx}:

a) $7y' + 5y = 0$

b) $2y'' + 7y' - 4y = 0$

c) $y'' - 3y' - 10y = 0$

d) $3y'' - 7y' - 6y = 0$

e) $4y'' + 3y' - y = 0$

***Exercise* 0.2.152:** *Verify that* $y = y(x)$ *solves the DE. Then find the constant* C *which satisfies the given initial condition.*

a) $3y' = 2y$, $y(0) = 5$; $y = Ce^{\frac{2}{3}x}$

b) $y' = (3x^2 + 1)y$, $y(1) = 1$; $y = Ce^{x^3+x}$

c) $y' = 8x^3(y^2 + 1)$, $y(0) = 1$; $y = \tan(2x^4 + C)$

d) $y' + 2y = 6$, $y(0) = -5$; $y = 3 + Ce^{-2x}$

0.3 Classification of differential equations

Note: less than 1 lecture or left as reading, §1.3 in [BD]

There are many types of differential equations, and we classify them into different categories based on their properties. Let us quickly go over the most basic classification. We already saw the distinction between ordinary and partial differential equations:

- *Ordinary differential equations* or (ODE) are equations where the derivatives are taken with respect to only one variable. That is, there is only one independent variable.
- *Partial differential equations* or (PDE) are equations that depend on partial derivatives of several variables. That is, there are several independent variables.

Let us see some examples of ordinary differential equations:

$$\begin{aligned}
\frac{dy}{dt} &= ky, && \text{(Exponential growth)} \\
\frac{dy}{dt} &= k(A-y), && \text{(Newton's law of cooling)} \\
m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx &= f(t). && \text{(Mechanical vibrations)}
\end{aligned}$$

And of partial differential equations:

$$\begin{aligned}
\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} &= 0, && \text{(Transport equation)} \\
\frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2}, && \text{(Heat equation)} \\
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. && \text{(Wave equation in 2 dimensions)}
\end{aligned}$$

If there are several equations working together, we have a so-called *system of differential equations*. For example,

$$y' = x, \qquad x' = y$$

is a simple system of ordinary differential equations. Maxwell's equations for electromagnetics,

$$\begin{aligned}
\nabla \cdot \vec{D} &= \rho, & \nabla \cdot \vec{B} &= 0, \\
\nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t}, & \nabla \times \vec{H} &= \vec{J} + \frac{\partial \vec{D}}{\partial t},
\end{aligned}$$

are a system of partial differential equations. The divergence operator $\nabla \cdot$ and the curl operator $\nabla \times$ can be written out in partial derivatives of the functions involved in the x, y, and z variables.

The next bit of information is the *order* of the equation (or system). The order is simply the order of the largest derivative that appears. If the highest derivative that appears is the first derivative, the equation is of first order. If the highest derivative that appears is the second derivative, then the equation is of second order. For example, Newton's law of cooling above is a first order equation, while the mechanical vibrations equation is a second order equation. The equation governing transversal vibrations in a beam,

$$a^4 \frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0,$$

is a fourth order partial differential equation. It is fourth order as at least one derivative is the fourth derivative. It does not matter that the derivative in t is only of second order.

In the first chapter, we will start attacking first order ordinary differential equations, that is, equations of the form $\frac{dy}{dx} = f(x, y)$. In general, lower order equations are easier to work with and have simpler behavior, which is why we start with them.

We also distinguish how the dependent variables appear in the equation (or system). In particular, we say an equation is *linear* if the dependent variable (or variables) and their derivatives appear linearly, that is only as first powers, they are not multiplied together, and no other functions of the dependent variables appear. In other words, the equation is a sum of terms, where each term is some function of the independent variables or some function of the independent variables multiplied by a dependent variable or its derivative. Otherwise, the equation is called *nonlinear*. For example, an ordinary differential equation is linear if it can be put into the form

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x) y = b(x). \tag{2}$$

The functions $a_0, a_1, \ldots, a_n$ are called the *coefficients*. The equation is allowed to depend arbitrarily on the independent variable. So

$$e^x \frac{d^2 y}{dx^2} + \sin(x) \frac{dy}{dx} + x^2 y = \frac{1}{x} \tag{3}$$

is still a linear equation as y and its derivatives only appear linearly.

All the equations and systems above as examples are linear. It may not be immediately obvious for Maxwell's equations unless you write out the divergence and curl in terms of partial derivatives. Let us see some nonlinear equations. For example Burger's equation,

$$\frac{\partial y}{\partial t} + y \frac{\partial y}{\partial x} = \nu \frac{\partial^2 y}{\partial x^2},$$

is a nonlinear second order partial differential equation. It is nonlinear because y and $\frac{\partial y}{\partial x}$ are multiplied together. The equation

$$\frac{dx}{dt} = x^2 \tag{4}$$

is a nonlinear first order differential equation as there is a second power of the dependent variable x.

A linear equation may further be called *homogeneous* if all terms depend on the dependent variable. That is, if no term is a function of the independent variables alone. Otherwise, the equation is called *nonhomogeneous* or *inhomogeneous*. For example, the exponential growth equation, the wave equation, or the transport equation above are homogeneous. The mechanical vibrations equation above is nonhomogeneous as long as $f(t)$ is not the zero function. Similarly, if the ambient temperature A is nonzero, Newton's law of cooling is nonhomogeneous. A homogeneous linear ODE can be put into the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0.$$

Compare to (2) and notice there is no function $b(x)$.

If the coefficients of a linear equation are actually constant functions, then the equation is said to have *constant coefficients*. The coefficients are the functions multiplying the dependent variable(s) or one of its derivatives, not the function $b(x)$ standing alone. A constant coefficient nonhomogeneous ODE is an equation of the form

$$a_n\frac{d^n y}{dx^n} + a_{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1\frac{dy}{dx} + a_0 y = b(x),$$

where $a_0, a_1, \ldots, a_n$ are all constants, but b may depend on the independent variable x. The mechanical vibrations equation above is a constant coefficient nonhomogeneous second order ODE. The same nomenclature applies to PDEs, so the transport equation, heat equation and wave equation are all examples of constant coefficient linear PDEs.

Finally, an equation (or system) is called *autonomous* if the equation does not depend on the independent variable. For autonomous ordinary differential equations, the independent variable is then thought of as time. Autonomous equation means an equation that does not change with time. For example, Newton's law of cooling is autonomous, so is equation (4). On the other hand, mechanical vibrations or (3) are not autonomous.

0.3.1 Exercises

***Exercise* 0.3.1:** *Classify the following equations. Are they ODE or PDE? Is it an equation or a system? What is the order? Is it linear or nonlinear, and if it is linear, is it homogeneous, constant coefficient? If it is an ODE, is it autonomous?*

a) $\sin(t)\frac{d^2x}{dt^2} + \cos(t)x = t^2$

b) $\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = xy$

c) $y'' + 3y + 5x = 0, \quad x'' + x - y = 0$

d) $\frac{\partial^2 u}{\partial t^2} + u\frac{\partial^2 u}{\partial s^2} = 0$

e) $x'' + tx^2 = t$

f) $\frac{d^4x}{dt^4} = 0$

Exercise* 0.3.2*: *If* $\vec{u} = (u_1, u_2, u_3)$ *is a vector, we have the divergence* $\nabla \cdot \vec{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$ *and curl* $\nabla \times \vec{u} = \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right)$. *Notice that curl of a vector is still a vector. Write out Maxwell's equations in terms of partial derivatives and classify the system.*

Exercise* 0.3.3*: *Suppose* F *is a linear function, that is,* $F(x, y) = ax + by$ *for constants* a *and* b. *What is the classification of equations of the form* $F(y', y) = 0$.

Exercise* 0.3.4*: *Write down an explicit example of a third order, linear, nonconstant coefficient, nonautonomous, nonhomogeneous system of two ODE such that every derivative that could appear, does appear.*

Exercise* 0.3.101*: *Classify the following equations. Are they ODE or PDE? Is it an equation or a system? What is the order? Is it linear or nonlinear, and if it is linear, is it homogeneous, constant coefficient? If it is an ODE, is it autonomous?*

a) $\dfrac{\partial^2 v}{\partial x^2} + 3\dfrac{\partial^2 v}{\partial y^2} = \sin(x)$

b) $\dfrac{dx}{dt} + \cos(t)x = t^2 + t + 1$

c) $\dfrac{d^7 F}{dx^7} = 3F(x)$

d) $y'' + 8y' = 1$

e) $x'' + tyx' = 0, \quad y'' + txy = 0$

f) $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial s^2} + u^2$

Exercise* 0.3.102*: *Write down the general* zero*th order linear ordinary differential equation. Write down the general solution.*

Exercise* 0.3.103*: *For which* k *is* $\frac{dx}{dt} + x^k = t^{k+2}$ *linear. Hint: there are two answers.*

Chapter 1

First order equations

1.1 Integrals as solutions

Note: 1 lecture (or less), §1.2 in [EP], covered in §1.2 and §2.1 in [BD]

A first order ODE is an equation of the form

$$\frac{dy}{dx} = f(x, y),$$

or just

$$y' = f(x, y).$$

In general, there is no simple formula or procedure one can follow to find solutions. In the next few lectures we will look at special cases where solutions are not difficult to obtain. In this section, let us assume that f is a function of x alone, that is, the equation is

$$y' = f(x). \tag{1.1}$$

We could just integrate (antidifferentiate) both sides with respect to x.

$$\int y'(x)\, dx = \int f(x)\, dx + C,$$

that is

$$y(x) = \int f(x)\, dx + C.$$

This $y(x)$ is actually the general solution. So to solve (1.1), we find some antiderivative of $f(x)$ and then we add an arbitrary constant to get the general solution.

Now is a good time to discuss a point about calculus notation and terminology. Calculus textbooks muddy the waters by talking about the integral as primarily the so-called indefinite integral. The indefinite integral is really the *antiderivative* (in fact the whole one-parameter family of antiderivatives). There really exists only one integral and that is the definite integral. The only reason for the indefinite integral notation is that we can always write an antiderivative as a (definite) integral. That is, by the fundamental theorem of calculus we can always write $\int f(x)\,dx + C$ as

$$\int_{x_0}^{x} f(t)\,dt + C.$$

Hence the terminology *to integrate* when we may really mean *to antidifferentiate*. Integration is just one way to compute the antiderivative (and it is a way that always works, see the following examples). Integration is defined as the area under the graph, it only happens to also compute antiderivatives. For sake of consistency, we will keep using the indefinite integral notation when we want an antiderivative, and you should *always* think of the definite integral as a way to write it.

Example 1.1.1: Find the general solution of $y' = 3x^2$.

Elementary calculus tells us that the general solution must be $y = x^3 + C$. Let us check by differentiating: $y' = 3x^2$. We got *precisely* our equation back.

Normally, we also have an initial condition such as $y(x_0) = y_0$ for some two numbers x_0 and y_0 (x_0 is usually 0, but not always). We can then write the solution as a definite integral in a nice way. Suppose our problem is $y' = f(x)$, $y(x_0) = y_0$. Then the solution is

$$y(x) = \int_{x_0}^{x} f(s)\,ds + y_0. \tag{1.2}$$

Let us check! We compute $y' = f(x)$, via the fundamental theorem of calculus, and by Jupiter, y is a solution. Is it the one satisfying the initial condition? Well, $y(x_0) = \int_{x_0}^{x_0} f(x)\,dx + y_0 = y_0$. It is!

Do note that the definite integral and the indefinite integral (antidifferentiation) are completely different beasts. The definite integral always evaluates to a number. Therefore, (1.2) is a formula we can plug into the calculator or a computer, and it will be happy to calculate specific values for us. We will easily be able to plot the solution and work with it just like with any other function. It is not so crucial to always find a closed form for the antiderivative.

Example 1.1.2: Solve

$$y' = e^{-x^2}, \qquad y(0) = 1.$$

By the preceding discussion, the solution must be

$$y(x) = \int_{0}^{x} e^{-s^2}\,ds + 1.$$

Here is a good way to make fun of your friends taking second semester calculus. Tell them to find the closed form solution. Ha ha ha (bad math joke). It is not possible (in closed form). There is absolutely nothing wrong with writing the solution as a definite integral. This particular integral is in fact very important in statistics.

Using this method, we can also solve equations of the form

$$y' = f(y).$$

Let us write the equation in Leibniz notation.

$$\frac{dy}{dx} = f(y).$$

Now we use the inverse function theorem from calculus to switch the roles of x and y to obtain

$$\frac{dx}{dy} = \frac{1}{f(y)}.$$

What we are doing seems like algebra with dx and dy. It is tempting to just do algebra with dx and dy as if they were numbers. And in this case it does work. Be careful, however, as this sort of hand-waving calculation can lead to trouble, especially when more than one independent variable is involved. At this point, we can simply integrate,

$$x(y) = \int \frac{1}{f(y)}\, dy + C.$$

Finally, we try to solve for y.

Example 1.1.3: Previously, we guessed $y' = ky$ (for some $k > 0$) has the solution $y = Ce^{kx}$. We can now find the solution without guessing. First we note that $y = 0$ is a solution. Henceforth, we assume $y \neq 0$. We write

$$\frac{dx}{dy} = \frac{1}{ky}.$$

We integrate to obtain

$$x(y) = x = \frac{1}{k} \ln |y| + D,$$

where D is an arbitrary constant. Now we solve for y (actually for $|y|$).

$$|y| = e^{kx-kD} = e^{-kD} e^{kx}.$$

If we replace e^{-kD} with an arbitrary constant C, we can get rid of the absolute value bars (which we can do as D was arbitrary). In this way, we also incorporate the solution $y = 0$. We get the same general solution as we guessed before, $y = Ce^{kx}$.

Example 1.1.4: Find the general solution of $y' = y^2$.

First we note that $y = 0$ is a solution. We can now assume that $y \neq 0$. Write

$$\frac{dx}{dy} = \frac{1}{y^2}.$$

We integrate to get

$$x = \frac{-1}{y} + C.$$

We solve for $y = \frac{1}{C-x}$. So the general solution is

$$y = \frac{1}{C - x} \qquad \text{or} \qquad y = 0.$$

Note the singularities of the solution. If for example $C = 1$, then the solution "blows up" as we approach $x = 1$. See Figure 1.1. Generally, it is hard to tell from just looking at the equation itself how the solution is going to behave. The equation $y' = y^2$ is very nice and defined everywhere, but the solution is only defined on some interval $(-\infty, C)$ or (C, ∞). Usually when this happens we only consider one of these the solution. For example if we impose a condition $y(0) = 1$, then the solution is $y = \frac{1}{1-x}$, and we would consider this solution only for x on the interval $(-\infty, 1)$. In the figure, it is the left side of the graph.

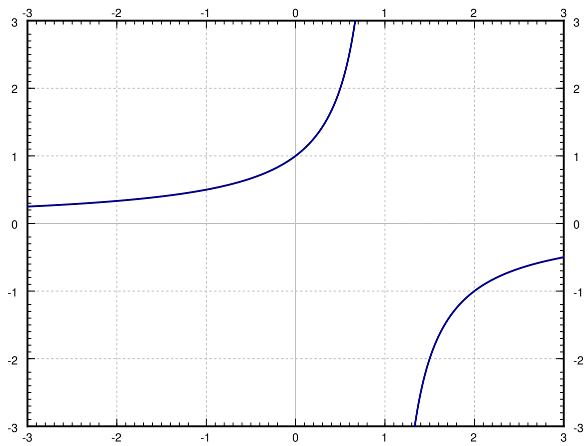


Figure 1.1: Plot of $y = \frac{1}{1-x}$.

Classical problems leading to differential equations solvable by integration are problems dealing with velocity, acceleration and distance. You have surely seen these problems before in your calculus class.

Example 1.1.5: Suppose a car drives at a speed $e^{t/2}$ meters per second, where t is time in seconds. How far did the car get in 2 seconds (starting at $t = 0$)? How far in 10 seconds?

Let x denote the distance the car traveled. The equation is

$$x' = e^{t/2}.$$

We just integrate this equation to get that

$$x(t) = 2e^{t/2} + C.$$

We still need to figure out C. We know that when $t = 0$, then $x = 0$. That is, $x(0) = 0$. So

$$0 = x(0) = 2e^{0/2} + C = 2 + C.$$

Thus $C = -2$ and

$$x(t) = 2e^{t/2} - 2.$$

Now we just plug in to get where the car is at 2 and at 10 seconds. We obtain

$$x(2) = 2e^{2/2} - 2 \approx 3.44 \text{ meters}, \qquad x(10) = 2e^{10/2} - 2 \approx 294 \text{ meters}.$$

Example 1.1.6: Suppose that the car accelerates at a rate of t^2 m/s². At time $t = 0$ the car is at the 1 meter mark and is traveling at 10 m/s. Where is the car at time $t = 10$.

Well this is actually a second order problem. If x is the distance traveled, then x' is the velocity, and x'' is the acceleration. The equation with initial conditions is

$$x'' = t^2, \qquad x(0) = 1, \qquad x'(0) = 10.$$

What if we say $x' = v$. Then we have the problem

$$v' = t^2, \qquad v(0) = 10.$$

Once we solve for v, we can integrate and find x.

***Exercise* 1.1.1:** *Solve for v, and then solve for x. Find $x(10)$ to answer the question.*

1.1.1 Exercises

***Exercise* 1.1.2:** *Solve* $\frac{dy}{dx} = x^2 + x$ *for* $y(1) = 3$.

***Exercise* 1.1.3:** *Solve* $\frac{dy}{dx} = \sin(5x)$ *for* $y(0) = 2$.

***Exercise* 1.1.4:** *Solve* $\frac{dy}{dx} = \frac{1}{x^2-1}$ *for* $y(0) = 0$.

***Exercise* 1.1.5:** *Solve* $y' = y^3$ *for* $y(0) = 1$.

***Exercise* 1.1.6** (little harder)**:** *Solve* $y' = (y-1)(y+1)$ *for* $y(0) = 3$.

***Exercise* 1.1.7:** *Solve* $\frac{dy}{dx} = \frac{1}{y+1}$ *for* $y(0) = 0$.

***Exercise* 1.1.8** (harder)**:** *Solve* $y'' = \sin x$ *for* $y(0) = 0$, $y'(0) = 2$.

***Exercise* 1.1.9:** *A spaceship is traveling at the speed* $2t^2 + 1$ km/s *(t is time in seconds). It is pointing directly away from earth and at time $t = 0$ it is 1000 kilometers from earth. How far from earth is it at one minute from time $t = 0$?*

***Exercise* 1.1.10:** *Solve* $\frac{dx}{dt} = \sin(t^2) + t$, $x(0) = 20$. *It is OK to leave your answer as a definite integral.*

***Exercise* 1.1.11:** *A dropped ball accelerates downwards at a constant rate* 9.8 *meters per second squared. Set up the differential equation for the height above ground* h *in meters. Then supposing* $h(0) = 100$ *meters, how long does it take for the ball to hit the ground.*

***Exercise* 1.1.12:** *Find the general solution of* $y' = e^x$*, and then* $y' = e^y$*.*

***Exercise* 1.1.51:** *Find the general solution of* $y' = \frac{1}{x^2-2x-8}$

***Exercise* 1.1.52:** *Find the general solution of* $y' = \frac{1}{x^4+4x^2}$

***Exercise* 1.1.101:** *Solve* $\frac{dy}{dx} = e^x + x$ *and* $y(0) = 10$.

***Exercise* 1.1.102:** *Solve* $x' = \frac{1}{x^2}$, $x(1) = 1$.

***Exercise* 1.1.103:** *Solve* $x' = \frac{1}{\cos(x)}$, $x(0) = \frac{\pi}{2}$.

***Exercise* 1.1.104:** *Sid is in a car traveling at speed* $10t + 70$ *miles per hour away from Las Vegas, where* t *is in hours. At* $t = 0$*, Sid is* 10 *miles away from Vegas. How far from Vegas is Sid* 2 *hours later?*

***Exercise* 1.1.105:** *Solve* $y' = y^n$, $y(0) = 1$, *where* n *is a positive integer. Hint: You have to consider different cases.*

***Exercise* 1.1.106:** *The rate of change of the volume of a snowball that is melting is proportional to the surface area of the snowball. Suppose the snowball is perfectly spherical. Then the volume (in centimeters cubed) of a ball of radius* r *centimeters is* $4/3\,\pi r^3$*. The surface area is* $4\pi r^2$*. Set up the differential equation for how* r *is changing. Then, suppose that at time* $t = 0$ *minutes, the radius is* 10 *centimeters. After* 5 *minutes, the radius is* 8 *centimeters. At what time* t *will the snowball be completely melted.*

***Exercise* 1.1.107:** *Find the general solution to* $y'''' = 0$*. How many distinct constants do you need?*

***Exercise* 1.1.151:** *Find the general solution to the following differential equations (DEs) by integration:*

a) $\frac{d^2y}{dx^2} = 4x^3 + e^{2x}$

b) $\frac{d^3y}{dx^3} = 6x^2 + 1$

c) $\frac{d^3y}{dx^3} = \cos(2x)$

***Exercise* 1.1.152:** *Find the particular solution to the following initial value problems (IVPs):*

a) $y'' = 6x + 2;\ y(1) = -1,\ y'(1) = 7$

b) $y'' = e^{\frac{1}{3}x};\ y(0) = -2,\ y'(0) = 4$

c) $y'' = \frac{1}{\sqrt{x+9}};\ y(0) = 10,\ y'(0) = 1$

1.2 Slope fields

Note: 1 lecture, §1.3 in [EP], §1.1 in [BD]

As we said, the general first order equation we are studying looks like

$$y' = f(x, y).$$

A lot of the time, we cannot simply solve these kinds of equations explicitly. It would be nice if we could at least figure out the shape and behavior of the solutions, or find approximate solutions.

1.2.1 Slope fields

The equation $y' = f(x, y)$ gives you a slope at each point in the (x, y)-plane. And this is the slope a solution $y(x)$ would have at x if its value was y. In other words, $f(x, y)$ is the slope of a solution whose graph runs through the point (x, y). At a point (x, y), we plot a short line with the slope $f(x, y)$. For example, if $f(x, y) = xy$, then at point $(2, 1.5)$ we draw a short line of slope $xy = 2 \times 1.5 = 3$. So, if $y(x)$ is a solution and $y(2) = 1.5$, then the equation mandates that $y'(2) = 3$. See Figure 1.2.

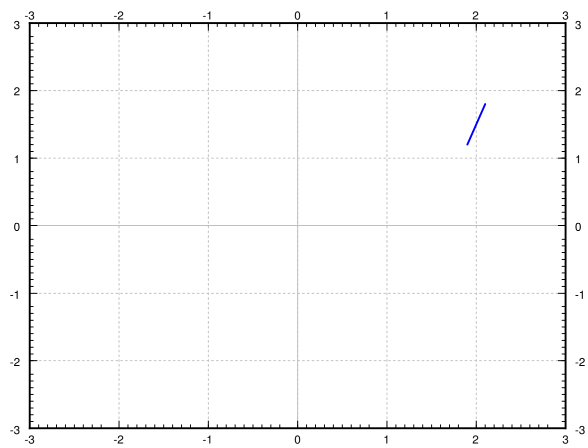


Figure 1.2: The slope $y' = xy$ at $(2, 1.5)$.

To get an idea of how solutions behave, we draw such lines at lots of points in the plane, not just the point $(2, 1.5)$. We would ideally want to see the slope at every point, but that is just not possible. Usually we pick a grid of points fine enough so that it shows the behavior, but not too fine so that we can still recognize the individual lines. We call this picture the *slope field* of the equation. See Figure 1.3 on the following page for the slope field of the equation $y' = xy$. Usually in practice, one does not do this by hand, but has a computer do the drawing.

Suppose we are given a specific initial condition $y(x_0) = y_0$. A solution, that is, the graph of the solution, would be a curve that follows the slopes we drew. For a few sample solutions, see Figure 1.4. It is easy to roughly sketch (or at least imagine) possible solutions in the slope field, just from looking at the slope field itself. You simply sketch a line that roughly fits the little line segments and goes through your initial condition.

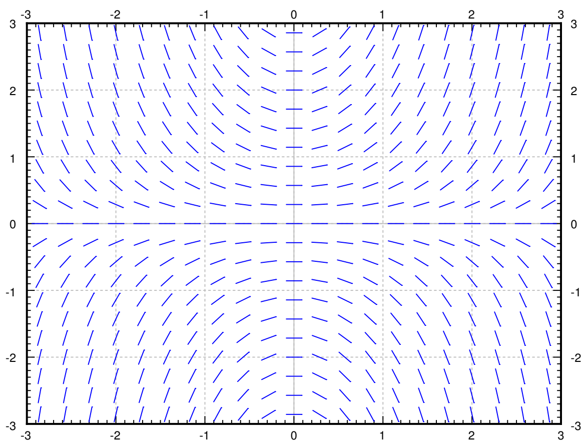


Figure 1.3: Slope field of $y' = xy$.

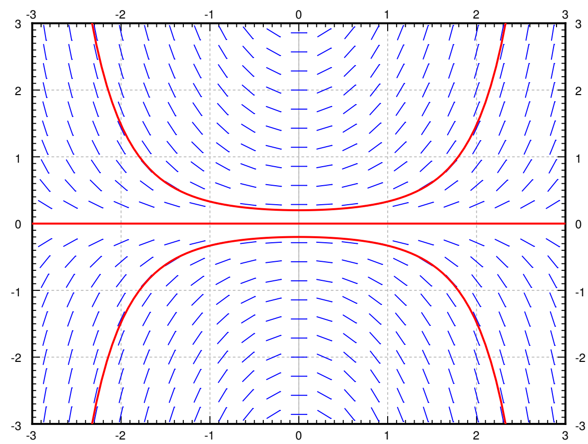


Figure 1.4: Slope field of $y' = xy$ with a graph of solutions satisfying $y(0) = 0.2$, $y(0) = 0$, and $y(0) = -0.2$.

By looking at the slope field we get a lot of information about the behavior of solutions without having to solve the equation. For example, in Figure 1.4 we see what the solutions do when the initial conditions are $y(0) > 0$, $y(0) = 0$ and $y(0) < 0$. A small change in the initial condition causes quite different behavior. We see this behavior just from the slope field and imagining what solutions ought to do.

We see a different behavior for the equation $y' = -y$. The slope field and a few solutions is in see Figure 1.5 on the next page. If we think of moving from left to right (perhaps x is time and time is usually increasing), then we see that no matter what $y(0)$ is, all solutions tend to zero as x tends to infinity. Again that behavior is clear from simply looking at the slope field itself.

1.2.2 Existence and uniqueness

We wish to ask two fundamental questions about the problem

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

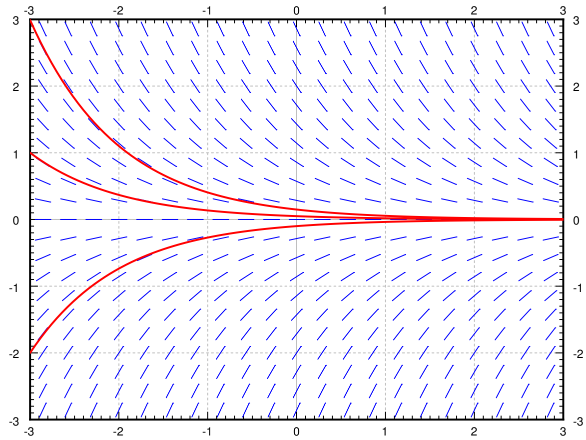


Figure 1.5: Slope field of $y' = -y$ with a graph of a few solutions.

What do you think is the answer? The answer seems to be yes to both does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

Example 1.2.1: Attempt to solve:

$$y' = \frac{1}{x}, \qquad y(0) = 0.$$

Integrate to find the general solution $y = \ln|x| + C$. The solution does not exist at $x = 0$. See Figure 1.6 on the following page. The equation may have been written as the seemingly harmless $xy' = 1$.

Example 1.2.2: Solve:

$$y' = 2\sqrt{|y|}, \qquad y(0) = 0.$$

See Figure 1.7 on the next page. Note that $y = 0$ is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

It is hard to tell by staring at the slope field that the solution is not unique. Is there any hope? Of course there is. We have the following theorem, known as Picard's theorem*.

*Named after the French mathematician Charles Émile Picard (1856–1941)

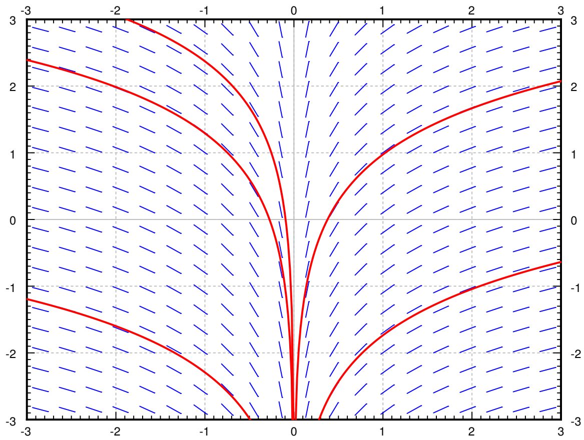


Figure 1.6: Slope field of $y' = {}^{1}\!/_{x}$.

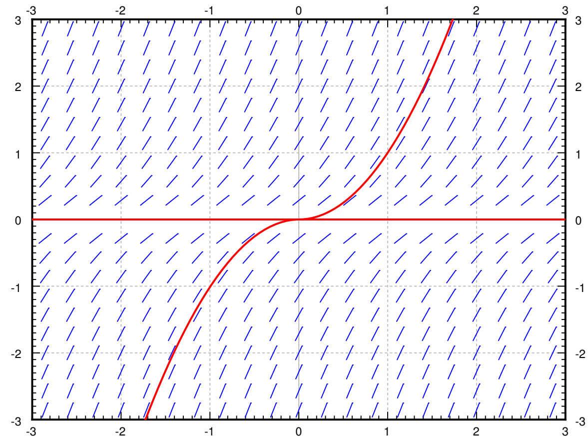


Figure 1.7: Slope field of $y' = 2\sqrt{|y|}$ *with two solutions satisfying* $y(0) = 0$.

Theorem 1.2.1 (Picard's theorem on existence and uniqueness)**.** *If* $f(x, y)$ *is continuous (as a function of two variables) and* $\frac{\partial f}{\partial y}$ *exists and is continuous near some* (x_0, y_0)*, then a solution to*

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

exists (at least for some small interval of x*'s) and is unique.*

Note that the problems $y' = {}^{1}\!/_{x}$, $y(0) = 0$ and $y' = 2\sqrt{|y|}$, $y(0) = 0$ do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

Example 1.2.3: For some constant A, solve:

$$y' = y^2, \qquad y(0) = A.$$

We know how to solve this equation. First assume that $A \neq 0$, so y is not equal to zero at least for some x near 0. So $x' = {}^{1}\!/_{y^2}$, so $x = {}^{-1}\!/_{y} + C$, so $y = \frac{1}{C-x}$. If $y(0) = A$, then $C = {}^{1}\!/_{A}$ so

$$y = \frac{1}{{}^{1}\!/_{A} - x}.$$

If $A = 0$, then $y = 0$ is a solution.

For example, when $A = 1$ the solution "blows up" at $x = 1$. Hence, the solution does not exist for all x even if the equation is nice everywhere. The equation $y' = y^2$ certainly looks nice.

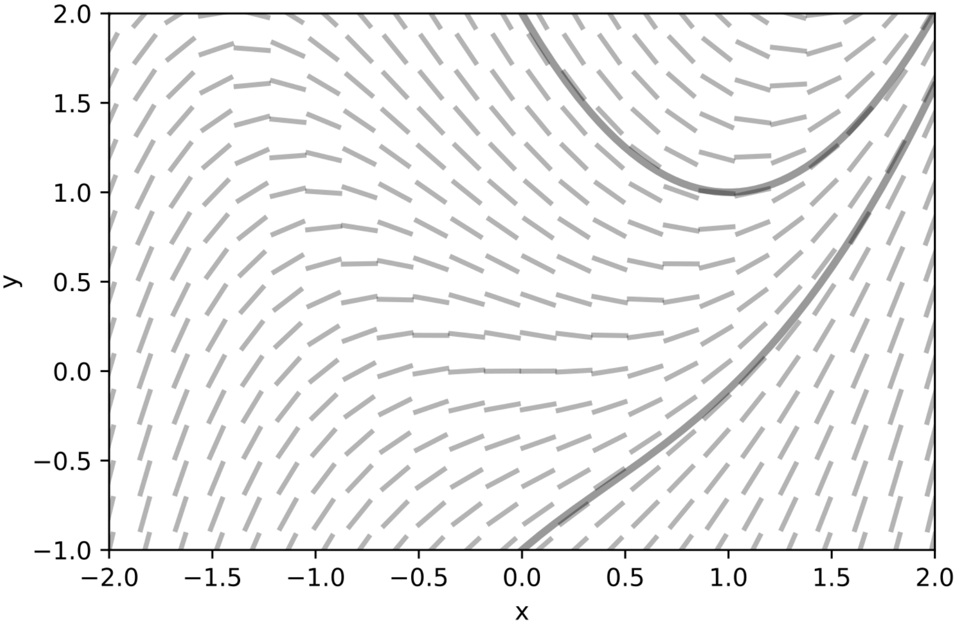
For most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds "globally" unlike for the equation $y' = y^2$.

1.2.3 Slope fields with Python

The *resources306* module provides a function *slopefieldplot*. You supply *slopefieldplot* with a function (called f in the example below) that returns the right hand side of your differential equation, the desired minimum and maximum values on the horizontal and vertical axes respectively, and the desired spacing between line segments on the horizontal axis. Any of the graphical options for *matplotlib.pyplot.plot()*, such as line color, line width, etc., can also be supplied to *slopefieldplot*. Here we create a slope field plot for the differential equation $\frac{dy}{dx} = x^2 - y$:

```
from resources306 import *
def f(x,y): return x**2 - y
slopefieldplot( f, -2,2, -1,2, .2 ,lw=2)
plt.xlabel('x')
plt.ylabel('y');
```

If using a Python environment that does not automatically display graphics, add the line "plt.show()".
This generates the slope field part of the picture below.



The *resources306* module also provides a function *expressionplot* as a simple way to plot a sympy expression. You supply *expressionplot* with the expression you want to plot, the (one and only) variable in the expression, the minimum and maximum values of that variable to be shown in the plot, and any graphical options you like. Below, we create two solutions of the differential equation $\frac{dy}{dx} = x^2 - y$ as sympy expressions:

```
x = sp.symbols('x')
y1 = x**2-2*x+2
y2 = ((x**2-2*x+2)*sp.exp(x)-3)*sp.exp(-x)
y1,y2
```

$$\left(x^2 - 2x + 2, \quad \left(\left(x^2 - 2x + 2\right) e^x - 3\right) e^{-x}\right)$$

and then with the code below add their graphs to the slope field plot.

```
slopefieldplot( f, -2,2, -1,2, .2 ,lw=2)
expressionplot(y1,x,-2,2,color='r',alpha=.4,lw=3)
expressionplot(y2,x,-1,2,color='b',alpha=.4,lw=3)
plt.xlabel('x')
plt.ylabel('y')
```

1.2.4 Exercises

***Exercise* 1.2.1:** *Sketch slope field for $y' = e^{x-y}$. How do the solutions behave as x grows? Can you guess a particular solution by looking at the slope field?*

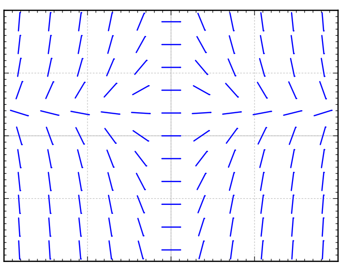
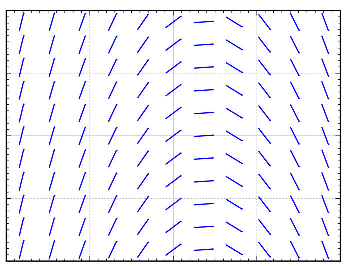
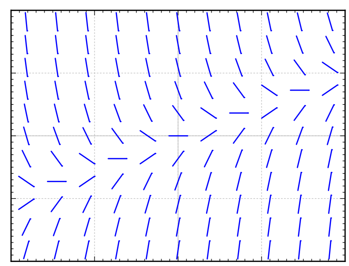
***Exercise* 1.2.2:** *Sketch slope field for $y' = x^2$.*

***Exercise* 1.2.3:** *Sketch slope field for $y' = y^2$.*

***Exercise* 1.2.4:** *Is it possible to solve the equation $y' = \frac{xy}{\cos x}$ for $y(0) = 1$? Justify.*

***Exercise* 1.2.5:** *Is it possible to solve the equation $y' = y\sqrt{|x|}$ for $y(0) = 0$? Is the solution unique? Justify.*

***Exercise* 1.2.6:** *Match equations $y' = 1 - x$, $y' = x - 2y$, $y' = x(1 - y)$ to slope fields. Justify.*

a)  *b)*  *c)* 

***Exercise* 1.2.7** (challenging)**:** *Take $y' = f(x, y)$, $y(0) = 0$, where $f(x, y) > 1$ for all x and y. If the solution exists for all x, can you say what happens to $y(x)$ as x goes to positive infinity? Explain.*

***Exercise* 1.2.8** (challenging)**:** *Take $(y - x)y' = 0$, $y(0) = 0$.*

a) *Find two distinct solutions.*

b) *Explain why this does not violate Picard's theorem.*

***Exercise* 1.2.9:** *Suppose $y' = f(x, y)$. What will the slope field look like, explain and sketch an example, if you know the following about $f(x, y)$:*

a) *f does not depend on y.*

b) *f does not depend on x.*

c) *$f(t, t) = 0$ for any number t.*

d) *$f(x, 0) = 0$ and $f(x, 1) = 1$ for all x.*

***Exercise* 1.2.10:** *Find a solution to $y' = |y|$, $y(0) = 0$. Does Picard's theorem apply?*

***Exercise* 1.2.11:** *Take an equation $y' = (y - 2x)g(x, y) + 2$ for some function $g(x, y)$. Can you solve the problem for the initial condition $y(0) = 0$, and if so what is the solution?*

***Exercise* 1.2.12** (challenging)**:** *Suppose $y' = f(x,y)$ is such that $f(x,1) = 0$ for every x, f is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous for every x and y.*

a) *Guess a solution given the initial condition $y(0) = 1$.*

b) *Can graphs of two solutions of the equation for different initial conditions ever intersect?*

c) *Given $y(0) = 0$, what can you say about the solution. In particular, can $y(x) > 1$ for any x? Can $y(x) = 1$ for any x? Why or why not?*

***Exercise* 1.2.51:** *Sketch the region of continuity for $f(x,y)$ on a set of axes and sketch the region of continuity for $\frac{\partial f}{\partial y}(x,y)$ on a separate set of axes. Apply Picard's Theorem to determine whether the solution exists and whether it is unique.*

a) $y' = 2x^2y + 3xy^2$, $y(1) = 2$

b) $y' = \frac{1}{x+y}$, $y(-1) = 3$

c) $y' = \sqrt{2x - 3y}$, $y(3) = 2$

d) $y' = \sqrt[3]{2y + 6x}$, $y(1) = -3$

e) $y' = x\ln(y)$, $y(2) = 3$

f) $y' = x^2e^y$, $y(-1) = 4$

g) $y' = \sqrt{5y + 10x}$, $y(1) = 0$

h) $y' = \sqrt{5y + 10x}$, $y(1) = -2$

i) $y' = 2y^{\frac{1}{3}}$, $y(0) = 0$

j) $y' = 2y^{\frac{1}{3}}$, $y(-1) = -2$

***Exercise* 1.2.52:** *Sketch, on one single set of axes, the region where both $f(x,y)$ and $\frac{\partial f}{\partial y}(x,y)$ are continuous. Determine the conditions on (x_0, y_0) for which Picard's Theorem guarantees that a unique solution exists.*

a) $y' = \sqrt{2x - y}$

b) $y' = \ln(y - x^2)$

c) $y' = \sqrt{y}\ln(x)$

d) $y' = \sqrt[3]{x^2 - y}$

e) $y' = \frac{3x+2y}{x^2-y^2}$

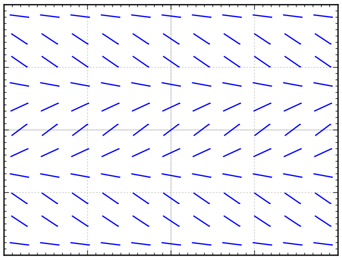
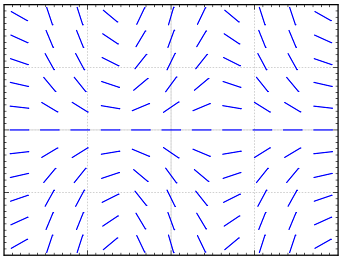
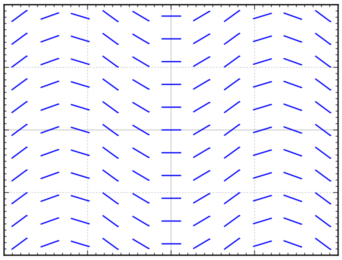
Exercise* 1.2.53*: *Use the following method to determine* $y(4)$ *approximately if* $y(0) = 0$ *and* $\frac{dy}{dt} = \cos y + y \sin t$ *for all* t*. Use Python to draw the slope field of the differential equation on the region where* t *runs from* 0 *to* 5 *and* y *runs from* -1 *to* 5*. Save the graphic you've created as a PNG image, and then open it in a program that allows you to draw on it, like Inkscape, GIMP, etc. On your slope field, sketch the curve* $y(t)$ *that has the following properties:* (i) $y(0) = 0$*, i.e., it starts at* $t = 0$*,* $y = 0$*,* (ii) *its slope agrees with the slope field at every point along it. Then from your picture, estimate* $y(4)$ *as accurately as you can.*

Exercise* 1.2.101*: *Sketch the slope field of* $y' = y^3$*. Can you visually find the solution that satisfies* $y(0) = 0$*?*

Exercise* 1.2.102*: *Is it possible to solve* $y' = xy$ *for* $y(0) = 0$*? Is the solution unique?*

Exercise* 1.2.103*: *Is it possible to solve* $y' = \frac{x}{x^2-1}$ *for* $y(1) = 0$*?*

Exercise* 1.2.104*: *Match equations* $y' = \sin x$*,* $y' = \cos y$*,* $y' = y\cos(x)$ *to slope fields. Justify.*

a)  *b*)  *c*) 

Exercise* 1.2.105** (tricky):*** *Suppose*

$$f(y) = \begin{cases} 0 & \text{if } y > 0, \\ 1 & \text{if } y \leq 0. \end{cases}$$

Does $y' = f(y)$*,* $y(0) = 0$ *have a continuously differentiable solution? Does Picard apply? Why, or why not?*

Exercise* 1.2.106*: *Consider an equation of the form* $y' = f(x)$ *for some continuous function* f*, and an initial condition* $y(x_0) = y_0$*. Does a solution exist for all* x*? Why or why not?*

1.3 Separable equations

Note: 1 lecture, §1.4 in [EP], §2.2 in [BD]

When a differential equation is of the form $y' = f(x)$, we can just integrate: $y = \int f(x)\,dx + C$. Unfortunately this method no longer works for the general form of the equation $y' = f(x, y)$. Integrating both sides yields

$$y = \int f(x, y)\,dx + C.$$

Notice the dependence on y in the integral.

1.3.1 Separable equations

We say a differential equation is *separable* if we can write it as

$$y' = f(x)g(y),$$

for some functions $f(x)$ and $g(y)$. Let us write the equation in the Leibniz notation

$$\frac{dy}{dx} = f(x)g(y).$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x)\,dx.$$

Both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x)\,dx + C.$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for y.

Example 1.3.1: Take the equation

$$y' = xy.$$

Note that $y = 0$ is a solution. We will remember that fact and assume $y \neq 0$ from now on, so that we can divide by y. Write the equation as $\frac{dy}{dx} = xy$. Then

$$\int \frac{dy}{y} = \int x\,dx + C.$$

We compute the antiderivatives to get

$$\ln|y| = \frac{x^2}{2} + C,$$

or

$$|y| = e^{\frac{x^2}{2}+C} = e^{\frac{x^2}{2}} e^C = D e^{\frac{x^2}{2}},$$

where $D > 0$ is some constant. Because $y = 0$ is also a solution and because of the absolute value we can write:

$$y = D e^{\frac{x^2}{2}},$$

for any number D (including zero or negative).

We check:

$$y' = D x e^{\frac{x^2}{2}} = x \left(D e^{\frac{x^2}{2}} \right) = xy.$$

Yay!

We should be a little bit more careful with this method. You may be worried that we integrated in two different variables. We seemingly did a different operation to each side. Let us work through this method more rigorously. Take

$$\frac{dy}{dx} = f(x) g(y).$$

We rewrite the equation as follows. Note that $y = y(x)$ is a function of x and so is $\frac{dy}{dx}$!

$$\frac{1}{g(y)} \frac{dy}{dx} = f(x).$$

We integrate both sides with respect to x:

$$\int \frac{1}{g(y)} \frac{dy}{dx} \, dx = \int f(x) \, dx + C.$$

We use the change of variables formula (substitution) on the left hand side:

$$\int \frac{1}{g(y)} \, dy = \int f(x) \, dx + C.$$

And we are done.

1.3.2 Implicit solutions

We sometimes get stuck even if we can do the integration. Consider the separable equation

$$y' = \frac{xy}{y^2+1}.$$

We separate variables,

$$\frac{y^2+1}{y} \, dy = \left(y + \frac{1}{y} \right) dy = x \, dx.$$

We integrate to get

$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + C,$$

or perhaps the easier looking expression (where $D = 2C$)

$$y^2 + 2\ln|y| = x^2 + D.$$

It is not easy to find the solution explicitly as it is hard to solve for y. We, therefore, leave the solution in this form and call it an *implicit solution*. It is still easy to check that an implicit solution satisfies the differential equation. In this case, we differentiate with respect to x, and remember that y is a function of x, to get

$$y'\left(2y + \frac{2}{y}\right) = 2x.$$

Multiply both sides by y and divide by $2(y^2+1)$ and you will get exactly the differential equation. We leave this computation to the reader.

If you have an implicit solution, and you want to compute values for y, you might have to be tricky. You might get multiple solutions y for each x, so you have to pick one. Sometimes you can graph x as a function of y, and then flip your paper. Sometimes you have to do more.

Computers are also good at some of these tricks. More advanced mathematical software usually has some way of plotting solutions to implicit equations. For example, for $C = 0$ if you plot all the points (x, y) that are solutions to $y^2 + 2\ln|y| = x^2$, you find the two curves in Figure 1.8 on the following page. This is not quite a graph of a function. For each x there are two choices of y. To find a function you would have to pick one of these two curves. You pick the one that satisfies your initial condition if you have one. For example, the top curve satisfies the condition $y(1) = 1$. So for each C we really got two solutions. As you can see, computing values from an implicit solution can be somewhat tricky. But sometimes, an implicit solution is the best we can do.

The equation above also has the solution $y = 0$. So the general solution is

$$y^2 + 2\ln|y| = x^2 + C, \qquad \text{and} \qquad y = 0.$$

These outlying solutions such as $y = 0$ are sometimes called *singular solutions*.

1.3.3 Examples of separable equations

Example 1.3.2: Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2$, $y(1) = 0$.

Factor the right-hand side

$$x^2y' = (1 - x^2)(1 + y^2).$$

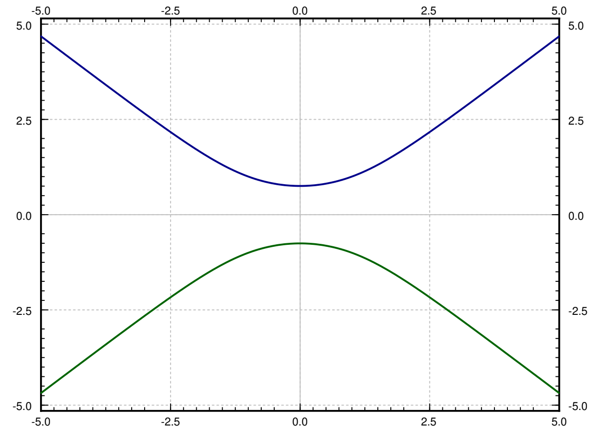


Figure 1.8: The implicit solution $y^2 + 2\ln|y| = x^2$ *to* $y' = \frac{xy}{y^2+1}$.

Separate variables, integrate, and solve for y:

$$\begin{aligned} \frac{y'}{1+y^2} &= \frac{1-x^2}{x^2}, \\ \frac{y'}{1+y^2} &= \frac{1}{x^2} - 1, \\ \arctan(y) &= \frac{-1}{x} - x + C, \\ y &= \tan\left(\frac{-1}{x} - x + C\right). \end{aligned}$$

Solve for the initial condition, $0 = \tan(-2 + C)$ to get $C = 2$ (or $C = 2 + \pi$, or $C = 2 + 2\pi$, etc.). The particular solution we seek is, therefore,

$$y = \tan\left(\frac{-1}{x} - x + 2\right).$$

Example 1.3.3: Bob made a cup of coffee, and Bob likes to drink coffee only once reaches 60 degrees Celsius and will not burn him. Initially at time $t = 0$ minutes, Bob measured the temperature and the coffee was 89 degrees Celsius. One minute later, Bob measured the coffee again and it had 85 degrees. The temperature of the room (the ambient temperature) is 22 degrees. When should Bob start drinking?

Let T be the temperature of the coffee in degrees Celsius, and let A be the ambient (room) temperature, also in degrees Celsius. Newton's law of cooling states that the rate at which the temperature of the coffee is changing is proportional to the difference between the ambient temperature and the temperature of the coffee. That is,

$$\frac{dT}{dt} = k(A - T),$$

for some constant k. For our setup $A = 22$, $T(0) = 89$, $T(1) = 85$. We separate variables and integrate (let C and D denote arbitrary constants):

$$\begin{aligned} \frac{1}{T-A}\,\frac{dT}{dt} &= -k, \\ \ln(T-A) &= -kt + C, \qquad \text{(note that } T - A > 0\text{)} \\ T - A &= D\,e^{-kt}, \\ T &= A + D\,e^{-kt}. \end{aligned}$$

That is, $T = 22 + D\,e^{-kt}$. We plug in the first condition: $89 = T(0) = 22 + D$, and hence $D = 67$. So $T = 22 + 67\,e^{-kt}$. The second condition says $85 = T(1) = 22 + 67\,e^{-k}$. Solving for k we get $k = -\ln\frac{85-22}{67} \approx 0.0616$. Now we solve for the time t that gives us a temperature of 60 degrees. Namely, we solve

$$60 = 22 + 67e^{-0.0616t}$$

to get $t = -\frac{\ln\frac{60-22}{67}}{0.0616} \approx 9.21$ minutes. So Bob can begin to drink the coffee at just over 9 minutes from the time Bob made it. That is probably about the amount of time it took us to calculate how long it would take. See Figure 1.9.

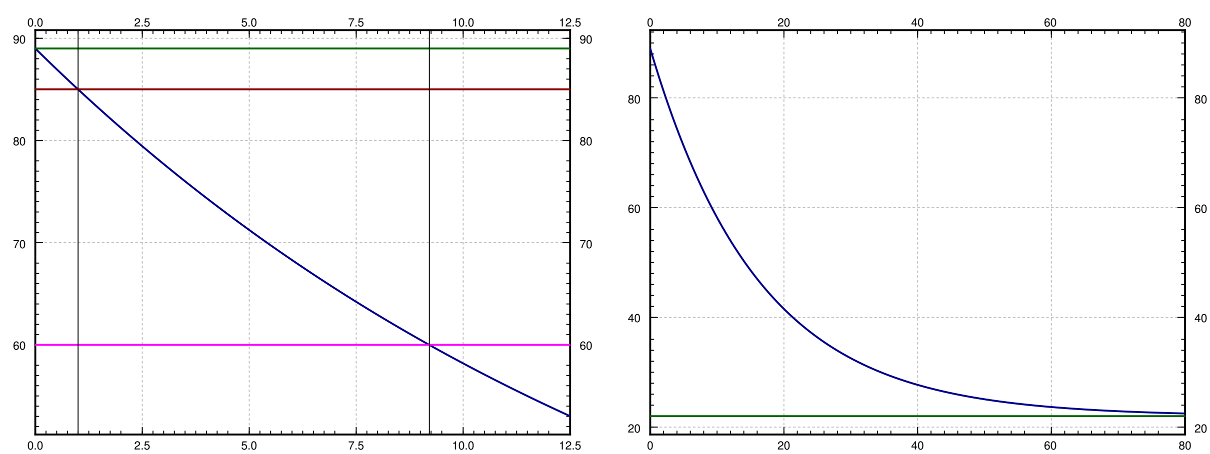


Figure 1.9: Graphs of the coffee temperature function $T(t)$. On the left, horizontal lines are drawn at temperatures 60, 85, and 89. Vertical lines are drawn at $t = 1$ and $t = 9.21$. Notice that the temperature of the coffee hits 85 at $t = 1$, and 60 at $t \approx 9.21$. On the right, the graph is over a longer period of time, with a horizontal line at the ambient temperature 22.

Example 1.3.4: Find the general solution to $y' = \frac{-xy^2}{3}$ (including singular solutions).

First note that $y = 0$ is a solution (a singular solution). Now assume that $y \neq 0$.

$$\begin{aligned} \frac{-3}{y^2}\,y' &= x, \\ \frac{3}{y} &= \frac{x^2}{2} + C, \end{aligned}$$

$$y = \frac{3}{x^2/2 + C} = \frac{6}{x^2 + 2C}.$$

So the general solution is,

$$y = \frac{6}{x^2 + 2C}, \qquad \text{and} \qquad y = 0.$$

Example 1.3.5: Exponential growth $x' = kx,\ k > 0$

A culture initially contains 20,000 bacteria. After 5 hours there are 400,000 bacteria. Determine the function $P(t)$ expressing population as a function of time t (in hours). What is the rate of growth when the population is 1 million bacteria? Round to the nearest 1000 bacteria/hour.

The population is given by the separable DE:

$$P' = kP$$

with the IC:

$$P(0) = 20{,}000.$$

Solving,

$$\begin{aligned} \frac{dP}{P} &= kdt \\ \ln P &= kt + C \\ P(t) &= P_0 e^{kt} \ \text{ where } P_0 = P(0) = 20{,}000. \end{aligned}$$

To find the growth constant k:

$$\begin{aligned} P(5) = 400{,}000 &= 20{,}000 e^{k(5)} \\ k &= \frac{\ln 20}{5}\,(\text{hr})^{-1} \end{aligned}$$

Then $P(t) = 20{,}000 e^{kt}$

At the time t when $P(t) = 10^6$ bacteria, the growth rate of the population is:

$$P' = kP = \frac{\ln 20}{5}(10^6) \approx 599{,}000 \text{ bacteria/hour}$$

Example 1.3.6: Exponential decay $x' = -\lambda x,\ \lambda > 0.$

A sandal made of a cedar was found in an archaeological excavation at Uruk in Mesopotamia. A radiochemical analysis showed that the sandal contained 54% of the radioactive isotope ^{14}C present in a living cedar tree. How old is the sandal? Round to the nearest century.

Let $S(t)$ = amount of ^{14}C present at t years after the sandal was made, and suppose $\lambda = 0.00012\,(\text{yr})^{-1}$ is the decay constant for ^{14}C. Then

$$\begin{aligned} S(t) &= S_0 e^{-0.00012t} \\ S(t) &= 0.54S_0 = S_0 e^{-0.00012t}, \\ t &= \frac{\ln 0.54}{-0.00012} \approx 5100 \text{ years.} \end{aligned}$$

1.3.4 Exercises

***Exercise* 1.3.1:** *Solve* $y' = x/y$.

***Exercise* 1.3.2:** *Solve* $y' = x^2 y$.

***Exercise* 1.3.3:** *Solve* $\frac{dx}{dt} = (x^2 - 1)\,t$, *for* $x(0) = 0$.

***Exercise* 1.3.4:** *Solve* $\frac{dx}{dt} = x \sin(t)$, *for* $x(0) = 1$.

***Exercise* 1.3.5:** *Solve* $\frac{dy}{dx} = xy + x + y + 1$. *Hint: Factor the right-hand side.*

***Exercise* 1.3.6:** *Solve* $xy' = y + 2x^2 y$, *where* $y(1) = 1$.

***Exercise* 1.3.7:** *Solve* $\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$, *for* $y(0) = 1$.

***Exercise* 1.3.8:** *Find an implicit solution for* $\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$, *for* $y(0) = 1$.

***Exercise* 1.3.9:** *Find an explicit solution for* $y' = xe^{-y}$, $y(0) = 1$.

***Exercise* 1.3.10:** *Find an explicit solution for* $xy' = e^{-y}$, *for* $y(1) = 1$.

***Exercise* 1.3.11:** *Find an explicit solution for* $y' = ye^{-x^2}$, $y(0) = 1$. *It is alright to leave a definite integral in your answer.*

***Exercise* 1.3.12:** *Suppose a cup of coffee is at* 100 *degrees Celsius at time* $t = 0$, *it is at* 70 *degrees at* $t = 10$ *minutes, and it is at* 50 *degrees at* $t = 20$ *minutes. Compute the ambient temperature.*

***Exercise* 1.3.51:** *Sixteen grams of a radioactive substance decays to twelve grams in* 500 *years. Let* $S(t)$ *be the number of grams remaining at time t years.*

a) *Find the decay constant of this substance. Determine the exact value and an approximation.*

b) *What is the half-life of the radioactive substance?*

c) *How much will remain after* 1000 *years? What is the* ***rate of disintegration*** *at this time?*

Exercise* 1.3.52*: *Let $P(t)$ be the population of a certain species of insect at time t, where t is measured in days. Suppose a population of* 30,000 *insects grows to* 150,000 *in* 3 *days.*

a) *Find the growth constant for this population.*

b) *Write the corresponding initial value problem (DE and IC) and its solution.*

c) *How long will it take for the population to triple?*

Exercise* 1.3.53*: *Suppose that* \$10,000 *is invested at* 5.5% *interest. Let $A(t)$ be the amount in this account at time t years.*

a) *Write the DE, IC and solution for this problem.*

b) *How much is in the account after* 5 *years? What is the rate of growth at this time?*

c) *When there is* \$20,000 *in the account, what is the rate of growth?*

d) *When will the original amount triple?*

Exercise* 1.3.54*: *Assume that the motion of a car is subject to a combined resistive force due to friction and wind resistance. Newton's Second Law gives $m\,v' = r\,v$ or $v' = -k\,v$.*

a) *The car is moving with a constant speed on a straight road at* 70 *mph when the engine suddenly stops. At* 1 *minute after the engine has stopped, the speed of the car is* 56 *mph. Solve the IVP for $v(t)$.*

b) *Determine the position function $x(t)$.*

c) *After the engine stops, how far does the car go before it stops moving?*

Exercise* 1.3.101*: *Solve $y' = 2xy$.*

Exercise* 1.3.102*: *Solve $x' = 3xt^2 - 3t^2$, $x(0) = 2$.*

Exercise* 1.3.103*: *Find an implicit solution for $x' = \frac{1}{3x^2+1}$, $x(0) = 1$.*

Exercise* 1.3.104*: *Find an explicit solution to $xy' = y^2$, $y(1) = 1$.*

Exercise* 1.3.105*: *Find an implicit solution to $y' = \frac{\sin(x)}{\cos(y)}$.*

Exercise* 1.3.106*: *Take Example 1.3.3 with the same numbers:* 89 *degrees at $t = 0$,* 85 *degrees at $t = 1$, and ambient temperature of* 22 *degrees. Suppose these temperatures were measured with precision of* ± 0.5 *degrees. Given this imprecision, the time it takes the coffee to cool to (exactly)* 60 *degrees is also only known in a certain range. Find this range. Hint: Think about what kind of error makes the cooling time longer and what shorter.*

***Exercise* 1.3.107:** *A population x of rabbits on an island is modeled by $x' = x - \left(1/1000\right)x^2$, where the independent variable is time in months. At time $t = 0$, there are 40 rabbits on the island.*

a) *Find the solution to the equation with the initial condition.*

b) *How many rabbits are on the island in 1 month, 5 months, 10 months, 15 months (round to the nearest integer).*

***Exercise* 1.3.151:** *In 1982, a local sponge diver discovered a shipwreck off the coast of Uluburun in south-western Turkey. Dendrochronology dated the ship in the late Bronze Age, about 1305 B.C.E. If a radiochemical assay had been performed in 2010 on a wooden writing tablet recovered from the shipwreck, what percentage of ^{14}C would have been found in the tablet? Round to the nearest percent. Use the value $\lambda = 0.00012\,(yr)^{-1}$ for the decay constant of ^{14}C.*

***Exercise* 1.3.152:** *A long-term investment account guarantees 3% annual interest.*

a) *How long would it take for the initial amount to double?*

b) *What should be invested initially, if one wishes to have \$250,000 in 25 years?*

***Exercise* 1.3.153:** *A radioactive substance decays to 4/5 of its original mass in 7 years. What is the half-life of this substance?*

***Exercise* 1.3.154:** *The population of a city was 0.9 million in 1995 and 1.2 million in 2000. Assuming an exponential model,*

a) *Find the growth constant k.*

b) *Letting $P(t)$ be the population with t in years since 1995, write the IVP and the solution.*

c) *What was the population in 2015?*

d) *What will the population be in 2025?*

***Exercise* 1.3.155:** *A long-term certificate of deposit (CD) with continuous compounding is opened at a bank. The terms of the CD specify no additional deposits and no withdrawals.*

a. *What is the interest rate r if the amount of the deposit grows by a factor of $\frac{4}{3}$ in 10 years?*

b. *What was the initial deposit if the amount in the account is \$40,000 after 8 years?*

***Exercise* 1.3.156:** *Determine the decay constant λ for a radioactive substance if the mass of this substance is m_1 at time t_1 and m_2 at time t_2, $0 < t_1 < t_2$ years.*

***Exercise* 1.3.157:** *A site close to a nuclear power plant was found to be contaminated by Strontium-90 (^{90}Sr), which has a half-life of 28.8 years. If the site has 40 times the maximum level considered safe for human habitation, how long should this site remain uninhabited?*

Find an explicit solution to each of the followings IVPs:

Exercise* 1.3.158*: $y' - 2xy = 3x^2y$, $y(1) = 1$.

Exercise* 1.3.159*: $(x^2 - 1)y' = 2y$, $y(2) = 3$.

Exercise* 1.3.160*: $yy' = \frac{x}{x^2+1}$, $y(0) = -2$.

Exercise* 1.3.161*: $\cot(x)y' = y$, $y(0) = 2$.

Exercise* 1.3.162*: $e^{-x}y' = \frac{x}{y}$, $y(0) = -5$.

1.4 Linear equations and the integrating factor

Note: 1 lecture, §1.5 in [EP], §2.1 in [BD]

One of the most important types of equations we will learn how to solve are the so-called *linear equations*. In fact, the majority of the course is about linear equations. In this section we focus on the *first order linear equation*. A first order equation is linear if we can put it into the form:

$$y' + p(x)y = f(x). \tag{1.3}$$

The word "linear" means linear in y and y'; no higher powers nor functions of y or y' appear. The dependence on x can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations.

The trick is to rewrite the left-hand side of (1.3) as a derivative of a product of y with another function. To this end we find a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}\Big[r(x)y\Big].$$

This is the left-hand side of (1.3) multiplied by $r(x)$. So if we multiply (1.3) by $r(x)$, we obtain

$$\frac{d}{dx}\Big[r(x)y\Big] = r(x)f(x).$$

Now we integrate both sides. The right-hand side does not depend on y and the left-hand side is written as a derivative of a function. Afterwards, we solve for y. The function $r(x)$ is called the *integrating factor* and the method is called the *integrating factor method*.

We are looking for a function $r(x)$, such that if we differentiate it, we get the same function back multiplied by $p(x)$. That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x)\,dx}.$$

We compute:

$$\begin{aligned}
y' + p(x)y &= f(x), \\
e^{\int p(x)\,dx} y' + e^{\int p(x)\,dx} p(x)y &= e^{\int p(x)\,dx} f(x), \\
\frac{d}{dx}\left[e^{\int p(x)\,dx} y\right] &= e^{\int p(x)\,dx} f(x), \\
e^{\int p(x)\,dx} y &= \int e^{\int p(x)\,dx} f(x)\,dx + C, \\
y &= e^{-\int p(x)\,dx}\left(\int e^{\int p(x)\,dx} f(x)\,dx + C\right).
\end{aligned}$$

Of course, to get a closed form formula for y, we need to be able to find a closed form formula for the integrals appearing above.

Example 1.4.1: Solve

$$y' + 2xy = e^{x-x^2}, \qquad y(0) = -1.$$

First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x)\,dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$ to get

$$\begin{aligned} e^{x^2} y' + 2xe^{x^2} y &= e^{x-x^2} e^{x^2}, \\ \frac{d}{dx}\left[e^{x^2} y\right] &= e^x. \end{aligned}$$

We integrate

$$\begin{aligned} e^{x^2} y &= e^x + C, \\ y &= e^{x-x^2} + Ce^{-x^2}. \end{aligned}$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

***Exercise* 1.4.1:** *Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above.*

Advice: Do not try to remember the formula itself, that is way too hard. It is easier to remember the process and repeat it.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \qquad y(x_0) = y_0.$$

Look at the solution and write the integrals as definite integrals.

$$\boxed{y(x) = e^{-\int_{x_0}^{x} p(s)\,ds} \left(\int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)\,ds} f(t)\,dt + y_0 \right).} \tag{1.4}$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

***Exercise* 1.4.2:** *Check that $y(x_0) = y_0$ in formula (1.4).*

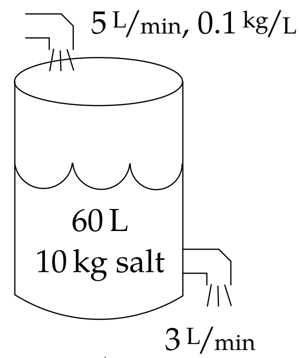
Exercise 1.4.3: *Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.*

$$y' + y = e^{x^2-x}, \qquad y(0) = 10.$$

Remark 1.4.1: Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem $y' + p(x)y = f(x)$, $y(x_0) = y_0$, there is always an explicit formula (1.4) for the solution. Second, it follows from the formula (1.4) that if $p(x)$ and $f(x)$ are continuous on some interval (a, b), then the solution $y(x)$ exists and is differentiable on (a, b). Compare with the simple nonlinear example we have seen previously, $y' = y^2$, and compare to Theorem 1.2.1.

Example 1.4.2: Let us discuss a common simple application of linear equations. This type of problem is used often in real life. For example, linear equations are used in figuring out the concentration of chemicals in bodies of water (rivers and lakes).

A 100 liter tank contains 10 kilograms of salt dissolved in 60 liters of water. Solution of water and salt (brine) with concentration of 0.1 kilograms per liter is flowing in at the rate of 5 liters a minute. The solution in the tank is well stirred and flows out at a rate of 3 liters a minute. How much salt is in the tank when the tank is full?



Let us come up with the equation. Let x denote the kilograms of salt in the tank, let t denote the time in minutes. For a small change Δt in time, the change in x (denoted Δx) is approximately

$$\Delta x \approx (\text{rate in} \times \text{concentration in})\Delta t - (\text{rate out} \times \text{concentration out})\Delta t.$$

Dividing through by Δt and taking the limit $\Delta t \to 0$ we see that

$$\frac{dx}{dt} = (\text{rate in} \times \text{concentration in}) - (\text{rate out} \times \text{concentration out}).$$

In our example, we have

$$\begin{aligned} \text{rate in} &= 5, \\ \text{concentration in} &= 0.1, \\ \text{rate out} &= 3, \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5-3)t}. \end{aligned}$$

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \times 0.1) - \left(3 \frac{x}{60 + 2t}\right).$$

Or in the form (1.3)

$$\frac{dx}{dt} + \frac{3}{60 + 2t} x = 0.5.$$

Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60+2t}\,dt\right) = \exp\left(\frac{3}{2}\ln(60+2t)\right) = (60+2t)^{3/2}.$$

We multiply both sides of the equation to get

$$\begin{aligned}
(60+2t)^{3/2}\frac{dx}{dt} + (60+2t)^{3/2}\frac{3}{60+2t}x &= 0.5(60+2t)^{3/2},\\
\frac{d}{dt}\left[(60+2t)^{3/2}x\right] &= 0.5(60+2t)^{3/2},\\
(60+2t)^{3/2}x &= \int 0.5(60+2t)^{3/2}\,dt + C,\\
x &= (60+2t)^{-3/2}\int \frac{(60+2t)^{3/2}}{2}\,dt + C(60+2t)^{-3/2},\\
x &= (60+2t)^{-3/2}\frac{1}{10}(60+2t)^{5/2} + C(60+2t)^{-3/2},\\
x &= \frac{60+2t}{10} + C(60+2t)^{-3/2}.
\end{aligned}$$

We need to find C. We know that at $t=0$, $x=10$. So

$$10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2},$$

or

$$C = 4(60^{3/2}) \approx 1859.03.$$

We are interested in x when the tank is full. The tank is full when $60+2t = 100$, or when $t = 20$. So

$$\begin{aligned}
x(20) &= \frac{60+40}{10} + C(60+40)^{-3/2}\\
&\approx 10 + 1859.03(100)^{-3/2} \approx 11.86.
\end{aligned}$$

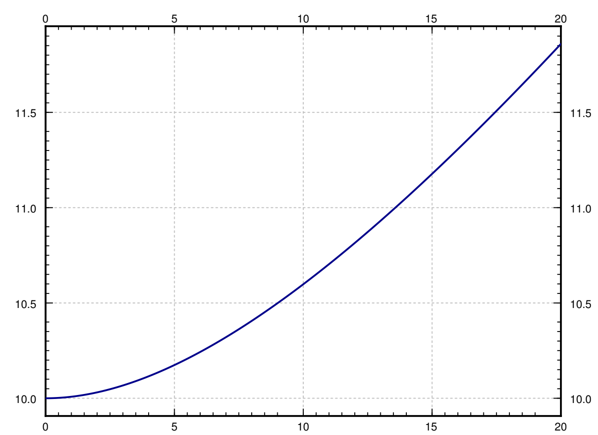


Figure 1.10: Graph of the solution x kilograms of salt in the tank at time t.

See Figure 1.10 for the graph of x over t.

The concentration when the tank is full is approximately 0.1186 kg/liter, and we started with 1/6 or 0.167 kg/liter.

Example 1.4.2: (cf. example 1.3.3), $T' = k(A-T)$:

An iron object at 400^o F is dropped into a large vat of water at $A = 60^o$ F. After 5 seconds, the temperature of the object is 300^o F.

a) Write and solve the IVP.

b) Find the constant k.

c) When will the temperature of the object be 150^o F? Round to the nearest second.

a) The IVP consists of the DE $T' = k(A - T)$ together with the IC $T(0) = 400$. Writing the DE in the standard form for a linear, first order equation, one obtains $T' + kT = 60k$. Applying the method of integrating factors gives:

$$\begin{aligned} r(t) &= e^{\int p(t)\,dt} = e^{kt} \\ T'e^{kt} + ke^{kt}T &= 60ke^{kt} \\ (e^{kt}T)' &= 60ke^{kt} \\ \text{So,} \qquad e^{kt}T &= 60e^{kt} + C \\ \text{or} \qquad T(t) &= 60 + Ce^{-kt}. \end{aligned}$$

Imposing the IC allows one to solve for C:

$$\begin{aligned} T(0) = 60 + C = 400,\ C &= 340 \\ T(t) &= 60 + 340e^{-kt}. \end{aligned}$$

b) The value of the constant k is determined by the temperature of the object at $t = 5$ seconds:

$$\begin{aligned} T(5) = 60 + 340e^{-k(5)} &= 300 \\ k = \frac{\ln \frac{12}{17}}{-5} &= \frac{1}{5}\ln\frac{17}{12}\ (\text{sec})^{-1} \end{aligned}$$

c) If the temperature of the object is 150^oF at some time t, then $T(t) = 150$:

$$\begin{aligned} 60 + 340e^{-kt} &= 150 \\ t = \frac{\ln \frac{9}{34}}{-k} &= \frac{\ln \frac{34}{9}}{k} \approx \frac{\ln \frac{34}{9}}{0.070} \approx 19\ \text{ seconds.} \end{aligned}$$

1.4.1 Exercises

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

***Exercise* 1.4.4:** *Solve* $y' + xy = x$.

***Exercise* 1.4.5:** *Solve* $y' + 6y = e^x$.

***Exercise* 1.4.6:** *Solve* $y' + 3x^2y = \sin(x)\,e^{-x^3}$, *with* $y(0) = 1$.

Exercise 1.4.7: *Solve* $y' + \cos(x)y = \cos(x)$.

Exercise 1.4.8: *Solve* $\frac{1}{x^2+1}\, y' + xy = 3$, *with* $y(0) = 0$.

Exercise 1.4.9: *Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is* 500 *liters per hour. The first lake contains* 100 *thousand liters of water and the second lake contains* 200 *thousand liters of water. A truck with* 500 *kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream.*

a) *Find the concentration of toxic substance as a function of time in both lakes.*

b) *When will the concentration in the first lake be below* 0.001 *kg per liter*?

c) *When will the concentration in the second lake be maximal*?

Exercise 1.4.10: *Newton's law of cooling states that* $\frac{dx}{dt} = -k(x - A)$ *where* x *is the temperature,* t *is time,* A *is the ambient temperature, and* $k > 0$ *is a constant. Suppose that* $A = A_0 \cos(\omega t)$ *for some constants* A_0 *and* ω. *That is, the ambient temperature oscillates* (*for example night and day temperatures*).

a) *Find the general solution.*

b) *In the long term, will the initial conditions make much of a difference*? *Why or why not*?

Exercise 1.4.11: *Initially* 5 *grams of salt are dissolved in* 20 *liters of water. Brine with concentration of salt* 2 *grams of salt per liter is added at a rate of* 3 *liters a minute. The tank is mixed well and is drained at* 3 *liters a minute. How long does the process have to continue until there are* 20 *grams of salt in the tank*?

Exercise 1.4.12: *Initially a tank contains* 10 *liters of pure water. Brine of unknown* (*but constant*) *concentration of salt is flowing in at* 1 *liter per minute. The water is mixed well and drained at* 1 *liter per minute. In* 20 *minutes there are* 15 *grams of salt in the tank. What is the concentration of salt in the incoming brine*?

Exercise 1.4.51: (*cf. example* 1.3.3), $T' = k(A - T)$: *The acceleration* $v'(t)$ *of a sports car is proportional to the difference between* 180 *miles/hr and the velocity* $v(t)$ *of the car. If this car can accelerate from rest to* 60*mph in* 5 *seconds*:

a) *Apply the method of integrating factors* (*IFs*) *to determine the particular solution to the IVP.*

b) *How fast was the car going at* 3 *seconds*?

c) *What is* $\lim_{t\to\infty} v(t)$?

d) *When will the velocity be* 5/9 *of the limiting velocity*?

***Exercise* 1.4.101:** *Solve* $y' + 3x^2 y = x^2$.

***Exercise* 1.4.102:** *Solve* $y' + 2\sin(2x)y = 2\sin(2x)$, $y(\pi/2) = 3$.

***Exercise* 1.4.103:** *Suppose a water tank is being pumped out at* 3 L/min. *The water tank starts at* 10 L *of clean water. Water with toxic substance is flowing into the tank at* 2 L/min, *with concentration* $20t$ g/L *at time* t. *When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?*

***Exercise* 1.4.104:** *Suppose we have bacteria on a plate and suppose that we are slowly adding a toxic substance such that the rate of growth is slowing down. That is, suppose that* $\frac{dP}{dt} = (2 - 0.1\,t)P$. *If* $P(0) = 1000$, *find the population at* $t = 5$.

***Exercise* 1.4.105:** *A cylindrical water tank has water flowing in at* I *cubic meters per second. Let* A *be the area of the cross section of the tank in meters. Suppose water is flowing from the bottom of the tank at a rate proportional to the height of the water level. Set up the differential equation for* h, *the height of the water, introducing and naming constants that you need. You should also give the units for your constants.*

Apply the method of IFs to solve the following problems.

***Exercise* 1.4.151:** *Before opening the parachute a skydiver jumping out of an airplane falls at an increasing rate. However, air resistance creates an upward force which balances the downward force of gravity, resulting in a constant terminal velocity* T. *If* $v(t)$ *the* <u>*downward*</u> *velocity of the skydiver at t seconds, then* $v' = k(T - v)$.
If the initial velocity is 10 *feet/second, the velocity after* 3 *seconds is* 40 *feet/second and the terminal velocity is* 70 *ft/sec,*

a) *Apply the method of IFs to determine the particular solution to the IVP, before the parachute opens.*

b) *When will the velocity of the skydiver be* 60 *ft/sec?*

***Exercise* 1.4.152:** *Suppose the spread of information by mass media is proportional to the difference between* 100% *and* $x(t)$, *the* <u>*percentage*</u> *of the population knowing the information after* t *hours. Suppose that* 10% *of the population knows the information initially and that* 30% *knows the information after* 4 *hours.*

a) *Solve the IVP.*

b) *What percentage of the population will know this information after* 7 *hours?*

c) *What is the rate of dissemination of this information (in % of the population/hour) at* 7 *hours?*

***Exercise* 1.4.153:** *A boat has mass* 200 *slugs (approximate weight* 6,435 *pounds). The boat motor provides a constant thrust of* 5,000 *pounds force. Assume the total resistive force due to water and air resistance is proportional to velocity and the coefficient of resistance is* 100 *pounds force per ft/sec of velocity. Applying Newton's Second Law gives,* $200v' = 5,000 - 100v$ *for velocity* $v(t)$ *ft/sec.*

a) *If the boat starts from rest, determine* $v(t)$.

b) *When will the velocity of the boat be* 80% *of its maximum velocity*?

c) *How far has the boat gone at the time specified in part* b)?

***Exercise* 1.4.154:** *An individual retires with a retirement account which yields* 4% *interest per year, compounded continuously. Online banking allows continuous withdrawals for living expenses at a rate of* \$30,000 *per year.*

a) *If the account balance is* \$300,000 *when this person retires, solve the IVP for the amount* $A(t)$ *in the retirement account at time* t *years.*

b) *How long will it take for this account to close due to a zero balance.*

c) *What initial amount* A_0 *must be in the account so that the rate of growth due to interest equals the rate of withdrawals*?

***Exercise* 1.4.155:** *A constant horizontal force of* 8N *is applied to a* 2kg *mass. The total resistive force of the level surface is proportional to velocity, with a coefficient of* $\frac{1}{5}\frac{N-s}{m}$.

a) *If the initial velocity is* 1 *m/s, solve the IVP for* $v(t)$.

b) *What was the initial velocity if the object is moving* 20 *m/s at* 5 *seconds*?

***Exercise* 1.4.156:** *The concentration* $C(t)$ *of solute inside a cell changes with time due to the passage of solute across the cell membrane. The rate of change is given by* $C' = k(M - C)$, *where* $C(t)$ *is the concentration of solute inside the cell,* M *is the concentration of solute outside the cell, assumed constant,* $k > 0$ *is a constant and* t *is the time measured in seconds.*

a) *Suppose that the initial concentration inside the cell is* $C(0) = 20$ *μg/mL, that* $M = 100$ *μg/mL, and that after* 3 *seconds the concentration inside the cell is* $C(3) = 35$ *μg/mL. Solve the IVP for* $C(t)$ *and determine the value of* k.

b) *How long will it take for the concentration* $C(t)$ *to increase from* 35 *μg/mL to* 45 *μg/mL*?

***Exercise* 1.4.157:** *A piece of iron heated to* $450^o F$ *is placed outside in order to create this problem. The outside temperature is a constant* $75^o F$. *At* 2 *PM the temperature of the iron is* $400^o F$, *and* 2 *minutes later temperature is* $375^o F$.

a) *When was this piece of iron placed outside*?

b) *When will its temperature be* $300^o F$?

Solve the following differential equations(DEs) by the method of integrating factors. If an initial condition(IC) is given, find the particular solution to the IVP.

Exercise* 1.4.158*: $3y' - 6y = 12,\ y(0) = 4.$

Exercise* 1.4.159*: $y' + 2xy = 2x.$

Exercise* 1.4.160*: $y' - 6x^2y = 0,\ y(1) = e.$

Exercise* 1.4.161*: $x^2y' = 3xy + 4x^6.$

Exercise* 1.4.162*: $xy' + 2y = 5\sqrt{x}.$

Exercise* 1.4.163*: $y' + (\cos x)y = 2\cos x,\ y(\pi) = 5.$

Exercise* 1.4.164*: $(x+2)y' - y = \frac{3}{x+2}.$

Exercise* 1.4.165*: $x^2y' + xy = 1\ (x > 0),\ y(1) = 3.$

Exercise* 1.4.166*: $y' + 3(\sin 3x)y = e^{\cos 3x},\ y(0) = e^2.$

Exercise* 1.4.167*: $(x^2+1)y' + 2xy = \frac{1}{x^2+1}.$

Exercise* 1.4.168*: $y' = x - 2y.$

1.5 Substitution

Note: 1 lecture, can safely be skipped, §1.6 in [EP], not in [BD]

Just as when solving integrals, one method to try is to change variables to end up with a simpler equation to solve.

1.5.1 Substitution

The equation
$$y' = (x - y + 1)^2$$
is neither separable nor linear. What can we do? How about trying to change variables, so that in the new variables the equation is simpler. We use another variable v, which we treat as a function of x. Let us try
$$v = x - y + 1.$$

We need to figure out y' in terms of v', v and x. We differentiate (in x) to obtain $v' = 1 - y'$. So $y' = 1 - v'$. We plug this into the equation to get
$$1 - v' = v^2.$$

In other words, $v' = 1 - v^2$. Such an equation we know how to solve by separating variables:
$$\frac{1}{1 - v^2}\, dv = dx.$$

So
$$\frac{1}{2} \ln \left| \frac{v+1}{v-1} \right| = x + C, \qquad \text{or} \qquad \left| \frac{v+1}{v-1} \right| = e^{2x+2C}, \qquad \text{or} \qquad \frac{v+1}{v-1} = De^{2x},$$
for some constant D. Note that $v = 1$ and $v = -1$ are also solutions.

Now we need to "unsubstitute" to obtain
$$\frac{x - y + 2}{x - y} = De^{2x},$$

and also the two solutions $x - y + 1 = 1$ or $y = x$, and $x - y + 1 = -1$ or $y = x + 2$. We solve the first equation for y.
$$\begin{aligned}
x - y + 2 &= (x - y)De^{2x}, \\
x - y + 2 &= Dxe^{2x} - yDe^{2x}, \\
-y + yDe^{2x} &= Dxe^{2x} - x - 2, \\
y\left(-1 + De^{2x}\right) &= Dxe^{2x} - x - 2, \\
y &= \frac{Dxe^{2x} - x - 2}{De^{2x} - 1}.
\end{aligned}$$

Note that $D = 0$ gives $y = x + 2$, but no value of D gives the solution $y = x$.

Substitution in differential equations is applied in much the same way that it is applied in calculus. You guess. Several different substitutions might work. There are some general patterns to look for. We summarize a few of these in a table.

When you see	Try substituting
yy'	$v = y^2$
y^2y'	$v = y^3$
$(\cos y)y'$	$v = \sin y$
$(\sin y)y'$	$v = \cos y$
$y'e^y$	$v = e^y$

Usually you try to substitute in the "most complicated" part of the equation with the hopes of simplifying it. The table above is just a rule of thumb. You might have to modify your guesses. If a substitution does not work (it does not make the equation any simpler), try a different one.

1.5.2 Bernoulli equations

There are some forms of equations where there is a general rule for substitution that always works. One such example is the so-called *Bernoulli equation**:

$$y' + p(x)y = q(x)y^n.$$

This equation looks a lot like a linear equation except for the y^n. If $n = 0$ or $n = 1$, then the equation is linear and we can solve it. Otherwise, the substitution $v = y^{1-n}$ transforms the Bernoulli equation into a linear equation. Note that n need not be an integer.

Example 1.5.1: Solve

$$xy' + y(x+1) + xy^5 = 0, \qquad y(1) = 1.$$

First, the equation is Bernoulli ($p(x) = (x+1)/x$ and $q(x) = -1$). We substitute

$$v = y^{1-5} = y^{-4}, \qquad v' = -4y^{-5}y'.$$

In other words, $(-1/4)\, y^5 v' = y'$. So

$$\begin{aligned} xy' + y(x+1) + xy^5 &= 0, \\ \frac{-xy^5}{4}v' + y(x+1) + xy^5 &= 0, \end{aligned}$$

*There are several things called Bernoulli equations, this is just one of them. The Bernoullis were a prominent Swiss family of mathematicians. These particular equations are named for Jacob Bernoulli (1654–1705).

$$\frac{-x}{4}v' + y^{-4}(x+1) + x = 0,$$
$$\frac{-x}{4}v' + v(x+1) + x = 0,$$

and finally

$$v' - \frac{4(x+1)}{x}v = 4.$$

The equation is now linear. We can use the integrating factor method. In particular, we use formula (1.4). Let us assume that $x > 0$ so $|x| = x$. This assumption is OK, as our initial condition is $x = 1$. Let us compute the integrating factor. Here $p(s)$ from formula (1.4) is $\frac{-4(s+1)}{s}$.

$$e^{\int_1^x p(s)\,ds} = \exp\left(\int_1^x \frac{-4(s+1)}{s}\,ds\right) = e^{-4x-4\ln(x)+4} = e^{-4x+4}x^{-4} = \frac{e^{-4x+4}}{x^4},$$
$$e^{-\int_1^x p(s)\,ds} = e^{4x+4\ln(x)-4} = e^{4x-4}x^4.$$

We now plug in to (1.4)

$$v(x) = e^{-\int_1^x p(s)\,ds}\left(\int_1^x e^{\int_1^t p(s)\,ds}4\,dt + 1\right)$$
$$= e^{4x-4}x^4\left(\int_1^x 4\frac{e^{-4t+4}}{t^4}\,dt + 1\right).$$

The integral in this expression is not possible to find in closed form. As we said before, it is perfectly fine to have a definite integral in our solution. Now "unsubstitute"

$$y^{-4} = e^{4x-4}x^4\left(4\int_1^x \frac{e^{-4t+4}}{t^4}\,dt + 1\right),$$
$$y = \frac{e^{-x+1}}{x\left(4\int_1^x \frac{e^{-4t+4}}{t^4}\,dt + 1\right)^{1/4}}.$$

1.5.3 Homogeneous equations

Another type of equations we can solve by substitution are the so-called *homogeneous equations*. Suppose that we can write the differential equation as

$$y' = F\left(\frac{y}{x}\right).$$

Here we try the substitutions

$$v = \frac{y}{x} \qquad \text{and therefore} \qquad y' = v + xv'.$$

We note that the equation is transformed into

$$v + xv' = F(v) \qquad \text{or} \qquad xv' = F(v) - v \qquad \text{or} \qquad \frac{v'}{F(v) - v} = \frac{1}{x}.$$

Hence an implicit solution is

$$\int \frac{1}{F(v) - v}\, dv = \ln |x| + C.$$

Example 1.5.2: Solve

$$x^2 y' = y^2 + xy, \qquad y(1) = 1.$$

We put the equation into the form $y' = (y/x)^2 + y/x$. We substitute $v = y/x$ to get the separable equation

$$xv' = v^2 + v - v = v^2,$$

which has a solution

$$\begin{aligned} \int \frac{1}{v^2}\, dv &= \ln |x| + C, \\ \frac{-1}{v} &= \ln |x| + C, \\ v &= \frac{-1}{\ln |x| + C}. \end{aligned}$$

We unsubstitute

$$\begin{aligned} \frac{y}{x} &= \frac{-1}{\ln |x| + C}, \\ y &= \frac{-x}{\ln |x| + C}. \end{aligned}$$

We want $y(1) = 1$, so

$$1 = y(1) = \frac{-1}{\ln |1| + C} = \frac{-1}{C}.$$

Thus $C = -1$ and the solution we are looking for is

$$y = \frac{-x}{\ln |x| - 1}.$$

1.5.4 Exercises

Hint: Answers need not always be in closed form.

***Exercise* 1.5.1:** *Solve* $y' + y(x^2 - 1) + xy^6 = 0$, *with* $y(1) = 1$.

***Exercise* 1.5.2:** *Solve* $2yy' + 1 = y^2 + x$, *with* $y(0) = 1$.

Exercise* 1.5.3*: *Solve* $y' + xy = y^4$, *with* $y(0) = 1$.

Exercise* 1.5.4*: *Solve* $yy' + x = \sqrt{x^2 + y^2}$.

Exercise* 1.5.5*: *Solve* $y' = (x + y - 1)^2$.

Exercise* 1.5.6*: *Solve* $y' = \frac{x^2 - y^2}{xy}$, *with* $y(1) = 2$.

Exercise* 1.5.101*: *Solve* $xy' + y + y^2 = 0$, $y(1) = 2$.

Exercise* 1.5.102*: *Solve* $xy' + y + x = 0$, $y(1) = 1$.

Exercise* 1.5.103*: *Solve* $y^2 y' = y^3 - 3x$, $y(0) = 2$.

Exercise* 1.5.104*: *Solve* $2yy' = e^{y^2 - x^2} + 2x$.

Find a closed-form solution to the following problems. If possible, solve explicitly for $y(x)$.

Exercise* 1.5.151*: $x^2 y' = y^2 + 3xy$.

Exercise* 1.5.152*: $y' = \sqrt{x + y - 5}$.

Exercise* 1.5.153*: $x^3 yy' + x^2 y^2 = y^5$.

Exercise* 1.5.154*: $y' = (x + y + 7)^2, \; y(0) = -6$.

Exercise* 1.5.155*: $xy' = 2x + 3y, \; y(-1) = 3$.

Exercise* 1.5.156*: $(\cos y) y' = e^{2x} + 1, \; y(0) = 0$.

Exercise* 1.5.157*: $(x^2 - y^2) y' = xy$.

Exercise* 1.5.158*: $y^4 y' = -3x^2 y^5 + x^2$.

Exercise* 1.5.159*: $e^{2y} y' = 1 - e^{2y}$

Exercise* 1.5.160*: $xy^2 y' = x^3 + 2y^3$.

Exercise* 1.5.161*: $x^2 y' - x^3 e^{-1/x} y^{2/3} = 3y$.

1.6 Autonomous equations

Note: 1 lecture, §2.2 in [EP], §2.5 in [BD]

Consider problems of the form

$$\frac{dx}{dt} = f(x),$$

where the derivative of solutions depends only on x (the dependent variable). Such equations are called *autonomous equations*. If we think of t as time, the naming comes from the fact that the equation is independent of time.

We return to the cooling coffee problem (Example 1.3.3). Newton's law of cooling says

$$\frac{dx}{dt} = k(A - x),$$

where x is the temperature, t is time, k is some positive constant, and A is the ambient temperature. See Figure 1.11 for an example with $k = 0.3$ and $A = 5$.

Note the solution $x = A$ (in the figure $x = 5$). We call these constant solutions the *equilibrium solutions*. The points on the x-axis where $f(x) = 0$ are called *critical points*. The point $x = A$ is a critical point. In fact, each critical point corresponds to an equilibrium solution. Note also, by looking at the graph, that the solution $x = A$ is "stable" in that small perturbations in x do not lead to substantially different solutions as t increases. In fact in this simple example if we change the initial condition a little bit, then as $t \to \infty$ we get $x(t) \to A$. If a critical point is not stable, we say it is *unstable*.

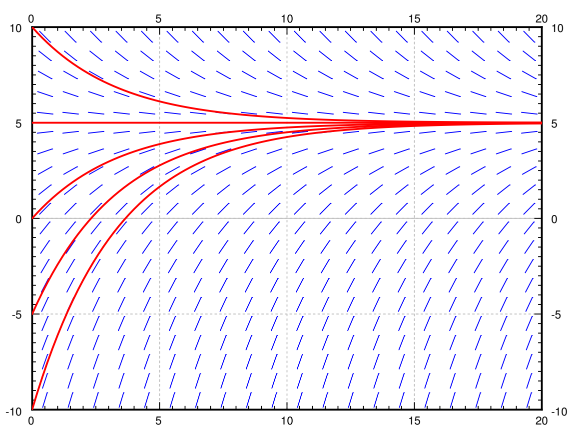


Figure 1.11: The slope field and some solutions of $x' = 0.3\,(5 - x)$.

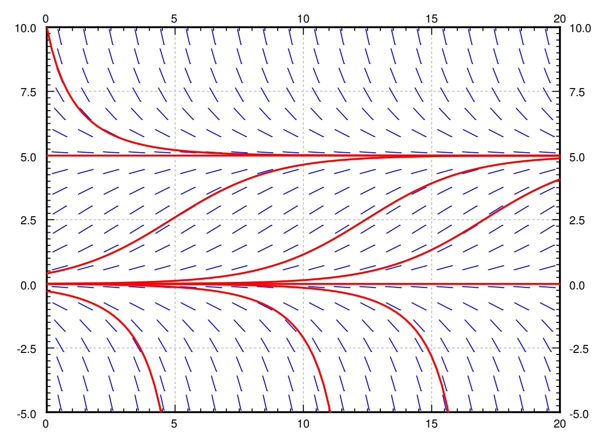


Figure 1.12: The slope field and some solutions of $x' = 0.1\,x\,(5 - x)$.

Consider now the *logistic equation*

$$\frac{dx}{dt} = kx(M - x),$$

for some positive k and M. This equation is commonly used to model population if we know the limiting population M, that is the maximum sustainable population. The logistic equation leads to less catastrophic predictions on world population than $x' = kx$. In the real world there is no such thing as negative population, but we will still consider negative x for the purposes of the math.

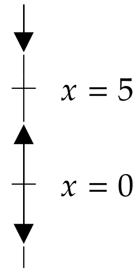
See Figure 1.12 on the preceding page for an example, $x' = 0.1x(5 - x)$. There are two critical points, $x = 0$ and $x = 5$. The critical point at $x = 5$ is stable, while the critical point at $x = 0$ is unstable.

It is not necessary to find the exact solutions to talk about the long term behavior of the solutions. From the slope field above of $x' = 0.1x(5 - x)$, we see that

$$\lim_{t\to\infty} x(t) = \begin{cases} 5 & \text{if } x(0) > 0, \\ 0 & \text{if } x(0) = 0, \\ \text{DNE or } -\infty & \text{if } x(0) < 0. \end{cases}$$

Here DNE means "does not exist." From just looking at the slope field we cannot quite decide what happens if $x(0) < 0$. It could be that the solution does not exist for t all the way to ∞. Think of the equation $x' = x^2$; we have seen that solutions only exist for some finite period of time. Same can happen here. In our example equation above it turns out that the solution does not exist for all time, but to see that we would have to solve the equation. In any case, the solution does go to $-\infty$, but it may get there rather quickly.

If we are interested only in the long term behavior of the solution, we would be doing unnecessary work if we solved the equation exactly. We could draw the slope field, but it is easier to just look at the *phase diagram* or *phase portrait*, which is a simple way to visualize the behavior of autonomous equations. In this case there is one dependent variable x. We draw the x-axis, we mark all the critical points, and then we draw arrows in between. Since x is the dependent variable we draw the axis vertically, as it appears in the slope field diagrams above. If $f(x) > 0$, we draw an up arrow. If $f(x) < 0$, we draw a down arrow. To figure this out, we could just plug in some x between the critical points, $f(x)$ will have the same sign at all x between two critical points as long $f(x)$ is continuous. For example, $f(6) = -0.6 < 0$, so $f(x) < 0$ for $x > 5$, and the arrow above $x = 5$ is a down arrow. Next, $f(1) = 0.4 > 0$, so $f(x) > 0$ whenever $0 < x < 5$, and the arrow points up. Finally, $f(-1) = -0.6 < 0$ so $f(x) < 0$ when $x < 0$, and the arrow points down.



Armed with the phase diagram, it is easy to sketch the solutions approximately: As time t moves from left to right, the graph of a solution goes up if the arrow is up, and it goes down if the arrow is down.

***Exercise* 1.6.1:** *Try sketching a few solutions simply from looking at the phase diagram. Check with the preceding graphs if you are getting the type of curves.*

Once we draw the phase diagram, we classify critical points as stable or unstable[*].

Since any mathematical model we cook up will only be an approximation to the real world, unstable points are generally bad news.

Let us think about the logistic equation with harvesting. Suppose an alien race really likes to eat humans. They keep a planet with humans on it and harvest the humans at a rate of h million humans per year. Suppose x is the number of humans in millions on the planet and t is time in years. Let M be the limiting population when no harvesting is done. The number $k > 0$ is a constant depending on how fast humans multiply. Our equation becomes

$$\frac{dx}{dt} = kx(M - x) - h.$$

We expand the right-hand side and set it to zero.

$$kx(M - x) - h = -kx^2 + kMx - h = 0.$$

Solving for the critical points, let us call them A and B, we get

$$A = \frac{kM + \sqrt{(kM)^2 - 4hk}}{2k}, \qquad B = \frac{kM - \sqrt{(kM)^2 - 4hk}}{2k}.$$

***Exercise* 1.6.2:** *Sketch a phase diagram for different possibilities. Note that these possibilities are $A > B$, or $A = B$, or A and B both complex* (*i.e. no real solutions*). *Hint*: *Fix some simple k and M and then vary h.*

For example, let $M = 8$ and $k = 0.1$. When $h = 1$, then A and B are distinct and positive. The slope field we get is in Figure 1.13 on the next page. As long as the population starts above B, which is approximately 1.55 million, then the population will not die out. It will in fact tend towards $A \approx 6.45$ million. If ever some catastrophe happens and the population drops below B, humans will die out, and the fast food restaurant serving them will go out of business.

[*] Unstable points with one of the arrows pointing towards the critical point are sometimes called *semistable*.

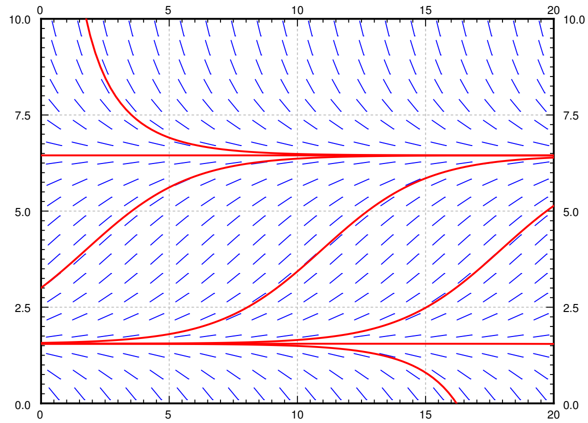


Figure 1.13: The slope field and some solutions of $x' = 0.1\,x\,(8 - x) - 1$.

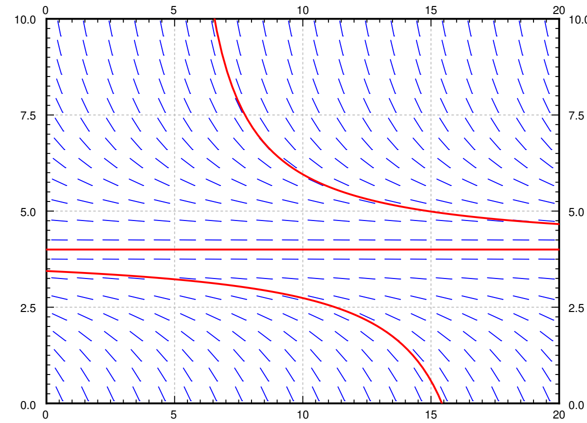


Figure 1.14: The slope field and some solutions of $x' = 0.1\,x\,(8 - x) - 1.6$.

When $h = 1.6$, then $A = B = 4$. There is only one critical point and it is unstable. When the population starts above 4 million it will tend towards 4 million. If it ever drops below 4 million, humans will die out on the planet. This scenario is not one that we (as the human fast food proprietor) want to be in. A small perturbation of the equilibrium state and we are out of business. There is no room for error. See Figure 1.14.

Finally if we are harvesting at 2 million humans per year, there are no critical points. The population will always plummet towards zero, no matter how well stocked the planet starts. See Figure 1.15.

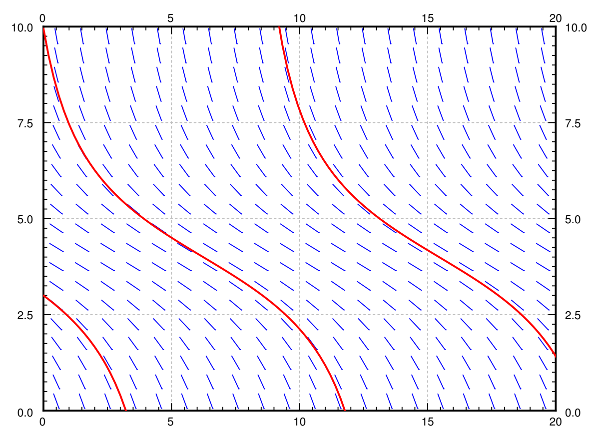


Figure 1.15: The slope field and some solutions of $x' = 0.1\,x\,(8 - x) - 2$.

1.6.1 Sketching qualitatively-different solutions to autonomous DEs

We are considering problems of the form:

$$x' = f(x)$$

where $f(x)$ and $\frac{df}{dx}(x)$ are continuous. The basic idea is that the sign of the function $f(x)$ tells us if the solution is increasing or decreasing. This allows one to sketch rough approximate solutions with the correct qualitative properties.

Example 1.6.1:

$$x' = f(x) = x^2 - 4x + 3 \tag{1.5}$$

Step 1) Draw 3 sets of axes:

(i) xx'-plane, for the graph of $x' = f(x)$,

(ii) a vertical x axis, for the phase line,

(iii) tx-plane, for the solution curves.

It is helpful to draw (i) "sideways" as in Figure 1.16 so that the x-axes of all 3 plots are aligned.

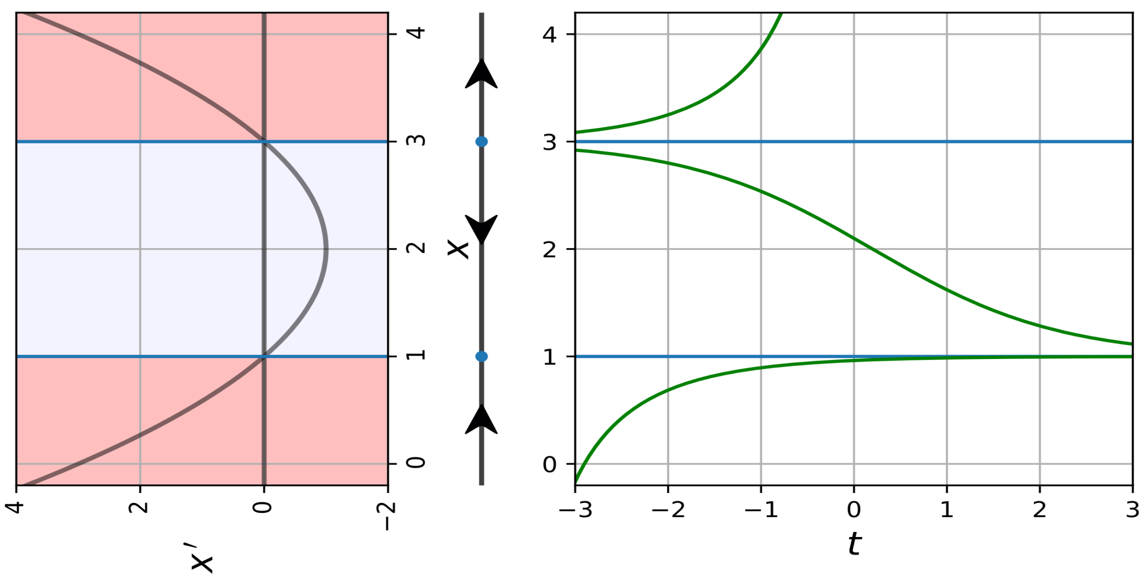


Figure 1.16

Step 2) Sketch $x' = f(x)$ in the xx'-plane. Note that each value $x = c$ such that $f(c) = 0$ corresponds to a constant solution $x(t) \equiv c$ for all x. The values $x = c$ are also called critical points or equilibria of the DE.

Step 3) Mark the critical points $x = c$ as dots on the x-axis of the phase line. These divide the phase line into subintervals.

Step 4) Draw an arrow on each subinterval, pointing up if $f > 0$ on the subinterval and down if $f < 0$.

Step 5) Sketch the constant solutions $x(t) \equiv c$ in the tx-plane. These lines divide the tx-plane into subregions corresponding to the subintervals of the phase line.

Step 6) Select one IC (t_0, x_0) for each different subregion and sketch the corresponding solution. Base the sketch on whether $x(t)$ is increasing or decreasing as shown by the arrows on the phase line. Each solution curve must lie entirely in one of the subregions. It can be shown by using the Picard existence/uniqueness theorem for first order ODEs (Theorem 1.2.1) that different solution curves cannot intersect (y and x of the Theorem correspond to x and t here, respectively). It follows that each non-constant solution $x(t)$ cannot cross or even touch any of the lines $x = c$ because these lines represent constant solutions $x(t) \equiv c$.

In the example shown in Figure 1.16 on the previous page, each of the three non-constant solution curves represents a class of solutions having the same qualitative behavior:

1) Any IC (t_0, x_0) with $x_0 < 1$ leads to a solution which is strictly increasing and asymptotic as $t \to \infty$ to $x \equiv 1$. For $t < t_0$, the solution takes on arbitrarily large negative values.

2) Any IC (t_0, x_0) with $1 < x_0 < 3$ leads to a solution which is strictly decreasing for all t and asymptotic as $t \to \infty$ to $x \equiv 1$. As $t \to -\infty$ the solution is asymptotic to $x \equiv 3$.

3) Any IC (t_0, x_0) with $x_0 > 3$ leads to a solution which is strictly increasing. For $t > t_0$, the solution takes on arbitrarily large positive values. As $t \to -\infty$, the solution is asymptotic to $x \equiv 3$.

1.6.2 Exercises

***Exercise* 1.6.3:** *Consider $x' = x^2$.*

a) *Draw the phase diagram, find the critical points, and mark them stable or unstable.*

b) *Sketch typical solutions of the equation.*

c) *Find $\lim\limits_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = -1$.*

***Exercise* 1.6.4:** *Consider $x' = \sin x$.*

a) *Draw the phase diagram for $-4\pi \leq x \leq 4\pi$. On this interval mark the critical points stable or unstable.*

b) *Sketch typical solutions of the equation.*

c) *Find $\lim\limits_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = 1$.*

***Exercise* 1.6.5:** *Suppose $f(x)$ is positive for $0 < x < 1$, it is zero when $x = 0$ and $x = 1$, and it is negative for all other x.*

a) *Draw the phase diagram for $x' = f(x)$, find the critical points, and mark them stable or unstable.*

b) *Sketch typical solutions of the equation.*

c) *Find $\lim_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = 0.5$.*

***Exercise* 1.6.6:** *Start with the logistic equation $\frac{dx}{dt} = kx(M-x)$. Suppose we modify our harvesting. That is we will only harvest an amount proportional to current population. In other words, we harvest hx per unit of time for some $h > 0$ (Similar to earlier example with h replaced with hx).*

a) *Construct the differential equation.*

b) *Show that if $kM > h$, then the equation is still logistic.*

c) *What happens when $kM < h$?*

***Exercise* 1.6.7:** *A disease is spreading through the country. Let x be the number of people infected. Let the constant S be the number of people susceptible to infection. The infection rate $\frac{dx}{dt}$ is proportional to the product of already infected people, x, and the number of susceptible but uninfected people, $S - x$.*

a) *Write down the differential equation.*

b) *Supposing $x(0) > 0$, that is, some people are infected at time $t = 0$, what is $\lim_{t\to\infty} x(t)$.*

c) *Does the solution to part b) agree with your intuition? Why or why not?*

Follow the steps in Example 1.6.1 to sketch the qualitative-different solution curves for the following autonomous DEs. Be sure to show all key features of the three graphs, zeros of $f(x)$, arrows of increase / decrease, ..., as in Figure 1.16 on page 63.

***Exercise* 1.6.51:** $x' = x^2 - 6x + 5$.

***Exercise* 1.6.52:** $x' = -x^2 + x + 2$.

***Exercise* 1.6.53:** $x' = x + 3$.

***Exercise* 1.6.54:** $x' = (x + 2)^2$.

***Exercise* 1.6.55:** $x' = -x^2 + 6x - 9$.

***Exercise* 1.6.56:** $x' = -2x + 2$.

***Exercise* 1.6.57:** $x' = x^2 - 4x + 4$.

***Exercise* 1.6.58:** $x' = -x^2 + 5x - 4$.

***Exercise* 1.6.59:** $x' = x^2 - 4$.

***Exercise* 1.6.60:** $x' = -x^2 + x + 6$.

***Exercise* 1.6.61:** $x' = (3 - x)^3$.

***Exercise* 1.6.101:** *Let* $x' = (x-1)(x-2)x^2$.

a) *Sketch the phase diagram and find critical points.*

b) *Classify the critical points.*

c) *If* $x(0) = 0.5$, *then find* $\lim_{t\to\infty} x(t)$.

***Exercise* 1.6.102:** *Let* $x' = e^{-x}$.

a) *Find and classify all critical points.*

b) *Find* $\lim_{t\to\infty} x(t)$ *given any initial condition.*

***Exercise* 1.6.103:** *Assume that a population of fish in a lake satisfies* $\frac{dx}{dt} = kx(M-x)$. *Now suppose that fish are continually added at A fish per unit of time.*

a) *Find the differential equation for* x.

b) *What is the new limiting population?*

***Exercise* 1.6.104:** *Suppose* $\frac{dx}{dt} = (x-\alpha)(x-\beta)$ *for two numbers* $\alpha < \beta$.

a) *Find the critical points, and classify them.*

For b), c), d), find $\lim_{t\to\infty} x(t)$ *based on the phase diagram.*

b) $x(0) < \alpha$,

c) $\alpha < x(0) < \beta$,

d) $\beta < x(0)$.

1.7 Numerical methods: Euler's method

Note: 1 lecture, can safely be skipped, §2.4 in [EP], §8.1 in [BD]

Unless $f(x, y)$ is of a special form, it is generally very hard if not impossible to get a nice formula for the solution of the problem

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

If the equation can be solved in closed form, we should do that. But what if we have an equation that cannot be solved in closed form? What if we want to find the value of the solution at some particular x? Or perhaps we want to produce a graph of the solution to inspect the behavior. In this section we will learn about the basics of numerical approximation of solutions.

The simplest method for approximating a solution is *Euler's method**. It works as follows: Take x_0 and compute the slope $k = f(x_0, y_0)$. The slope is the change in y per unit change in x. Follow the line for an interval of length h on the x-axis. Hence if $y = y_0$ at x_0, then we say that y_1 (the approximate value of y at $x_1 = x_0 + h$) is $y_1 = y_0 + hk$. Rinse, repeat! Let $k = f(x_1, y_1)$, and then compute $x_2 = x_1 + h$, and $y_2 = y_1 + hk$. Now compute x_3 and y_3 using x_2 and y_2, etc. Consider the equation $y' = {}^{y^2}\!/_3$, $y(0) = 1$, and $h = 1$. Then $x_0 = 0$ and $y_0 = 1$. We compute

$$x_1 = x_0 + h = 0 + 1 = 1, \qquad y_1 = y_0 + h\,f(x_0, y_0) = 1 + 1 \cdot {}^1\!/_3 = {}^4\!/_3 \approx 1.333,$$
$$x_2 = x_1 + h = 1 + 1 = 2, \qquad y_2 = y_1 + h\,f(x_1, y_1) = {}^4\!/_3 + 1 \cdot \frac{({}^4\!/_3)^2}{3} = {}^{52}\!/_{27} \approx 1.926.$$

We then draw an approximate graph of the solution by connecting the points (x_0, y_0), (x_1, y_1), (x_2, y_2),. . . . For the first two steps of the method see Figure 1.17 on the next page.

More abstractly, for any $i = 0, 1, 2, 3, \ldots$, we compute

$$x_{i+1} = x_i + h, \qquad y_{i+1} = y_i + h\,f(x_i, y_i).$$

The line segments we get are an approximate graph of the solution. Generally it is not exactly the solution. See Figure 1.18 on the following page for the plot of the real solution and the approximation.

We continue with the equation $y' = {}^{y^2}\!/_3$, $y(0) = 1$. Let us try to approximate $y(2)$ using Euler's method. In Figures 1.17 and 1.18 we have graphically approximated $y(2)$ with step size 1. With step size 1, we have $y(2) \approx 1.926$. The real answer is 3. We are approximately 1.074 off. Let us halve the step size. Computing y_4 with $h = 0.5$, we find that $y(2) \approx 2.209$, so an error of about 0.791. Table 1.1 on page 69 gives the values computed for various parameters.

***Exercise* 1.7.1:** *Solve this equation exactly and show that $y(2) = 3$.*

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*Named after the Swiss mathematician Leonhard Paul Euler (1707–1783). The correct pronunciation of the name sounds more like "oiler."

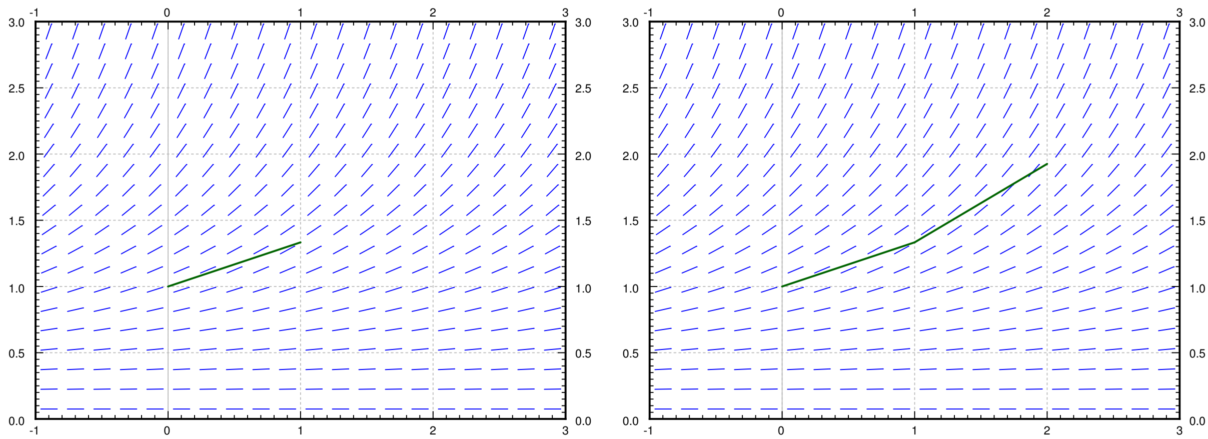


Figure 1.17: First two steps of Euler's method with $h = 1$ for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

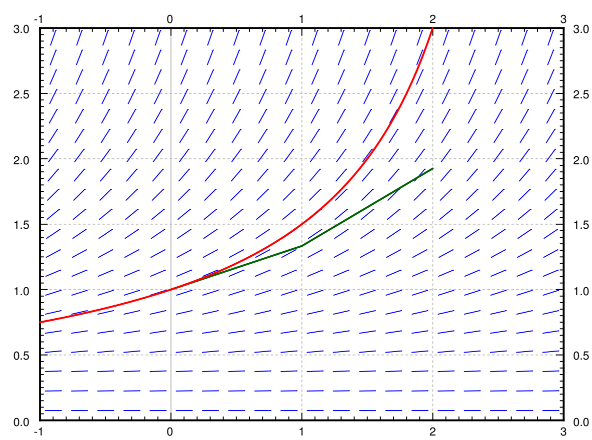


Figure 1.18: Two steps of Euler's method (step size 1) and the exact solution for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

The difference between the actual solution and the approximate solution is called the error. We usually talk about just the size of the error and we do not care much about its sign. The point is, we usually do not know the real solution, so we only have a vague understanding of the error. If we knew the error exactly . . . what is the point of doing the approximation?

Notice that except for the first few times, every time we halved the interval the error approximately halved. This halving of the error is a general feature of Euler's method as it is a *first order method*. There exists an improved Euler method, see the exercises, which is a second order method. A second order method reduces the error to approximately one quarter every time we halve the interval. The meaning of "second" order is the squaring in

h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
0.015625	2.95035	0.04965	0.51779
0.0078125	2.97472	0.02528	0.50913

Table 1.1: Euler's method approximation of $y(2)$ where of $y' = y^2/3$, $y(0) = 1$.

$1/4 = 1/2 \times 1/2 = (1/2)^2$.

To get the error to be within 0.1 of the answer we had to already do 64 steps. To get it to within 0.01 we would have to halve another three or four times, meaning doing 512 to 1024 steps. That is quite a bit to do by hand. The improved Euler method from the exercises should quarter the error every time we halve the interval, so we would have to approximately do half as many "halvings" to get the same error. This reduction can be a big deal. With 10 halvings (starting at $h = 1$) we have 1024 steps, whereas with 5 halvings we only have to do 32 steps, assuming that the error was comparable to start with. A computer may not care about this difference for a problem this simple, but suppose each step would take a second to compute (the function may be substantially more difficult to compute than $y^2/3$). Then the difference is 32 seconds versus about 17 minutes. We are not being altogether fair, a second order method would probably double the time to do each step. Even so, it is 1 minute versus 17 minutes. Next, suppose that we have to repeat such a calculation for different parameters a thousand times. You get the idea.

Note that in practice we do not know how large the error is! How do we know what is the right step size? Well, essentially we keep halving the interval, and if we are lucky, we can estimate the error from a few of these calculations and the assumption that the error goes down by a factor of one half each time (if we are using standard Euler).

***Exercise* 1.7.2:** *In the table above, suppose you do not know the error. Take the approximate values of the function in the last two lines, assume that the error goes down by a factor of 2. Can you estimate the error in the last time from this? Does it (approximately) agree with the table? Now do it for the first two rows. Does this agree with the table?*

Let us talk a little bit more about the example $y' = y^2/3$, $y(0) = 1$. Suppose that instead of the value $y(2)$ we wish to find $y(3)$. The results of this effort are listed in Table 1.2 on the following page for successive halvings of h. What is going on here? Well, you should solve the equation exactly and you will notice that the solution does not exist at $x = 3$. In fact, the solution goes to infinity when you approach $x = 3$.

h	Approximate $y(3)$
1	3.16232
0.5	4.54329
0.25	6.86079
0.125	10.80321
0.0625	17.59893
0.03125	29.46004
0.015625	50.40121
0.0078125	87.75769

Table 1.2: Attempts to use Euler's to approximate $y(3)$ where of $y' = y^2/3$, $y(0) = 1$.

Another case where things go bad is if the solution oscillates wildly near some point. The solution may exist at all points, but even a much better numerical method than Euler would need an insanely small step size to approximate the solution with reasonable precision. And computers might not be able to easily handle such a small step size.

In real applications we would not use a simple method such as Euler's. The simplest method that would probably be used in a real application is the standard Runge–Kutta method (see exercises). That is a fourth order method, meaning that if we halve the interval, the error generally goes down by a factor of 16 (it is fourth order as $1/16 = 1/2 \times 1/2 \times 1/2 \times 1/2$).

Choosing the right method to use and the right step size can be very tricky. There are several competing factors to consider.

- Computational time: Each step takes computer time. Even if the function f is simple to compute, we do it many times over. Large step size means faster computation, but perhaps not the right precision.

- Roundoff errors: Computers only compute with a certain number of significant digits. Errors introduced by rounding numbers off during our computations become noticeable when the step size becomes too small relative to the quantities we are working with. So reducing step size may in fact make errors worse. There is a certain optimum step size such that the precision increases as we approach it, but then starts getting worse as we make our step size smaller still. Trouble is: this optimum may be hard to find.

- Stability: Certain equations may be numerically unstable. What may happen is that the numbers never seem to stabilize no matter how many times we halve the interval. We may need a ridiculously small interval size, which may not be practical due to roundoff errors or computational time considerations. Such problems are sometimes called *stiff*. In the worst case, the numerical computations might be giving us bogus numbers that look like a correct answer. Just because the numbers seem to have stabilized after successive halving, does not mean that we must have the right answer.

We have seen just the beginnings of the challenges that appear in real applications. Numerical approximation of solutions to differential equations is an active research area for engineers and mathematicians. For example, the general purpose method used for the ODE solver in Matlab and Octave (as of this writing) is a method that appeared in the literature only in the 1980s.

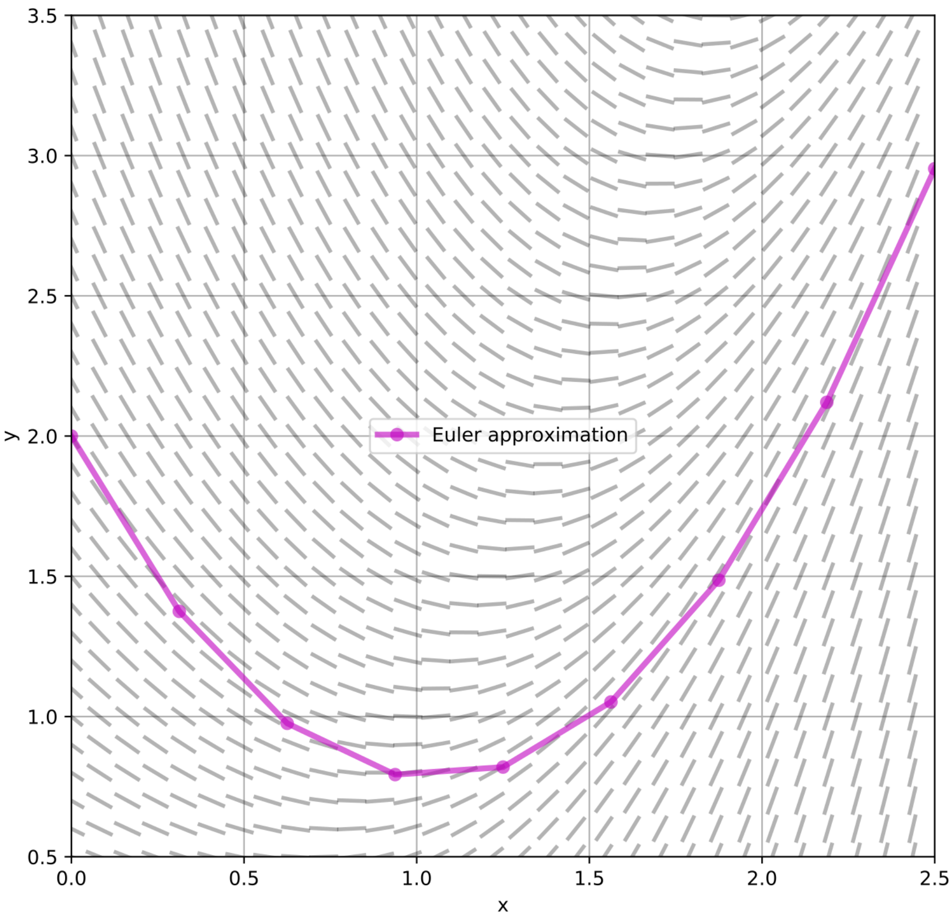
1.7.1 Euler's method with Python

Below we show an implementation of Euler's method applied to the initial value problem $y' = x^2 - y$, $y(0) = 2$. We plot the resulting Euler approximation on top of the slope field.

```
from resources306 import *

def f(x,y):
    return x**2 - y

plt.figure(figsize=(8,8))
slopefieldplot( f, 0,2.5, 0.5,3.5, .1 ,lw=2)

y = 2.0             # This is the initial value of y.
x = 0.0             # This is the initial time.
xfinal = 2.5        # This is the value of x we want to get to.
n = 8               # Here we say how many steps we want to take.
h =(xfinal-x)/n     # This is our step-size.

xlist = [x]         # Initialize lists to store the data in
ylist = [y]         # for later plotting.

for i in range(n):  # Take n steps
    slope = f(x,y)  # Compute the slope at the current location with DE
    y = y + h*slope # Take the Euler step to the new value of y.
    x = x + h       # Advance x by one step.
    xlist.append(x) # Tack the new values at the ends of the lists.
    ylist.append(y)

for x,y in zip(xlist,ylist):
    print(f'{x:8.4f} {y:12.8f}')

plt.plot( xlist, ylist, 'mo-', lw=3, alpha=0.6, label='Euler approximation' )
plt.xlabel('x')
plt.ylabel('y')
```

```
plt.grid()
plt.legend(loc='center')
```

```
0.0000   2.00000000
0.3125   1.37500000
0.6250   0.97583008
0.9375   0.79295349
1.2500   0.81981373
1.5625   1.05190319
1.8750   1.48612290
2.1875   2.12034230
2.5000   2.95309666
```



1.7.2 Exercises

***Exercise* 1.7.3:** *Consider* $\frac{dx}{dt} = (2t - x)^2$, $x(0) = 2$. *Use Euler's method with step size* $h = 0.5$ *to approximate* $x(1)$.

***Exercise* 1.7.4:** *Consider* $\frac{dx}{dt} = t - x$, $x(0) = 1$.

a) *Use Euler's method with step sizes* $h = 1, 1/2, 1/4, 1/8$ *to approximate* $x(1)$.

b) *Solve the equation exactly.*

c) *Describe what happens to the errors for each h you used. That is, find the factor by which the error changed each time you halved the interval.*

Exercise 1.7.5: *Approximate the value of e by looking at the initial value problem $y' = y$ with $y(0) = 1$ and approximating $y(1)$ using Euler's method with a step size of 0.2.*

Exercise 1.7.6: *Example of numerical instability: Take $y' = -5y$, $y(0) = 1$. We know that the solution should decay to zero as x grows. Using Euler's method, start with $h = 1$ and compute y_1, y_2, y_3, y_4 to try to approximate $y(4)$. What happened? Now halve the interval. Keep halving the interval and approximating $y(4)$ until the numbers you are getting start to stabilize (that is, until they start going towards zero). Note: You might want to use a calculator.*

The simplest method used in practice is the *Runge–Kutta method*. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, and a step size h. Everything is the same as in Euler's method, except the computation of y_{i+1} and x_{i+1}.

$$
\begin{aligned}
k_1 &= f(x_i, y_i), \\
k_2 &= f\bigl(x_i + h/2,\, y_i + k_1(h/2)\bigr), & x_{i+1} &= x_i + h, \\
k_3 &= f\bigl(x_i + h/2,\, y_i + k_2(h/2)\bigr), & y_{i+1} &= y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6}\, h, \\
k_4 &= f(x_i + h,\, y_i + k_3 h).
\end{aligned}
$$

Exercise 1.7.7: *Consider $\dfrac{dy}{dx} = yx^2$, $y(0) = 1$.*

a) *Use Runge–Kutta (see above) with step sizes $h = 1$ and $h = 1/2$ to approximate $y(1)$.*

b) *Use Euler's method with $h = 1$ and $h = 1/2$.*

c) *Solve exactly, find the exact value of $y(1)$, and compare.*

Exercise 1.7.101: *Let $x' = \sin(xt)$, and $x(0) = 1$. Approximate $x(1)$ using Euler's method with step sizes $1, 0.5, 0.25$. Use a calculator and compute up to 4 decimal digits.*

Exercise 1.7.102: *Let $x' = 2t$, and $x(0) = 0$.*

a) *Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.*

b) *Solve exactly, and compute the errors.*

c) *Compute the factor by which the errors changed.*

Exercise 1.7.103: *Let $x' = xe^{xt+1}$, and $x(0) = 0$.*

a) *Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.*

b) *Guess an exact solution based on part a) and compute the errors.*

There is a simple way to improve Euler's method to make it a second order method by doing just one extra step. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, and a step size h. What we do is to pretend we compute the next step as in Euler, that is, we start with (x_i, y_i), we compute a slope $k_1 = f(x_i, y_i)$, and then look at the point $(x_i + h, y_i + k_1 h)$. Instead of letting our new point be $(x_i + h, y_i + k_1 h)$, we compute the slope at that point, call it k_2, and then take the average of k_1 and k_2, hoping that the average is going to be closer to the actual slope on the interval from x_i to $x_i + h$. And we are correct, if we halve the step, the error should go down by a factor of $2^2 = 4$. To summarize, the setup is the same as for regular Euler, except the computation of y_{i+1} and x_{i+1}.

$$
\begin{aligned}
k_1 &= f(x_i, y_i), & x_{i+1} &= x_i + h, \\
k_2 &= f(x_i + h, y_i + k_1 h), & y_{i+1} &= y_i + \frac{k_1 + k_2}{2}\, h.
\end{aligned}
$$

***Exercise* 1.7.104:** *Consider* $\frac{dy}{dx} = x + y$, $y(0) = 1$.

a) *Use the improved Euler's method (see above) with step sizes* $h = 1/4$ *and* $h = 1/8$ *to approximate* $y(1)$.

b) *Use Euler's method with* $h = 1/4$ *and* $h = 1/8$.

c) *Solve exactly, find the exact value of* $y(1)$.

d) *Compute the errors, and the factors by which the errors changed.*

1.8 Exact equations

Note: 1–2 lectures, can safely be skipped, §1.6 in [EP], §2.6 in [BD]

Another type of equation that comes up quite often in physics and engineering is an *exact equation*. Suppose $F(x, y)$ is a function of two variables, which we call the *potential function*. The naming should suggest potential energy, or electric potential. Exact equations and potential functions appear when there is a conservation law at play, such as conservation of energy. Let us make up a simple example. Let

$$F(x, y) = x^2 + y^2.$$

We are interested in the lines of constant energy, that is lines where the energy is conserved; we want curves where $F(x, y) = C$, for some constant C. In our example, the curves $x^2 + y^2 = C$ are circles. See Figure 1.19.

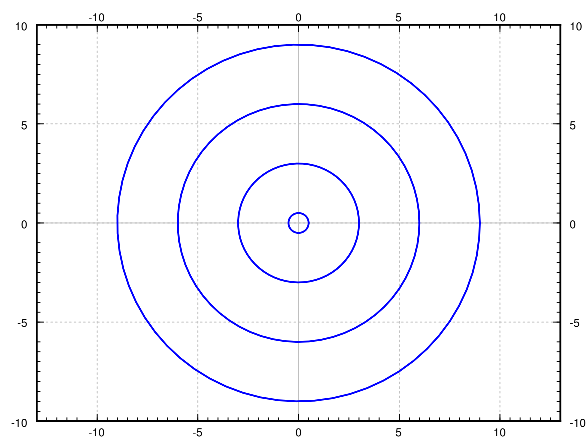


Figure 1.19: Solutions to $F(x, y) = x^2 + y^2 = C$ for various C.

We take the *total derivative* of F:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$$

For convenience, we will make use of the notation of $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$. In our example,

$$dF = 2x\, dx + 2y\, dy.$$

We apply the total derivative to $F(x, y) = C$, to find the differential equation $dF = 0$. The differential equation we obtain in such a way has the form

$$M\, dx + N\, dy = 0, \qquad \text{or} \qquad M + N \frac{dy}{dx} = 0.$$

An equation of this form is called *exact* if it was obtained as $dF = 0$ for some potential function F. In our simple example, we obtain the equation

$$2x\, dx + 2y\, dy = 0, \qquad \text{or} \qquad 2x + 2y \frac{dy}{dx} = 0.$$

Since we obtained this equation by differentiating $x^2 + y^2 = C$, the equation is exact. We often wish to solve for y in terms of x. In our example,

$$y = \pm\sqrt{C^2 - x^2}.$$

An interpretation of the setup is that at each point $\vec{v} = (M, N)$ is a vector in the plane, that is, a direction and a magnitude. As M and N are functions of (x, y), we have a *vector field*. The particular vector field $\vec{v}$ that comes from an exact equation is a so-called *conservative vector field*, that is, a vector field that comes with a potential function $F(x, y)$, such that

$$\vec{v} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right).$$

Let γ be a path in the plane starting at (x_1, y_1) and ending at (x_2, y_2). If we think of $\vec{v}$ as force, then the work required to move along γ is

$$\int_\gamma \vec{v}(\vec{r}) \cdot d\vec{r} = \int_\gamma M\,dx + N\,dy = F(x_2, y_2) - F(x_1, y_1).$$

That is, the work done only depends on endpoints, that is where we start and where we end. For example, suppose F is gravitational potential. The derivative of F given by $\vec{v}$ is the gravitational force. What we are saying is that the work required to move a heavy box from the ground floor to the roof, only depends on the change in potential energy. That is, the work done is the same no matter what path we took; if we took the stairs or the elevator. Although if we took the elevator, the elevator is doing the work for us. The curves $F(x, y) = C$ are those where no work need be done, such as the heavy box sliding along without accelerating or breaking on a perfectly flat roof, on a cart with incredibly well oiled wheels.

An exact equation is a conservative vector field, and the implicit solution of this equation is the potential function.

1.8.1 Solving exact equations

Now you, the reader, should ask: Where did we solve a differential equation? Well, in applications we generally know M and N, but we do not know F. That is, we may have just started with $2x + 2y\frac{dy}{dx} = 0$, or perhaps even

$$x + y\frac{dy}{dx} = 0.$$

It is up to us to find some potential F that works. Many different F will work; adding a constant to F does not change the equation. Once we have a potential function F, the equation $F\bigl(x, y(x)\bigr) = C$ gives an implicit solution of the ODE.

Example 1.8.1: Let us find the general solution to $2x + 2y\frac{dy}{dx} = 0$. Forget we knew what F was.

If we know that this is an exact equation, we start looking for a potential function F. We have $M = 2x$ and $N = 2y$. If F exists, it must be such that $F_x(x, y) = 2x$. Integrate in the x variable to find

$$F(x, y) = x^2 + A(y), \tag{1.6}$$

for some function $A(y)$. The function A is the "constant of integration", though it is only constant as far as x is concerned, and may still depend on y. Now differentiate (1.6) in y

and set it equal to N, which is what F_y is supposed to be:

$$2y = F_y(x, y) = A'(y).$$

Integrating, we find $A(y) = y^2$. We could add a constant of integration if we wanted to, but there is no need. We found $F(x, y) = x^2 + y^2$. Next for a constant C, we solve

$$F\bigl(x, y(x)\bigr) = C.$$

for y in terms of x. In this case, we obtain $y = \pm\sqrt{C^2 - x^2}$ as we did before.

***Exercise* 1.8.1:** *Why did we not need to add a constant of integration when integrating $A'(y) = 2y$? Add a constant of integration, say 3, and see what F you get. What is the difference from what we got above, and why does it not matter?*

The procedure, once we know that the equation is exact, is:

(i) Integrate $F_x = M$ in x resulting in $F(x, y) = \text{something} + A(y)$.

(ii) Differentiate this F in y, and set that equal to N, so that we may find $A(y)$ by integration.

The procedure can also be done by first integrating in y and then differentiating in x. Pretty easy huh? Let's try this again.

Example 1.8.2: Consider now $2x + y + xy\frac{dy}{dx} = 0$.

OK, so $M = 2x + y$ and $N = xy$. We try to proceed as before. Suppose F exists. Then $F_x(x, y) = 2x + y$. We integrate:

$$F(x, y) = x^2 + xy + A(y)$$

for some function $A(y)$. Differentiate in y and set equal to N:

$$N = xy = F_y(x, y) = x + A'(y).$$

But there is no way to satisfy this requirement! The function xy cannot be written as x plus a function of y. The equation is not exact; no potential function F exists.

Is there an easier way to check for the existence of F, other than failing in trying to find it? Turns out there is. Suppose $M = F_x$ and $N = F_y$. Then as long as the second derivatives are continuous,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

Let us state it as a theorem. Usually this is called the Poincaré Lemma*.

Theorem 1.8.1 (Poincaré)**.** *If M and N are continuously differentiable functions of (x, y), and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then near any point there is a function $F(x, y)$ such that $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$.*

*Named for the French polymath Jules Henri Poincaré (1854–1912).

The theorem doesn't give us a global F defined everywhere. In general, we can only find the potential locally, near some initial point. By this time, we have come to expect this from differential equations.

Let us return to the example above where $M = 2x + y$ and $N = xy$. Notice $M_y = 1$ and $N_x = y$, which are clearly not equal. The equation is not exact.

Example 1.8.3: Solve

$$\frac{dy}{dx} = \frac{-2x - y}{x - 1}, \qquad y(0) = 1.$$

We write the equation as

$$(2x + y) + (x - 1)\frac{dy}{dx} = 0,$$

so $M = 2x + y$ and $N = x - 1$. Then

$$M_y = 1 = N_x.$$

The equation is exact. Integrating M in x, we find

$$F(x, y) = x^2 + xy + A(y).$$

Differentiating in y and setting to N, we find

$$x - 1 = x + A'(y).$$

So $A'(y) = -1$, and $A(y) = -y$ will work. Take $F(x, y) = x^2 + xy - y$. We wish to solve $x^2 + xy - y = C$. First let us find C. As $y(0) = 1$ then $F(0, 1) = C$. Therefore $0^2 + 0 \times 1 - 1 = C$, so $C = -1$. Now we solve $x^2 + xy - y = -1$ for y to get

$$y = \frac{-x^2 - 1}{x - 1}.$$

Example 1.8.4: Solve

$$-\frac{y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy = 0, \qquad y(1) = 2.$$

We leave to the reader to check that $M_y = N_x$.

This vector field (M, N) is not conservative if considered as a vector field of the entire plane minus the origin. The problem is that if the curve γ is a circle around the origin, say starting at $(1, 0)$ and ending at $(1, 0)$ going counterclockwise, then if F existed we would expect

$$0 = F(1, 0) - F(1, 0) = \int_\gamma F_x\, dx + F_y\, dy = \int_\gamma \frac{-y}{x^2 + y^2}\, dx + \frac{x}{x^2 + y^2}\, dy = 2\pi.$$

That is nonsense! We leave the computation of the path integral to the interested reader, or you can consult your multivariable calculus textbook. So there is no potential function F defined everywhere outside the origin $(0, 0)$.

If we think back to the theorem, it does not guarantee such a function anyway. It only guarantees a potential function locally, that is only in some region near the initial point. As $y(1) = 2$ we start at the point $(1, 2)$. Considering $x > 0$ and integrating M in x or N in y, we find

$$F(x, y) = \arctan\big(y/x\big).$$

The implicit solution is $\arctan\big(y/x\big) = C$. Solving, $y = \tan(C)x$. That is, the solution is a straight line. Solving $y(1) = 2$ gives us that $\tan(C) = 2$, and so $y = 2x$ is the desired solution. See Figure 1.20, and note that the solution only exists for $x > 0$.

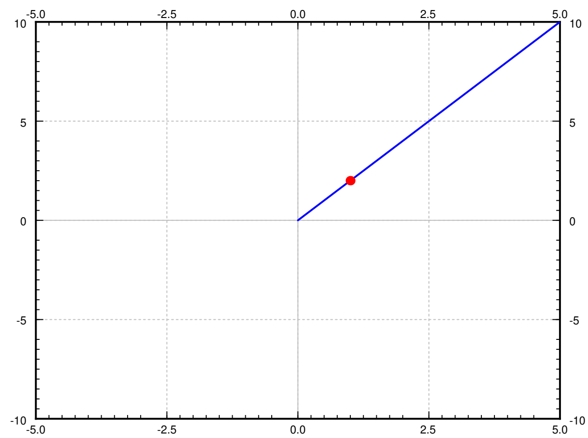


Figure 1.20: Solution to $-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 0$, $y(1) = 2$, *with initial point marked.*

Example 1.8.5: Solve

$$x^2 + y^2 + 2y(x + 1)\frac{dy}{dx} = 0.$$

The reader should check that this equation is exact. Let $M = x^2 + y^2$ and $N = 2y(x + 1)$. We follow the procedure for exact equations

$$F(x, y) = \frac{1}{3}x^3 + xy^2 + A(y),$$

and

$$2y(x + 1) = 2xy + A'(y).$$

Therefore $A'(y) = 2y$ or $A(y) = y^2$ and $F(x, y) = \frac{1}{3}x^3 + xy^2 + y^2$. We try to solve $F(x, y) = C$. We easily solve for y^2 and then just take the square root:

$$y^2 = \frac{C - (1/3)x^3}{x + 1}, \qquad \text{so} \qquad y = \pm\sqrt{\frac{C - (1/3)x^3}{x + 1}}.$$

When $x = -1$, the term in front of $\frac{dy}{dx}$ vanishes. You can also see that our solution is not valid in that case. However, one could in that case try to solve for x in terms of y starting from the implicit solution $\frac{1}{3}x^3 + xy^2 + y^2 = C$. The solution is somewhat messy and we leave it as implicit.

1.8.2 Integrating factors

Sometimes an equation $M\,dx + N\,dy = 0$ is not exact, but it can be made exact by multiplying with a function $u(x, y)$. That is, perhaps for some nonzero function $u(x, y)$,

$$u(x, y)M(x, y)\,dx + u(x, y)N(x, y)\,dy = 0$$

is exact. Any solution to this new equation is also a solution to $M\,dx + N\,dy = 0$.

In fact, a linear equation

$$\frac{dy}{dx} + p(x)y = f(x), \qquad \text{or} \qquad \big(p(x)y - f(x)\big)\,dx + dy = 0$$

is always such an equation. Let $r(x) = e^{\int p(x)\,dx}$ be the integrating factor for a linear equation. Multiply the equation by $r(x)$ and write it in the form of $M + N\frac{dy}{dx} = 0$.

$$r(x)p(x)y - r(x)f(x) + r(x)\frac{dy}{dx} = 0.$$

Then $M = r(x)p(x)y - r(x)f(x)$, so $M_y = r(x)p(x)$, while $N = r(x)$, so $N_x = r'(x) = r(x)p(x)$. In other words, we have an exact equation. Integrating factors for linear functions are just a special case of integrating factors for exact equations.

But how do we find the integrating factor u? Well, given an equation

$$M\,dx + N\,dy = 0,$$

u should be a function such that

$$\frac{\partial}{\partial y}\big[uM\big] = u_y M + uM_y = \frac{\partial}{\partial x}\big[uN\big] = u_x N + uN_x.$$

Therefore,

$$(M_y - N_x)u = u_x N - u_y M.$$

At first it may seem we replaced one differential equation by another. True, but all hope is not lost.

A strategy that often works is to look for a u that is a function of x alone, or a function of y alone. If u is a function of x alone, that is $u(x)$, then we write $u'(x)$ instead of u_x, and u_y is just zero. Then

$$\frac{M_y - N_x}{N}u = u'.$$

In particular, $\frac{M_y - N_x}{N}$ ought to be a function of x alone (not depend on y). If so, then we have a linear equation

$$u' - \frac{M_y - N_x}{N}u = 0.$$

Letting $P(x) = \frac{M_y - N_x}{N}$, we solve using the standard integrating factor method, to find $u(x) = Ce^{\int P(x)\,dx}$. The constant in the solution is not relevant, we need any nonzero solution, so we take $C = 1$. Then $u(x) = e^{\int P(x)\,dx}$ is the integrating factor.

Similarly we could try a function of the form $u(y)$. Then

$$\frac{M_y - N_x}{M} u = -u'.$$

In particular, $\frac{M_y - N_x}{M}$ ought to be a function of y alone. If so, then we have a linear equation

$$u' + \frac{M_y - N_x}{M} u = 0.$$

Letting $Q(y) = \frac{M_y - N_x}{M}$, we find $u(y) = Ce^{-\int Q(y)\,dy}$. We take $C = 1$. So $u(y) = e^{-\int Q(y)\,dy}$ is the integrating factor.

Example 1.8.6: Solve

$$\frac{x^2 + y^2}{x+1} + 2y\frac{dy}{dx} = 0.$$

Let $M = \frac{x^2+y^2}{x+1}$ and $N = 2y$. Compute

$$M_y - N_x = \frac{2y}{x+1} - 0 = \frac{2y}{x+1}.$$

As this is not zero, the equation is not exact. We notice

$$P(x) = \frac{M_y - N_x}{N} = \frac{2y}{x+1}\frac{1}{2y} = \frac{1}{x+1}$$

is a function of x alone. We compute the integrating factor

$$e^{\int P(x)\,dx} = e^{\ln(x+1)} = x + 1.$$

We multiply our given equation by $(x+1)$ to obtain

$$x^2 + y^2 + 2y(x+1)\frac{dy}{dx} = 0,$$

which is an exact equation that we solved in Example 1.8.5. The solution was

$$y = \pm\sqrt{\frac{C - (1/3)x^3}{x+1}}.$$

Example 1.8.7: Solve

$$y^2 + (xy + 1)\frac{dy}{dx} = 0.$$

First compute
$$M_y - N_x = 2y - y = y.$$
As this is not zero, the equation is not exact. We observe
$$Q(y) = \frac{M_y - N_x}{M} = \frac{y}{y^2} = \frac{1}{y}$$
is a function of y alone. We compute the integrating factor
$$e^{-\int Q(y)\,dy} = e^{-\ln y} = \frac{1}{y}.$$
Therefore we look at the exact equation
$$y + \frac{xy+1}{y}\frac{dy}{dx} = 0.$$
The reader should double check that this equation is exact. We follow the procedure for exact equations
$$F(x,y) = xy + A(y),$$
and
$$\frac{xy+1}{y} = x + \frac{1}{y} = x + A'(y). \tag{1.7}$$
Consequently $A'(y) = \frac{1}{y}$ or $A(y) = \ln y$. Thus $F(x,y) = xy + \ln y$. It is not possible to solve $F(x,y) = C$ for y in terms of elementary functions, so let us be content with the implicit solution:
$$xy + \ln y = C.$$
We are looking for the general solution and we divided by y above. We should check what happens when $y = 0$, as the equation itself makes perfect sense in that case. We plug in $y = 0$ to find the equation is satisfied. So $y = 0$ is also a solution.

1.8.3 Exercises

***Exercise* 1.8.2:** *Solve the following exact equations, implicit general solutions will suffice*:

a) $(2xy + x^2)\,dx + (x^2 + y^2 + 1)\,dy = 0$

b) $x^5 + y^5\frac{dy}{dx} = 0$

c) $e^x + y^3 + 3xy^2\frac{dy}{dx} = 0$

d) $(x+y)\cos(x) + \sin(x) + \sin(x)y' = 0$

***Exercise* 1.8.3:** *Find the integrating factor for the following equations making them into exact equations*:

a) $e^{xy}\,dx + \frac{y}{x}e^{xy}\,dy = 0$

b) $\frac{e^x+y^3}{y^2}\,dx + 3x\,dy = 0$

c) $4(y^2+x)\,dx + \frac{2x+2y^2}{y}\,dy = 0$

d) $2\sin(y)\,dx + x\cos(y)\,dy = 0$

Exercise* 1.8.4*: *Suppose you have an equation of the form:* $f(x) + g(y)\frac{dy}{dx} = 0$.

a) *Show it is exact.*

b) *Find the form of the potential function in terms of* f *and* g.

Exercise* 1.8.5*: *Suppose that we have the equation* $f(x)\,dx - dy = 0$.

a) *Is this equation exact?*

b) *Find the general solution using a definite integral.*

Exercise* 1.8.6*: *Find the potential function* $F(x, y)$ *of the exact equation* $\frac{1+xy}{x}\,dx + \left(1/y + x\right)dy = 0$ *in two different ways.*

a) *Integrate* M *in terms of* x *and then differentiate in* y *and set to* N.

b) *Integrate* N *in terms of* y *and then differentiate in* x *and set to* M.

Exercise* 1.8.7*: *A function* $u(x, y)$ *is said to be a* harmonic function *if* $u_{xx} + u_{yy} = 0$.

a) *Show that* $-u_y\,dx + u_x\,dy = 0$ *is an exact equation. Therefore there exists (at least locally) the so-called* harmonic conjugate *function* $v(x, y)$ *such that* $v_x = -u_y$ *and* $v_y = u_x$.

Verify that the following u *are harmonic and find the corresponding harmonic conjugates* v:

b) $u = 2xy$ *c*) $u = e^x \cos y$ *d*) $u = x^3 - 3xy^2$

Exercise* 1.8.101*: *Solve the following exact equations, implicit general solutions will suffice:*

a) $\cos(x) + ye^{xy} + xe^{xy}y' = 0$

b) $(2x + y)\,dx + (x - 4y)\,dy = 0$

c) $e^x + e^y\frac{dy}{dx} = 0$

d) $(3x^2 + 3y)\,dx + (3y^2 + 3x)\,dy = 0$

Exercise* 1.8.102*: *Find the integrating factor for the following equations making them into exact equations:*

a) $\frac{1}{y}\,dx + 3y\,dy = 0$

b) $dx - e^{-x-y}\,dy = 0$

c) $\left(\frac{\cos(x)}{y^2} + \frac{1}{y}\right)dx + \frac{x}{y^2}\,dy = 0$

d) $\left(2y + \frac{y^2}{x}\right)dx + (2y + x)\,dy = 0$

Exercise* 1.8.103*:

a) *Show that every separable equation* $y' = f(x)g(y)$ *can be written as an exact equation, and verify that it is indeed exact.*

b) *Using this rewrite* $y' = xy$ *as an exact equation, solve it and verify that the solution is the same as it was in Example* 1.3.1.

Chapter 2

Higher order linear ODEs

2.1 Second order linear ODEs

Note: 1 lecture, reduction of order optional, first part of §3.1 in [EP], parts of §3.1 and §3.2 in [BD]

Let us consider the general *second order linear differential equation*

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

We usually divide through by $A(x)$ to get

$$y'' + p(x)y' + q(x)y = f(x), \tag{2.1}$$

where $p(x) = {}^{B(x)}/_{A(x)}$, $q(x) = {}^{C(x)}/_{A(x)}$, and $f(x) = {}^{F(x)}/_{A(x)}$. The word *linear* means, for example, that the equation contains no quadratic or higher powers of y, y', and y''.

In the special case when $f(x) = 0$, we have a so-called *homogeneous* equation

$$y'' + p(x)y' + q(x)y = 0. \tag{2.2}$$

We have already seen some second order linear homogeneous equations.

$$\begin{aligned} &y'' + k^2 y = 0 \qquad \text{Two solutions are:} \quad y_1 = \cos(kx), \quad y_2 = \sin(kx). \\ &y'' - k^2 y = 0 \qquad \text{Two solutions are:} \quad y_1 = e^{kx}, \quad y_2 = e^{-kx}. \end{aligned}$$

If we know two solutions of a linear homogeneous equation, we know many more of them.

Theorem 2.1.1 (Superposition)**.** *Suppose y_1 and y_2 are two solutions of the homogeneous equation* (2.2)*. Then*

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

also solves (2.2) *for arbitrary constants C_1 and C_2.*

That is, we can add solutions together and multiply them by constants to obtain new and different solutions. We call the expression $C_1 y_1 + C_2 y_2$ a *linear combination* of y_1 and y_2. Let us prove this theorem; the proof is very enlightening and illustrates how linear equations work.

Proof: Let $y = C_1 y_1 + C_2 y_2$. Then

$$\begin{aligned} y'' + py' + qy &= (C_1 y_1 + C_2 y_2)'' + p(C_1 y_1 + C_2 y_2)' + q(C_1 y_1 + C_2 y_2) \\ &= C_1 y_1'' + C_2 y_2'' + C_1 p y_1' + C_2 p y_2' + C_1 q y_1 + C_2 q y_2 \\ &= C_1 (y_1'' + p y_1' + q y_1) + C_2 (y_2'' + p y_2' + q y_2) \\ &= C_1 \cdot 0 + C_2 \cdot 0 = 0. \quad \square \end{aligned}$$

The proof becomes even simpler to state if we use the operator notation. An *operator* is an object that eats functions and spits out functions (kind of like what a function is, but a function eats numbers and spits out numbers). Define the operator L by

$$Ly = y'' + py' + qy.$$

The differential equation now becomes $Ly = 0$. The operator (and the equation) L being *linear* means that $L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2$. The proof above becomes

$$Ly = L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2 = C_1 \cdot 0 + C_2 \cdot 0 = 0.$$

Two different solutions to the second equation $y'' - k^2 y = 0$ are $y_1 = \cosh(kx)$ and $y_2 = \sinh(kx)$. Let us remind ourselves of the definition, $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$. Therefore, these are solutions by superposition as they are linear combinations of the two exponential solutions.

The functions sinh and cosh are sometimes more convenient to use than the exponential. Let us review some of their properties:

$$\begin{aligned} &\cosh 0 = 1, && \sinh 0 = 0, \\ &\frac{d}{dx}\Big[\cosh x\Big] = \sinh x, && \frac{d}{dx}\Big[\sinh x\Big] = \cosh x, \\ &\cosh^2 x - \sinh^2 x = 1. && \end{aligned}$$

Exercise* 2.1.1*: *Derive these properties using the definitions of* sinh *and* cosh *in terms of exponentials.*

Linear equations have nice and simple answers to the existence and uniqueness question.

Theorem 2.1.2 (Existence and uniqueness)**.** *Suppose p, q, f are continuous functions on some interval I, a is a number in I, and a, b_0, b_1 are constants. The equation*

$$y'' + p(x)y' + q(x)y = f(x),$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \qquad y'(a) = b_1.$$

For example, the equation $y'' + k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx).$$

The equation $y'' - k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cosh(kx) + \frac{b_1}{k} \sinh(kx).$$

Using cosh and sinh in this solution allows us to solve for the initial conditions in a cleaner way than if we have used the exponentials.

The initial conditions for a second order ODE consist of two equations. Common sense tells us that if we have two arbitrary constants and two equations, then we should be able to solve for the constants and find a solution to the differential equation satisfying the initial conditions.

Question: Suppose we find two different solutions y_1 and y_2 to the homogeneous equation (2.2). Can every solution be written (using superposition) in the form $y = C_1 y_1 + C_2 y_2$?

Answer is affirmative! Provided that y_1 and y_2 are different enough in the following sense. We say y_1 and y_2 are *linearly independent* if one is not a constant multiple of the other.

Theorem 2.1.3. *Let p, q be continuous functions. Let y_1 and y_2 be two linearly independent solutions to the homogeneous equation* (2.2)*. Then every other solution is of the form*

$$y = C_1 y_1 + C_2 y_2.$$

That is, $y = C_1 y_1 + C_2 y_2$ is the general solution.

For example, we found the solutions $y_1 = \sin x$ and $y_2 = \cos x$ for the equation $y'' + y = 0$. It is not hard to see that sine and cosine are not constant multiples of each other. If $\sin x = A \cos x$ for some constant A, we let $x = 0$ and this would imply $A = 0$. But then $\sin x = 0$ for all x, which is preposterous. So y_1 and y_2 are linearly independent. Hence,

$$y = C_1 \cos x + C_2 \sin x$$

is the general solution to $y'' + y = 0$.

For two functions, checking linear independence is rather simple. Let us see another example. Consider $y'' - 2x^{-2} y = 0$. Then $y_1 = x^2$ and $y_2 = {}^1\!/_x$ are solutions. To see that they are linearly indepedent, suppose one is a multple of the other: $y_1 = A y_2$, we just have to find out that A cannot be a constant. In this case we have $A = {}^{y_1}\!/_{y_2} = x^3$, this most decidedly not a constant. So $y = C_1 x^2 + C_2 {}^1\!/_x$ is the general solution.

If you have one solution to a second order linear homogeneous equation, then you can find another one. This is the *reduction of order method*. The idea is that if we somehow found y_1 as a solution of $y'' + p(x)y' + q(x)y = 0$ we try a second solution of the form $y_2(x) = y_1(x)v(x)$. We just need to find v. We plug y_2 into the equation:

$$\begin{aligned} 0 = y_2'' + p(x)y_2' + q(x)y_2 &= y_1''v + 2y_1'v' + y_1v'' + p(x)(y_1'v + y_1v') + q(z)y_1v \\ &= y_1v'' + (2y_1' + p(x)y_1)v' + \cancelto{0}{\big(y_1'' + p(x)y_1' + q(x)y_1\big)}v. \end{aligned}$$

In other words, $y_1v'' + (2y_1' + p(x)y_1)v' = 0$. Using $w = v'$ we have the first order linear equation $y_1w' + (2y_1' + p(x)y_1)w = 0$. After solving this equation for w (integrating factor), we find v by antidifferentiating w. We then form y_2 by computing y_1v. For example, suppose we somehow know $y_1 = x$ is a solution to $y'' + x^{-1}y' - x^{-2}y = 0$. The equation for w is then $xw' + 3w = 0$. We find a solution, $w = Cx^{-3}$, and we find an antiderivative $v = \frac{-C}{2x^2}$. Hence $y_2 = y_1v = \frac{-C}{2x}$. Any C works and so $C = -2$ makes $y_2 = {}^1/x$. Thus, the general solution is $y = C_1x + C_2{}^1/x$.

Since we have a formula for the solution to the first order linear equation, we can write a formula for y_2:

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{\big(y_1(x)\big)^2}\,dx$$

Although it is much easier to remember that we just need to try $y_2(x) = y_1(x)v(x)$ and find $v(x)$ as we did above. Also, the technique works for higher order equations too: you get to reduce the order for each solution you find. So it is better to remember how to do it rather than a specific formula.

We will study the solution of nonhomogeneous equations in § 2.5. We will first focus on finding general solutions to homogeneous equations.

2.1.1 Exercises

***Exercise* 2.1.2:** *Show that $y = e^x$ and $y = e^{2x}$ are linearly independent.*

***Exercise* 2.1.3:** *Take $y'' + 5y = 10x + 5$. Find (guess!) a solution.*

***Exercise* 2.1.4:** *Prove the superposition principle for nonhomogeneous equations. Suppose that y_1 is a solution to $Ly_1 = f(x)$ and y_2 is a solution to $Ly_2 = g(x)$ (same linear operator L). Show that $y = y_1 + y_2$ solves $Ly = f(x) + g(x)$.*

***Exercise* 2.1.5:** *For the equation $x^2y'' - xy' = 0$, find two solutions, show that they are linearly independent and find the general solution. Hint: Try $y = x^r$.*

Equations of the form $ax^2y'' + bxy' + cy = 0$ are called *Euler's equations* or *Cauchy–Euler equations*. They are solved by trying $y = x^r$ and solving for r (assume that $x \geq 0$ for simplicity).

***Exercise* 2.1.6:** *Suppose that* $(b-a)^2 - 4ac > 0$.

a) *Find a formula for the general solution of* $ax^2y'' + bxy' + cy = 0$. *Hint: Try* $y = x^r$ *and find a formula for* r.

b) *What happens when* $(b-a)^2 - 4ac = 0$ *or* $(b-a)^2 - 4ac < 0$?

We will revisit the case when $(b-a)^2 - 4ac < 0$ later.

***Exercise* 2.1.7:** *Same equation as in Exercise 2.1.6. Suppose* $(b-a)^2 - 4ac = 0$. *Find a formula for the general solution of* $ax^2y'' + bxy' + cy = 0$. *Hint: Try* $y = x^r \ln x$ *for the second solution.*

***Exercise* 2.1.8** (reduction of order)**:** *Suppose* y_1 *is a solution to* $y'' + p(x)y' + q(x)y = 0$. *By directly plugging into the equation, show that*

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{\big(y_1(x)\big)^2}\,dx$$

is also a solution.

***Exercise* 2.1.9** (Chebyshev's equation of order 1)**:** *Take* $(1-x^2)y'' - xy' + y = 0$.

a) *Show that* $y = x$ *is a solution.*

b) *Use reduction of order to find a second linearly independent solution.*

c) *Write down the general solution.*

***Exercise* 2.1.10** (Hermite's equation of order 2)**:** *Take* $y'' - 2xy' + 4y = 0$.

a) *Show that* $y = 1 - 2x^2$ *is a solution.*

b) *Use reduction of order to find a second linearly independent solution.*

c) *Write down the general solution.*

***Exercise* 2.1.101:** *Are* $\sin(x)$ *and* e^x *linearly independent? Justify.*

***Exercise* 2.1.102:** *Are* e^x *and* e^{x+2} *linearly independent? Justify.*

***Exercise* 2.1.103:** *Guess a solution to* $y'' + y' + y = 5$.

***Exercise* 2.1.104:** *Find the general solution to* $xy'' + y' = 0$. *Hint: It is a first order ODE in* y'.

***Exercise* 2.1.105:** *Write down an equation (guess) for which we have the solutions* e^x *and* e^{2x}. *Hint: Try an equation of the form* $y'' + Ay' + By = 0$ *for constants* A *and* B, *plug in both* e^x *and* e^{2x} *and solve for* A *and* B.

In the following exercises, show that the given functions y_1 and y_2 are solutions to the DE. Then show that y_1 and y_2 are linearly independent. Applying Theorem 2.1.3, write the general solution. Impose the given ICs to find the particular solution to the IVP.

Exercise* 2.1.151*: $y'' + 25y = 0$; $y_1 = \cos 5x,\ y_2 = \sin 5x$; $y(0) = -2,\ y'(0) = 3$.

Exercise* 2.1.152*: $y'' + y' - 6y = 0$; $y_1 = e^{2x},\ y_2 = e^{-3x}$; $y(0) = 4,\ y'(0) = -2$.

Exercise* 2.1.153*: $y'' + 8y' + 16y = 0$; $y_1 = e^{-4x},\ y_2 = xe^{-4x}$; $y(0) = -1,\ y'(0) = -4$.

Exercise* 2.1.154*: $y'' + 6y' + 8y = 0$; $y_1 = e^{-2x},\ y_2 = e^{-4x}$; $y(0) = 3,\ y'(0) = -5$.

Exercise* 2.1.155*: $y'' + 6y' + 9y = 0$; $y_1 = e^{-3x},\ y_2 = xe^{-3x}$; $y(0) = 2,\ y'(0) = 5$.

2.2 Constant coefficient second order linear ODEs

Note: more than 1 lecture, second part of §3.1 in [EP], §3.1 in [BD]

2.2.1 Solving constant coefficient equations

Consider the problem

$$y'' - 6y' + 8y = 0, \qquad y(0) = -2, \qquad y'(0) = 6.$$

This is a second order linear homogeneous equation with constant coefficients. *Constant coefficients* means that the functions in front of y'', y', and y are constants, they do not depend on x.

To guess a solution, think of a function that stays essentially the same when we differentiate it, so that we can take the function and its derivatives, add some multiples of these together, and end up with zero. Yes, we are talking about the exponential.

Let us try* a solution of the form $y = e^{rx}$. Then $y' = re^{rx}$ and $y'' = r^2e^{rx}$. Plug in to get

$$\begin{aligned} y'' - 6y' + 8y &= 0, \\ \underbrace{r^2e^{rx}}_{y''} - 6\underbrace{re^{rx}}_{y'} + 8\underbrace{e^{rx}}_{y} &= 0, \\ r^2 - 6r + 8 &= 0 \qquad \text{(divide through by } e^{rx}\text{)}, \\ (r-2)(r-4) &= 0. \end{aligned}$$

Hence, if $r = 2$ or $r = 4$, then e^{rx} is a solution. So let $y_1 = e^{2x}$ and $y_2 = e^{4x}$.

Exercise* 2.2.1*: *Check that y_1 and y_2 are solutions.*

The functions e^{2x} and e^{4x} are linearly independent. If they were not linearly independent, we could write $e^{4x} = Ce^{2x}$ for some constant C, implying that $e^{2x} = C$ for all x, which is clearly not possible. Hence, we can write the general solution as

$$y = C_1e^{2x} + C_2e^{4x}.$$

We need to solve for C_1 and C_2. To apply the initial conditions, we first find $y' = 2C_1e^{2x} + 4C_2e^{4x}$. We plug $x = 0$ into y and y' and solve.

$$\begin{aligned} -2 &= y(0) = C_1 + C_2, \\ 6 &= y'(0) = 2C_1 + 4C_2. \end{aligned}$$

*Making an educated guess with some parameters to solve for is such a central technique in differential equations, that people sometimes use a fancy name for such a guess: *ansatz*, German for "initial placement of a tool at a work piece." Yes, the Germans have a word for that.

Either apply some matrix algebra, or just solve these by high school math. For example, divide the second equation by 2 to obtain $3 = C_1 + 2C_2$, and subtract the two equations to get $5 = C_2$. Then $C_1 = -7$ as $-2 = C_1 + 5$. Hence, the solution we are looking for is

$$y = -7e^{2x} + 5e^{4x}.$$

Let us generalize this example into a method. Suppose that we have an equation

$$ay'' + by' + cy = 0, \tag{2.3}$$

where a, b, c are constants. Try the solution $y = e^{rx}$ to obtain

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0.$$

Divide by e^{rx} to obtain the so-called *characteristic equation* of the ODE:

$$ar^2 + br + c = 0.$$

Solve for the r by using the quadratic formula.

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

So e^{r_1x} and e^{r_2x} are solutions. There is still a difficulty if $r_1 = r_2$, but it is not hard to overcome.

Theorem 2.2.1. *Suppose that r_1 and r_2 are the roots of the characteristic equation.*

(*i*) *If r_1 and r_2 are distinct and real (when $b^2 - 4ac > 0$), then (2.3) has the general solution*

$$y = C_1e^{r_1x} + C_2e^{r_2x}.$$

(*ii*) *If $r_1 = r_2$ (happens when $b^2 - 4ac = 0$), then (2.3) has the general solution*

$$y = (C_1 + C_2x)\, e^{r_1x}.$$

Example 2.2.1: Solve

$$y'' - k^2y = 0.$$

The characteristic equation is $r^2 - k^2 = 0$ or $(r - k)(r + k) = 0$. Consequently, e^{-kx} and e^{kx} are the two linearly independent solutions, and the general solution is

$$y = C_1e^{kx} + C_2e^{-kx}.$$

Since $\cosh s = \frac{e^s+e^{-s}}{2}$ and $\sinh s = \frac{e^s-e^{-s}}{2}$, we can also write the general solution as

$$y = D_1 \cosh(kx) + D_2 \sinh(kx).$$

Example 2.2.2: Find the general solution of

$$y'' - 8y' + 16y = 0.$$

The characteristic equation is $r^2 - 8r + 16 = (r - 4)^2 = 0$. The equation has a double root $r_1 = r_2 = 4$. The general solution is, therefore,

$$y = (C_1 + C_2x)\, e^{4x} = C_1e^{4x} + C_2xe^{4x}.$$

***Exercise* 2.2.2:** *Check that e^{4x} and xe^{4x} are linearly independent.*

That e^{4x} solves the equation is clear. If xe^{4x} solves the equation, then we know we are done. Let us compute $y' = e^{4x} + 4xe^{4x}$ and $y'' = 8e^{4x} + 16xe^{4x}$. Plug in

$$y'' - 8y' + 16y = 8e^{4x} + 16xe^{4x} - 8(e^{4x} + 4xe^{4x}) + 16xe^{4x} = 0.$$

In some sense, a doubled root rarely happens. If coefficients are picked randomly, a doubled root is unlikely. There are, however, some natural phenomena (such as resonance as we will see) where a doubled root does happen, so we cannot just dismiss this case.

Let us give a short argument for why the solution xe^{rx} works when the root is doubled. This case is really a limiting case of when the two roots are distinct and very close. Note that $\frac{e^{r_2x}-e^{r_1x}}{r_2-r_1}$ is a solution when the roots are distinct. When we take the limit as r_1 goes to r_2, we are really taking the derivative of e^{rx} using r as the variable. Therefore, the limit is xe^{rx}, and hence this is a solution in the doubled root case.

2.2.2 Complex numbers and Euler's formula

A polynomial may have complex roots. The equation $r^2 + 1 = 0$ has no real roots, but it does have two complex roots. Here we review some properties of complex numbers.

Complex numbers may seem a strange concept, especially because of the terminology. There is nothing imaginary or really complicated about complex numbers. A complex number is simply a pair of real numbers, (a, b). Think of a complex number as a point in the plane. We add complex numbers in the straightforward way: $(a, b) + (c, d) = (a + c, b + d)$. We define multiplication by

$$(a, b) \times (c, d) \overset{\text{def}}{=} (ac - bd, ad + bc).$$

It turns out that with this multiplication rule, all the standard properties of arithmetic hold. Further, and most importantly $(0, 1) \times (0, 1) = (-1, 0)$.

Generally we write (a, b) as $a + ib$, and we treat i as if it were an unknown. When b is zero, then $(a, 0)$ is just the number a. We do arithmetic with complex numbers just as we would with polynomials. The property we just mentioned becomes $i^2 = -1$. So whenever we see i^2, we replace it by -1. For example,

$$(2 + 3i)(4i) - 5i = (2 \times 4)i + (3 \times 4)i^2 - 5i = 8i + 12(-1) - 5i = -12 + 3i.$$

The numbers i and $-i$ are the two roots of $r^2 + 1 = 0$. Engineers often use the letter j instead of i for the square root of -1. We use the mathematicians' convention and use i.

***Exercise* 2.2.3:** *Make sure you understand (that you can justify) the following identities:*

a) $i^2 = -1$, $i^3 = -i$, $i^4 = 1$,

b) $\dfrac{1}{i} = -i$,

c) $(3 - 7i)(-2 - 9i) = \cdots = -69 - 13i$,

d) $(3-2i)(3+2i) = 3^2-(2i)^2 = 3^2+2^2 = 13$,

e) $\frac{1}{3-2i} = \frac{1}{3-2i}\frac{3+2i}{3+2i} = \frac{3+2i}{13} = \frac{3}{13} + \frac{2}{13}i$.

We also define the exponential e^{a+ib} of a complex number. We do this by writing down the Taylor series and plugging in the complex number. Because most properties of the exponential can be proved by looking at the Taylor series, these properties still hold for the complex exponential. For example the very important property: $e^{x+y} = e^x e^y$. This means that $e^{a+ib} = e^a e^{ib}$. Hence if we can compute e^{ib}, we can compute e^{a+ib}. For e^{ib} we use the so-called *Euler's formula*.

Theorem 2.2.2 (Euler's formula)**.**

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta \qquad \textit{and} \qquad e^{-i\theta} = \cos\theta - i\sin\theta.}$$

In other words, $e^{a+ib} = e^a\big(\cos(b) + i\sin(b)\big) = e^a\cos(b) + ie^a\sin(b)$.

***Exercise* 2.2.4:** *Using Euler's formula, check the identities*:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \textit{and} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$

***Exercise* 2.2.5:** *Double angle identities*: *Start with $e^{i(2\theta)} = \big(e^{i\theta}\big)^2$. Use Euler on each side and deduce*:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \qquad \textit{and} \qquad \sin(2\theta) = 2\sin\theta\cos\theta.$$

For a complex number $a + ib$ we call a the *real part* and b the *imaginary part* of the number. Often the following notation is used,

$$\operatorname{Re}(a + ib) = a \qquad \text{and} \qquad \operatorname{Im}(a + ib) = b.$$

2.2.3 Complex roots

Suppose the equation $ay'' + by' + cy = 0$ has the characteristic equation $ar^2 + br + c = 0$ that has complex roots. By the quadratic formula, the roots are $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. These roots are complex if $b^2 - 4ac < 0$. In this case the roots are

$$r_1, r_2 = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}.$$

As you can see, we always get a pair of roots of the form $\alpha \pm i\beta$. In this case we can still write the solution as

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}.$$

However, the exponential is now complex-valued. We need to allow C_1 and C_2 to be complex numbers to obtain a real-valued solution (which is what we are after). While there is nothing particularly wrong with this approach, it can make calculations harder and it is generally preferred to find two real-valued solutions.

Here we can use Euler's formula. Let

$$y_1 = e^{(\alpha+i\beta)x} \qquad \text{and} \qquad y_2 = e^{(\alpha-i\beta)x}.$$

Then

$$y_1 = e^{\alpha x}\cos(\beta x) + ie^{\alpha x}\sin(\beta x),$$
$$y_2 = e^{\alpha x}\cos(\beta x) - ie^{\alpha x}\sin(\beta x).$$

Linear combinations of solutions are also solutions. Hence,

$$y_3 = \frac{y_1 + y_2}{2} = e^{\alpha x}\cos(\beta x),$$
$$y_4 = \frac{y_1 - y_2}{2i} = e^{\alpha x}\sin(\beta x),$$

are also solutions. Furthermore, they are real-valued. It is not hard to see that they are linearly independent (not multiples of each other). Therefore, we have the following theorem.

Theorem 2.2.3. *Take the equation*

$$ay'' + by' + cy = 0.$$

If the characteristic equation has the roots $\alpha \pm i\beta$ (when $b^2 - 4ac < 0$), then the general solution is

$$y = C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x).$$

Example 2.2.3: Find the general solution of $y'' + k^2 y = 0$, for a constant $k > 0$.

The characteristic equation is $r^2 + k^2 = 0$. Therefore, the roots are $r = \pm ik$, and by the theorem, we have the general solution

$$y = C_1\cos(kx) + C_2\sin(kx).$$

Example 2.2.4: Find the solution of $y'' - 6y' + 13y = 0$, $y(0) = 0$, $y'(0) = 10$.

The characteristic equation is $r^2 - 6r + 13 = 0$. By completing the square we get $(r-3)^2 + 2^2 = 0$ and hence the roots are $r = 3 \pm 2i$. By the theorem we have the general solution

$$y = C_1 e^{3x}\cos(2x) + C_2 e^{3x}\sin(2x).$$

To find the solution satisfying the initial conditions, we first plug in zero to get

$$0 = y(0) = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 = C_1.$$

Hence, $C_1 = 0$ and $y = C_2 e^{3x}\sin(2x)$. We differentiate,

$$y' = 3C_2 e^{3x}\sin(2x) + 2C_2 e^{3x}\cos(2x).$$

We again plug in the initial condition and obtain $10 = y'(0) = 2C_2$, or $C_2 = 5$. The solution we are seeking is

$$y = 5e^{3x}\sin(2x).$$

2.2.4 Exercises

***Exercise* 2.2.6:** *Find the general solution of* $2y'' + 2y' - 4y = 0$.

***Exercise* 2.2.7:** *Find the general solution of* $y'' + 9y' - 10y = 0$.

***Exercise* 2.2.8:** *Solve* $y'' - 8y' + 16y = 0$ *for* $y(0) = 2$, $y'(0) = 0$.

***Exercise* 2.2.9:** *Solve* $y'' + 9y' = 0$ *for* $y(0) = 1$, $y'(0) = 1$.

***Exercise* 2.2.10:** *Find the general solution of* $2y'' + 50y = 0$.

***Exercise* 2.2.11:** *Find the general solution of* $y'' + 6y' + 13y = 0$.

***Exercise* 2.2.12:** *Find the general solution of* $y'' = 0$ *using the methods of this section.*

***Exercise* 2.2.13:** *The method of this section applies to equations of other orders than two. We will see higher orders later. Try to solve the first order equation* $2y' + 3y = 0$ *using the methods of this section.*

***Exercise* 2.2.14:** *Let us revisit the Cauchy–Euler equations of Exercise 2.1.6 on page 88. Suppose now that* $(b - a)^2 - 4ac < 0$. *Find a formula for the general solution of* $ax^2y'' + bxy' + cy = 0$. *Hint: Note that* $x^r = e^{r \ln x}$.

***Exercise* 2.2.15:** *Find the solution to* $y'' - (2\alpha)y' + \alpha^2 y = 0$, $y(0) = a$, $y'(0) = b$, *where* α, a, *and* b *are real numbers.*

***Exercise* 2.2.16:** *Construct an equation such that* $y = C_1 e^{-2x} \cos(3x) + C_2 e^{-2x} \sin(3x)$ *is the general solution.*

***Exercise* 2.2.51:** a) *If* $r_1 \neq r_2$, *show that* $e^{r_1 x}$ *and* $e^{r_2 x}$ *are linearly independent.*

b) *Show that* e^{rx} *and* xe^{rx} *are linearly independent.*

***Exercise* 2.2.52:** *Find the general solution to each of the following DEs:*

a) $2y'' + 8y' + 8y = 0$

b) $y'' + 2y' + 2y = 0$

c) $2y'' + 5y' - 3y = 0$

d) $y'' - 4y' + 13y = 0$

e) $9y'' - 6y' + y = 0$

f) $y'' + 4y' + 5y = 0$

***Exercise* 2.2.101:** *Find the general solution to* $y'' + 4y' + 2y = 0$.

***Exercise* 2.2.102:** *Find the general solution to* $y'' - 6y' + 9y = 0$.

Exercise* 2.2.103*: *Find the solution to* $2y'' + y' + y = 0$, $y(0) = 1$, $y'(0) = -2$.

Exercise* 2.2.104*: *Find the solution to* $2y'' + y' - 3y = 0$, $y(0) = a$, $y'(0) = b$.

Exercise* 2.2.105*: *Find the solution to* $z''(t) = -2z'(t) - 2z(t)$, $z(0) = 2$, $z'(0) = -2$.

Exercise* 2.2.106*: *Find the solution to* $y'' - (\alpha + \beta)y' + \alpha\beta y = 0$, $y(0) = a$, $y'(0) = b$, *where* α, β, a, *and* b *are real numbers, and* $\alpha \neq \beta$.

Exercise* 2.2.107*: *Construct an equation such that* $y = C_1e^{3x} + C_2e^{-2x}$ *is the general solution.*

Exercise* 2.2.151*: *Construct the DE for each of the following general solutions*:

a) $y = C_1e^{-3x} + C_2e^{-5x}$

b) $y = C_1 + C_2e^{4x}$

c) $y = C_1e^{-2x} + C_2xe^{-2x}$

d) $y = C_1e^{-x}\cos x + C_2e^{-x}\sin x$

e) $y = C_1 + C_2x$

Exercise* 2.2.152*: *Find the general solution to each of the following Cauchy-Euler equations, for* $x > 0$.

a) $x^2y'' + 3xy' + 2y = 0$

b) $x^2y'' - 3xy' + 4y = 0$

c) $x^2y'' + 3xy' + 5y = 0$

d) $3x^2y'' + 14xy' - 4y = 0$

e) $4x^2y'' + 8xy' + y = 0$

f) $4x^2y'' + 8xy' + 5y = 0$

2.3 Higher order linear ODEs

Note: somewhat more than 1 lecture, §3.2 and §3.3 in [EP], §4.1 and §4.2 in [BD]

We briefly study higher order equations. Equations appearing in applications tend to be second order. Higher order equations do appear from time to time, but generally the world around us is "second order."

The basic results about linear ODEs of higher order are essentially the same as for second order equations, with 2 replaced by n. The important concept of linear independence is somewhat more complicated when more than two functions are involved. For higher order constant coefficient ODEs, the methods developed are also somewhat harder to apply, but we will not dwell on these complications. It is also possible to use the methods for systems of linear equations from chapter 3 to solve higher order constant coefficient equations.

Let us start with a general homogeneous linear equation

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0. \tag{2.4}$$

Theorem 2.3.1 (Superposition). *Suppose $y_1, y_2, \ldots, y_n$ are solutions of the homogeneous equation (2.4). Then*

$$y(x) = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x)$$

also solves (2.4) for arbitrary constants $C_1, C_2, \ldots, C_n$.

In other words, a *linear combination* of solutions to (2.4) is also a solution to (2.4). We also have the existence and uniqueness theorem for nonhomogeneous linear equations.

Theorem 2.3.2 (Existence and uniqueness). *Suppose p_0 through p_{n-1}, and f are continuous functions on some interval I, a is a number in I, and $b_0, b_1, \ldots, b_{n-1}$ are constants. The equation*

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = f(x)$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1, \quad \ldots, \quad y^{(n-1)}(a) = b_{n-1}.$$

2.3.1 Linear independence

When we had two functions y_1 and y_2 we said they were linearly independent if one was not the multiple of the other. Same idea holds for n functions. In this case it is easier to state as follows. The functions $y_1, y_2, \ldots, y_n$ are *linearly independent* if the equation

$$c_1y_1 + c_2y_2 + \cdots + c_ny_n = 0$$

has only the trivial solution $c_1 = c_2 = \cdots = c_n = 0$, where the equation must hold for all x. If we can solve equation with some constants where for example $c_1 \neq 0$, then we can solve for y_1 as a linear combination of the others. If the functions are not linearly independent, they are *linearly dependent*.

Example 2.3.1: Show that e^x, e^{2x}, e^{3x} are linearly independent.

Let us give several ways to show this fact. Many textbooks (including [EP] and [F]) introduce Wronskians, but it is difficult to see why they work and they are not really necessary here.

Let us write down

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0.$$

We use rules of exponentials and write $z = e^x$. Hence $z^2 = e^{2x}$ and $z^3 = e^{3x}$. Then we have

$$c_1 z + c_2 z^2 + c_3 z^3 = 0.$$

The left-hand side is a third degree polynomial in z. It is either identically zero, or it has at most 3 zeros. Therefore, it is identically zero, $c_1 = c_2 = c_3 = 0$, and the functions are linearly independent.

Let us try another way. As before we write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0.$$

This equation has to hold for all x. We divide through by e^{3x} to get

$$c_1 e^{-2x} + c_2 e^{-x} + c_3 = 0.$$

As the equation is true for all x, let $x \to \infty$. After taking the limit we see that $c_3 = 0$. Hence our equation becomes

$$c_1 e^x + c_2 e^{2x} = 0.$$

Rinse, repeat!

How about yet another way. We again write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0.$$

We can evaluate the equation and its derivatives at different values of x to obtain equations for c_1, c_2, and c_3. Let us first divide by e^x for simplicity.

$$c_1 + c_2 e^x + c_3 e^{2x} = 0.$$

We set $x = 0$ to get the equation $c_1 + c_2 + c_3 = 0$. Now differentiate both sides

$$c_2 e^x + 2c_3 e^{2x} = 0.$$

We set $x = 0$ to get $c_2 + 2c_3 = 0$. We divide by e^x again and differentiate to get $2c_3 e^x = 0$. It is clear that c_3 is zero. Then c_2 must be zero as $c_2 = -2c_3$, and c_1 must be zero because $c_1 + c_2 + c_3 = 0$.

There is no one best way to do it. All of these methods are perfectly valid. The important thing is to understand why the functions are linearly independent.

Example 2.3.2: On the other hand, the functions e^x, e^{-x}, and $\cosh x$ are linearly dependent. Simply apply definition of the hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{or} \qquad 2\cosh x - e^x - e^{-x} = 0.$$

2.3.2 Constant coefficient higher order ODEs

When we have a higher order constant coefficient homogeneous linear equation, the song and dance is exactly the same as it was for second order. We just need to find more solutions. If the equation is n^{th} order, we need to find n linearly independent solutions. It is best seen by example.

Example 2.3.3: Find the general solution to

$$y''' - 3y'' - y' + 3y = 0. \tag{2.5}$$

Try: $y = e^{rx}$. We plug in and get

$$\underbrace{r^3e^{rx}}_{y'''} - 3\underbrace{r^2e^{rx}}_{y''} - \underbrace{re^{rx}}_{y'} + 3\underbrace{e^{rx}}_{y} = 0.$$

We divide through by e^{rx}. Then

$$r^3 - 3r^2 - r + 3 = 0.$$

The trick now is to find the roots. There is a formula for the roots of degree 3 and 4 polynomials but it is very complicated. There is no formula for higher degree polynomials. That does not mean that the roots do not exist. There are always n roots for an n^{th} degree polynomial. They may be repeated and they may be complex. Computers are pretty good at finding roots approximately for reasonable size polynomials.

A good place to start is to plot the polynomial and check where it is zero. We can also simply try plugging in. We just start plugging in numbers $r = -2, -1, 0, 1, 2, \ldots$ and see if we get a hit (we can also try complex numbers). Even if we do not get a hit, we may get an indication of where the root is. For example, we plug $r = -2$ into our polynomial and get -15; we plug in $r = 0$ and get 3. That means there is a root between $r = -2$ and $r = 0$, because the sign changed. If we find one root, say r_1, then we know $(r - r_1)$ is a factor of our polynomial. Polynomial long division can then be used.

A good strategy is to begin with $r = 0$, 1, or -1. These are easy to compute. Our polynomial has two such roots, $r_1 = -1$ and $r_2 = 1$. There should be 3 roots and the last root is reasonably easy to find. The constant term in a monic* polynomial such as this is the multiple of the negations of all the roots because $r^3 - 3r^2 - r + 3 = (r - r_1)(r - r_2)(r - r_3)$. So

$$3 = (-r_1)(-r_2)(-r_3) = (1)(-1)(-r_3) = r_3.$$

You should check that $r_3 = 3$ really is a root. Hence e^{-x}, e^x and e^{3x} are solutions to (2.5). They are linearly independent as can easily be checked, and there are 3 of them, which happens to be exactly the number we need. So the general solution is

$$y = C_1e^{-x} + C_2e^x + C_3e^{3x}.$$

*The word monic means that the coefficient of the top degree r^d, in our case r^3, is 1.

Suppose we were given some initial conditions $y(0) = 1$, $y'(0) = 2$, and $y''(0) = 3$. Then

$$\begin{aligned} 1 = y(0) &= C_1 + C_2 + C_3, \\ 2 = y'(0) &= -C_1 + C_2 + 3C_3, \\ 3 = y''(0) &= C_1 + C_2 + 9C_3. \end{aligned}$$

It is possible to find the solution by high school algebra, but it would be a pain. The sensible way to solve a system of equations such as this is to use matrix algebra, see § 3.2. For now we note that the solution is $C_1 = -1/4$, $C_2 = 1$, and $C_3 = 1/4$. The specific solution to the ODE is

$$y = \frac{-1}{4}\, e^{-x} + e^x + \frac{1}{4}\, e^{3x}.$$

Next, suppose that we have real roots, but they are repeated. Let us say we have a root r repeated k times. In the spirit of the second order solution, and for the same reasons, we have the solutions

$$e^{rx}, \quad xe^{rx}, \quad x^2e^{rx}, \quad \ldots, \quad x^{k-1}e^{rx}.$$

We take a linear combination of these solutions to find the general solution.

Example 2.3.4: Solve

$$y^{(4)} - 3y''' + 3y'' - y' = 0.$$

We note that the characteristic equation is

$$r^4 - 3r^3 + 3r^2 - r = 0.$$

By inspection we note that $r^4 - 3r^3 + 3r^2 - r = r(r-1)^3$. Hence the roots given with multiplicity are $r = 0, 1, 1, 1$. Thus the general solution is

$$y = \underbrace{(C_1 + C_2x + C_3x^2)\, e^x}_{\text{terms coming from } r=1} + \underbrace{C_4}_{\text{from } r=0}.$$

The case of complex roots is similar to second order equations. Complex roots always come in pairs $r = \alpha \pm i\beta$. Suppose we have two such complex roots, each repeated k times. The corresponding solution is

$$(C_0 + C_1x + \cdots + C_{k-1}x^{k-1})\, e^{\alpha x}\cos(\beta x) + (D_0 + D_1x + \cdots + D_{k-1}x^{k-1})\, e^{\alpha x}\sin(\beta x).$$

where $C_0, \ldots, C_{k-1}, D_0, \ldots, D_{k-1}$ are arbitrary constants.

Example 2.3.5: Solve

$$y^{(4)} - 4y''' + 8y'' - 8y' + 4y = 0.$$

The characteristic equation is

$$\begin{aligned} r^4 - 4r^3 + 8r^2 - 8r + 4 &= 0, \\ (r^2 - 2r + 2)^2 &= 0, \\ \big((r-1)^2 + 1\big)^2 &= 0. \end{aligned}$$

Hence the roots are $1 \pm i$, both with multiplicity 2. Hence the general solution to the ODE is

$$y = (C_1 + C_2 x)\, e^x \cos x + (C_3 + C_4 x)\, e^x \sin x.$$

The way we solved the characteristic equation above is really by guessing or by inspection. It is not so easy in general. We could also have asked a computer or an advanced calculator for the roots.

2.3.3 Exercises

***Exercise* 2.3.1:** *Find the general solution for* $y''' - y'' + y' - y = 0$.

***Exercise* 2.3.2:** *Find the general solution for* $y^{(4)} - 5y''' + 6y'' = 0$.

***Exercise* 2.3.3:** *Find the general solution for* $y''' + 2y'' + 2y' = 0$.

***Exercise* 2.3.4:** *Suppose the characteristic equation for an ODE is* $(r-1)^2(r-2)^2 = 0$.

a) *Find such a differential equation.*

b) *Find its general solution.*

***Exercise* 2.3.5:** *Suppose that a fourth order equation has a solution* $y = 2e^{4x} x \cos x$.

a) *Find such an equation.*

b) *Find the initial conditions that the given solution satisfies.*

***Exercise* 2.3.6:** *Find the general solution for the equation of Exercise 2.3.5.*

***Exercise* 2.3.7:** *Let* $f(x) = e^x - \cos x$, $g(x) = e^x + \cos x$, *and* $h(x) = \cos x$. *Are* $f(x)$, $g(x)$, *and* $h(x)$ *linearly independent? If so, show it, if not, find a linear combination that works.*

***Exercise* 2.3.8:** *Let* $f(x) = 0$, $g(x) = \cos x$, *and* $h(x) = \sin x$. *Are* $f(x)$, $g(x)$, *and* $h(x)$ *linearly independent? If so, show it, if not, find a linear combination that works.*

***Exercise* 2.3.9:** *Are* x, x^2, *and* x^4 *linearly independent? If so, show it, if not, find a linear combination that works.*

***Exercise* 2.3.10:** *Are* e^x, xe^x, *and* x^2e^x *linearly independent? If so, show it, if not, find a linear combination that works.*

***Exercise* 2.3.11:** *Find an equation such that* $y = xe^{-2x}\sin(3x)$ *is a solution.*

***Exercise* 2.3.101:** *Find the general solution of* $y^{(5)} - y^{(4)} = 0$.

***Exercise* 2.3.102:** *Suppose that the characteristic equation of a third order differential equation has roots* $\pm 2i$ *and* 3.

a) *What is the characteristic equation?*

b) *Find the corresponding differential equation.*

c) *Find the general solution.*

***Exercise* 2.3.103:** *Solve* $1001y''' + 3.2y'' + \pi y' - \sqrt{4}y = 0$, $y(0) = 0$, $y'(0) = 0$, $y''(0) = 0$.

***Exercise* 2.3.104:** *Are* e^x, e^{x+1}, e^{2x}, $\sin(x)$ *linearly independent? If so, show it, if not find a linear combination that works.*

***Exercise* 2.3.105:** *Are* $\sin(x)$, x, $x\sin(x)$ *linearly independent? If so, show it, if not find a linear combination that works.*

***Exercise* 2.3.106:** *Find an equation such that* $y = \cos(x)$, $y = \sin(x)$, $y = e^x$ *are solutions.*

***Exercise* 2.3.151:** *Find the general solution to the following DEs:*

a) $4y^{(4)} + 7y''' - 2y'' = 0$

b) $y^{(4)} - 16y = 0$

c) $y^{(4)} - 5y'' + 4y = 0$

d) $y^{(5)} + 6y''' + 9y' = 0$

e) $y^{(4)} + 8y'' - 9y = 0$

f) $y^{(5)} + y^{(4)} - 12y''' = 0$

g) $y^{(4)} + 3y''' + 3y'' + y' = 0$

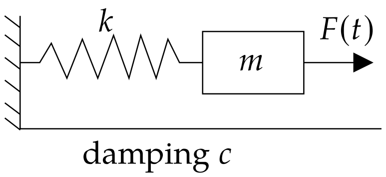
h) $y^{(4)} - 8y'' + 16y = 0$

2.4 Mechanical vibrations

Note: 2 lectures, §3.4 in [EP], §3.7 in [BD]

Let us look at some applications of linear second order constant coefficient equations.

2.4.1 Some examples



Our first example is a mass on a spring. Suppose we have a mass $m > 0$ (in kilograms) connected by a spring with spring constant $k > 0$ (in newtons per meter) to a fixed wall. There may be some external force $F(t)$ (in newtons) acting on the mass. Finally, there is some friction measured by $c \geq 0$ (in newton-seconds per meter) as the mass slides along the floor (or perhaps a damper is connected).

Let x be the displacement of the mass ($x = 0$ is the rest position), with x growing to the right (away from the wall). The force exerted by the spring is proportional to the compression of the spring by Hooke's law. Therefore, it is kx in the negative direction. Similarly the amount of force exerted by friction is proportional to the velocity of the mass. By Newton's second law we know that force equals mass times acceleration and hence $mx'' = F(t) - cx' - kx$ or

$$mx'' + cx' + kx = F(t).$$

This is a linear second order constant coefficient ODE. We say the motion is

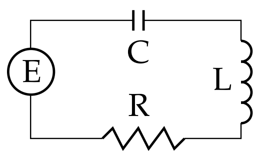
(i) *forced*, if $F \not\equiv 0$ (if F is not identically zero),

(ii) *unforced* or *free*, if $F \equiv 0$ (if F is identically zero),

(iii) *damped*, if $c > 0$, and

(iv) *undamped*, if $c = 0$.

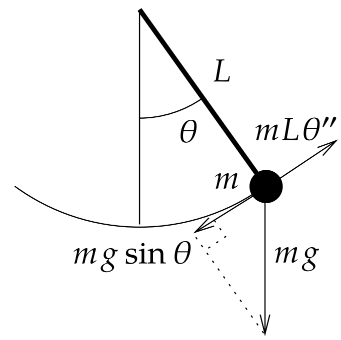
This system appears in lots of applications even if it does not at first seem like it. Many real-world scenarios can be simplified to a mass on a spring. For example, a bungee jump setup is essentially a mass and spring system (you are the mass). It would be good if someone did the math before you jump off the bridge, right? Let us give two other examples.



Here is an example for electrical engineers. Consider the pictured RLC circuit. There is a resistor with a resistance of R ohms, an inductor with an inductance of L henries, and a capacitor with a capacitance of C farads. There is also an electric source (such as a battery) giving a voltage of $E(t)$ volts at time t (measured in seconds). Let $Q(t)$ be the charge in coulombs on the capacitor and $I(t)$ be the current in the circuit. The relation between the two is $Q' = I$. By elementary principles we find $LI' + RI + Q/C = E$. We differentiate to get

$$LI''(t) + RI'(t) + \frac{1}{C}I(t) = E'(t).$$

This is a nonhomogeneous second order constant coefficient linear equation. As L, R, and C are all positive, this system behaves just like the mass and spring system. Position of the mass is replaced by current. Mass is replaced by inductance, damping is replaced by resistance, and the spring constant is replaced by one over the capacitance. The change in voltage becomes the forcing function—for constant voltage this is an unforced motion.



Our next example behaves like a mass and spring system only approximately. Suppose a mass m hangs on a pendulum of length L. We seek an equation for the angle $\theta(t)$ (in radians). Let g be the force of gravity. Elementary physics mandates that the equation is

$$\theta'' + \frac{g}{L}\sin\theta = 0.$$

Let us derive this equation using Newton's second law: force equals mass times acceleration. The acceleration is $L\theta''$ and mass is m. So $mL\theta''$ has to be equal to the tangential component of the force given by the gravity, which is $mg\sin\theta$ in the opposite direction. So $mL\theta'' = -mg\sin\theta$. The m curiously cancels from the equation.

Now we make our approximation. For small θ we have that approximately $\sin\theta \approx \theta$. This can be seen by looking at the graph. In Figure 2.1 we can see that for approximately $-0.5 < \theta < 0.5$ (in radians) the graphs of $\sin\theta$ and θ are almost the same.

Therefore, when the swings are small, θ is small and we can model the behavior by the simpler linear equation

$$\theta'' + \frac{g}{L}\theta = 0.$$

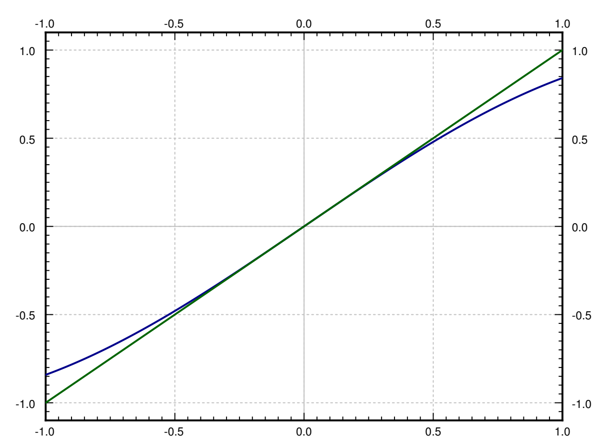


Figure 2.1: The graphs of $\sin\theta$ *and* θ *(in radians).*

The errors from this approximation build up. So after a long time, the state of the real-world system might be substantially different from our solution. Also we will see that in a mass-spring system, the amplitude is independent of the period. This is not true for a pendulum. Nevertheless, for reasonably short periods of time and small swings (that is, only small angles θ), the approximation is reasonably good.

In real-world problems it is often necessary to make these types of simplifications. We must understand both the mathematics and the physics of the situation to see if the simplification is valid in the context of the questions we are trying to answer.

2.4.2 Free undamped motion

In this section we only consider free or unforced motion, as we do not know yet how to solve nonhomogeneous equations. Let us start with undamped motion where $c = 0$. The equation is

$$mx'' + kx = 0.$$

We divide by m and let $\omega_0 = \sqrt{k/m}$ to rewrite the equation as

$$x'' + \omega_0^2 x = 0.$$

The general solution to this equation is

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t).$$

By a trigonometric identity

$$A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \gamma),$$

for two different constants C and γ. It is not hard to compute that $C = \sqrt{A^2 + B^2}$ and $\tan\gamma = B/A$. Therefore, we let C and γ be our arbitrary constants and write $x(t) = C\cos(\omega_0 t - \gamma)$.

***Exercise* 2.4.1:** *Justify the identity* $A\cos(\omega_0 t) + B\sin(\omega_0 t) = C\cos(\omega_0 t - \gamma)$ *and verify the equations for* C *and* γ*. Hint: Start with* $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$ *and multiply by* C*. Then what should* α *and* β *be?*

While it is generally easier to use the first form with A and B to solve for the initial conditions, the second form is much more natural. The constants C and γ have nice physical interpretation. Write the solution as

$$x(t) = C\cos(\omega_0 t - \gamma).$$

This is a pure-frequency oscillation (a sine wave). The *amplitude* is C, ω_0 is the (angular) *frequency*, and γ is the so-called *phase shift*. The phase shift just shifts the graph left or right. We call ω_0 the *natural (angular) frequency*. This entire setup is called *simple harmonic motion*.

Let us pause to explain the word *angular* before the word *frequency*. The units of ω_0 are radians per unit time, not cycles per unit time as is the usual measure of frequency. Because one cycle is 2π radians, the usual frequency is given by $\frac{\omega_0}{2\pi}$. It is simply a matter of where we put the constant 2π, and that is a matter of taste.

The *period* of the motion is one over the frequency (in cycles per unit time) and hence $\frac{2\pi}{\omega_0}$. That is the amount of time it takes to complete one full cycle.

Example 2.4.1: Suppose that $m = 2\,\text{kg}$ and $k = 8\,\text{N/m}$. The whole mass and spring setup is sitting on a truck that was traveling at $1\,\text{m/s}$. The truck crashes and hence stops. The mass was held in place 0.5 meters forward from the rest position. During the crash the mass gets loose. That is, the mass is now moving forward at $1\,\text{m/s}$, while the other end of the spring is held in place. The mass therefore starts oscillating. What is the frequency of the resulting oscillation? What is the amplitude? The units are the mks units (meters-kilograms-seconds).

The setup means that the mass was at half a meter in the positive direction during the crash and relative to the wall the spring is mounted to, the mass was moving forward (in the positive direction) at 1 m/s. This gives us the initial conditions.

So the equation with initial conditions is

$$2x'' + 8x = 0, \qquad x(0) = 0.5, \qquad x'(0) = 1.$$

We directly compute $\omega_0 = \sqrt{k/m} = \sqrt{4} = 2$. Hence the angular frequency is 2. The usual frequency in Hertz (cycles per second) is $2/2\pi = 1/\pi \approx 0.318$.

The general solution is

$$x(t) = A\cos(2t) + B\sin(2t).$$

Letting $x(0) = 0.5$ means $A = 0.5$. Then $x'(t) = -2(0.5)\sin(2t) + 2B\cos(2t)$. Letting $x'(0) = 1$ we get $B = 0.5$. Therefore, the amplitude is $C = \sqrt{A^2 + B^2} = \sqrt{0.25 + 0.25} = \sqrt{0.5} \approx 0.707$. The solution is

$$x(t) = 0.5\cos(2t) + 0.5\sin(2t).$$

A plot of $x(t)$ is shown in Figure 2.2.

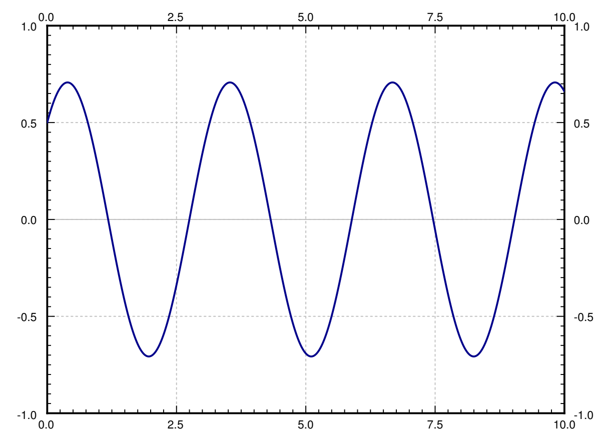


Figure 2.2: Simple undamped oscillation.

In general, for free undamped motion, a solution of the form

$$x(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

corresponds to the initial conditions $x(0) = A$ and $x'(0) = \omega_0 B$. Therefore, it is easy to figure out A and B from the initial conditions. The amplitude and the phase shift can then be computed from A and B. In the example, we have already found the amplitude C. Let us compute the phase shift. We know that $\tan\gamma = B/A = 1$. We take the arctangent of 1 and get $\pi/4$ or approximately 0.785. We still need to check if this γ is in the correct quadrant (and add π to γ if it is not). Since both A and B are positive, then γ should be in the first quadrant, $\pi/4$ radians is in the first quadrant, so $\gamma = \pi/4$.

Note: Many calculators and computer software have not only the `atan` function for arctangent, but also what is sometimes called `atan2`. This function takes two arguments, B and A, and returns a γ in the correct quadrant for you.

2.4.3 Free damped motion

Let us now focus on damped motion. Let us rewrite the equation

$$mx'' + cx' + kx = 0,$$

as

$$x'' + 2px' + \omega_0^2 x = 0,$$

where

$$\omega_0 = \sqrt{\frac{k}{m}}, \qquad p = \frac{c}{2m}.$$

The characteristic equation is

$$r^2 + 2pr + \omega_0^2 = 0.$$

Using the quadratic formula we get that the roots are

$$r = -p \pm \sqrt{p^2 - \omega_0^2}.$$

The form of the solution depends on whether we get complex or real roots. We get real roots if and only if the following number is nonnegative:

$$p^2 - \omega_0^2 = \left(\frac{c}{2m}\right)^2 - \frac{k}{m} = \frac{c^2 - 4km}{4m^2}.$$

The sign of $p^2 - \omega_0^2$ is the same as the sign of $c^2 - 4km$. Thus we get real roots if and only if $c^2 - 4km$ is nonnegative, or in other words if $c^2 \geq 4km$.

Overdamping

When $c^2 - 4km > 0$, the system is *overdamped*. In this case, there are two distinct real roots r_1 and r_2. Both roots are negative: As $\sqrt{p^2 - \omega_0^2}$ is always less than p, then $-p \pm \sqrt{p^2 - \omega_0^2}$ is negative in either case.

The solution is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

Since r_1, r_2 are negative, $x(t) \to 0$ as $t \to \infty$. Thus the mass will tend towards the rest position as time goes to infinity. For a few sample plots for different initial conditions, see Figure 2.3.

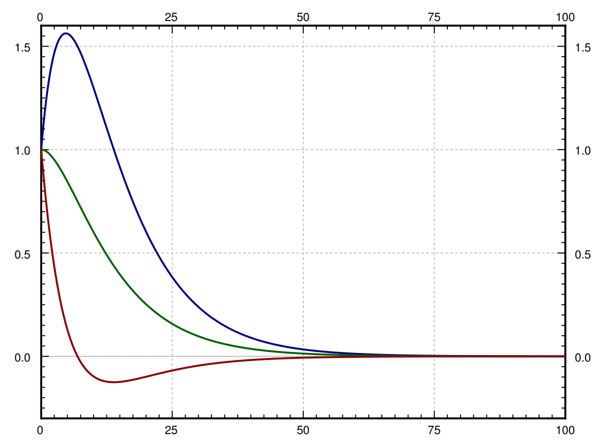


Figure 2.3: Overdamped motion for several different initial conditions.

No oscillation happens. In fact, the graph crosses the x-axis at most once. To see why, we try to solve $0 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. Therefore, $C_1 e^{r_1 t} = -C_2 e^{r_2 t}$ and using laws of exponents we obtain

$$\frac{-C_1}{C_2} = e^{(r_2 - r_1)t}.$$

This equation has at most one solution $t \geq 0$. For some initial conditions the graph never crosses the x-axis, as is evident from the sample graphs.

Example 2.4.2: Suppose the mass is released from rest. That is $x(0) = x_0$ and $x'(0) = 0$. Then

$$x(t) = \frac{x_0}{r_1 - r_2}\left(r_1 e^{r_2 t} - r_2 e^{r_1 t}\right).$$

It is not hard to see that this satisfies the initial conditions.

Critical damping

When $c^2 - 4km = 0$, the system is *critically damped*. In this case, there is one root of multiplicity 2 and this root is $-p$. Our solution is

$$x(t) = C_1 e^{-pt} + C_2 t e^{-pt}.$$

The behavior of a critically damped system is very similar to an overdamped system. After all a critically damped system is in some sense a limit of overdamped systems. Since these equations are really only an approximation to the real world, in reality we are never critically damped, it is a place we can only reach in theory. We are always a little bit underdamped or a little bit overdamped. It is better not to dwell on critical damping.

Underdamping

When $c^2 - 4km < 0$, the system is *underdamped*. In this case, the roots are complex.

$$\begin{aligned} r &= -p \pm \sqrt{p^2 - \omega_0^2} \\ &= -p \pm \sqrt{-1}\sqrt{\omega_0^2 - p^2} \\ &= -p \pm i\omega_1, \end{aligned}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. Our solution is

$$x(t) = e^{-pt}\left(A\cos(\omega_1 t) + B\sin(\omega_1 t)\right),$$

or

$$x(t) = Ce^{-pt}\cos(\omega_1 t - \gamma).$$

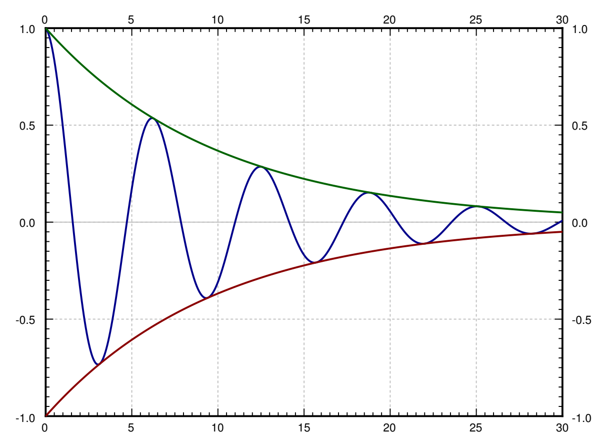


Figure 2.4: Underdamped motion with the envelope curves shown.

An example plot is given in Figure 2.4. Note that we still have that $x(t) \to 0$ as $t \to \infty$.

The figure also shows the *envelope curves* Ce^{-pt} and $-Ce^{-pt}$. The solution is the oscillating line between the two envelope curves. The envelope curves give the maximum amplitude of the oscillation at any given point in time. For example, if you are bungee jumping, you are really interested in computing the envelope curve as not to hit the concrete with your head.

The phase shift γ shifts the oscillation left or right, but within the envelope curves (the envelope curves do not change if γ changes).

Notice that the angular *pseudo-frequency** becomes smaller when the damping c (and hence p) becomes larger. This makes sense. When we change the damping just a little bit, we do not expect the behavior of the solution to change dramatically. If we keep making c larger, then at some point the solution should start looking like the solution for critical damping or overdamping, where no oscillation happens. So if c^2 approaches $4km$, we want ω_1 to approach 0.

On the other hand, when c gets smaller, ω_1 approaches ω_0 (ω_1 is always smaller than ω_0), and the solution looks more and more like the steady periodic motion of the undamped case. The envelope curves become flatter and flatter as c (and hence p) goes to 0.

2.4.4 Exercises

***Exercise* 2.4.2:** *Consider a mass and spring system with a mass $m = 2$, spring constant $k = 3$, and damping constant $c = 1$.*

a) *Set up and find the general solution of the system.*

b) *Is the system underdamped, overdamped or critically damped?*

c) *If the system is not critically damped, find a c that makes the system critically damped.*

***Exercise* 2.4.3:** *Do Exercise 2.4.2 for $m = 3$, $k = 12$, and $c = 12$.*

***Exercise* 2.4.4:** *Using the mks units (meters-kilograms-seconds), suppose you have a spring with spring constant 4 N/m. You want to use it to weigh items. Assume no friction. You place the mass on the spring and put it in motion.*

a) *You count and find that the frequency is 0.8 Hz (cycles per second). What is the mass?*

b) *Find a formula for the mass m given the frequency ω in Hz.*

***Exercise* 2.4.5:** *Suppose we add possible friction to Exercise 2.4.4. Further, suppose you do not know the spring constant, but you have two reference weights 1 kg and 2 kg to calibrate your setup. You put each in motion on your spring and measure the frequency. For the 1 kg weight you measured 1.1 Hz, for the 2 kg weight you measured 0.8 Hz.*

a) *Find k (spring constant) and c (damping constant).*

b) *Find a formula for the mass in terms of the frequency in Hz.* Note that there may be more than one possible mass for a given frequency.

c) *For an unknown object you measured 0.2 Hz, what is the mass of the object? Suppose that you know that the mass of the unknown object is more than a kilogram.*

*We do not call ω_1 a frequency since the solution is not really a periodic function.

Exercise 2.4.6: *Suppose you wish to measure the friction a mass of* 0.1 *kg experiences as it slides along a floor (you wish to find* c*). You have a spring with spring constant* $k = 5\,\mathrm{N/m}$*. You take the spring, you attach it to the mass and fix it to a wall. Then you pull on the spring and let the mass go. You find that the mass oscillates with frequency* 1 *Hz. What is the friction?*

Exercise 2.4.101: *A mass of* 2 *kilograms is on a spring with spring constant* k *newtons per meter with no damping. Suppose the system is at rest and at time* $t = 0$ *the mass is kicked and starts traveling at* 2 *meters per second. How large does* k *have to be to so that the mass does not go further than* 3 *meters from the rest position?*

Exercise 2.4.102: *Suppose we have an RLC circuit with a resistor of* 100 *milliohms* (0.1 *ohms*)*, inductor of inductance of* 50 *millihenries* (0.05 *henries*)*, and a capacitor of* 5 *farads, with constant voltage.*

a) *Set up the ODE equation for the current* I*.*

b) *Find the general solution.*

c) *Solve for* $I(0) = 10$ *and* $I'(0) = 0$*.*

Exercise 2.4.103: *A* 5000 *kg railcar hits a bumper (a spring) at* $1\,\mathrm{m/s}$*, and the spring compresses by* 0.1 *m. Assume no damping.*

a) *Find* k*.*

b) *How far does the spring compress when a* 10000 *kg railcar hits the spring at the same speed?*

c) *If the spring would break if it compresses further than* 0.3 *m, what is the maximum mass of a railcar that can hit it at* $1\,\mathrm{m/s}$*?*

d) *What is the maximum mass of a railcar that can hit the spring without breaking at* $2\,\mathrm{m/s}$*?*

Exercise 2.4.104: *A mass of* m *kg is on a spring with* $k = 3\,\mathrm{N/m}$ *and* $c = 2\,\mathrm{Ns/m}$*. Find the mass* m_0 *for which there is critical damping. If* $m < m_0$*, does the system oscillate or not, that is, is it underdamped or overdamped?*

Exercise 2.4.151: *Solve the following IVPs for position* $x(t)$*.*

If the system is overdamped or critically damped, solve a second IVP for the undamped system. Re-write the solution for the undamped system in the form $x_u(t) = C\cos(\omega_0 t - \gamma)$*.*

If the system is underdamped, re-write the solution in the form $x(t) = Ce^{-pt}\cos(\omega_1 t - \gamma)$ *where* ω_1 *is the pseudo-frequency of the damped oscillation.*

a) $2x'' + 5x' + 2x = 0$; $x(0) = 2,\ x'(0) = 1$

b) $x'' + 8x' + 20x = 0$; $x(0) = 2,\ x'(0) = -4$

c) $x'' + 6x' + 9x = 0$; $x(0) = 2,\ x'(0) = 4$

d) $4x'' + 4x' + 17x = 0$; $x(0) = 4,\ x'(0) = 4$

e) $x'' + 6x' + 25x = 0$; $x(0) = 2,\ x'(0) = -2$

2.5 Nonhomogeneous equations

Note: 2 lectures, §3.5 in [EP], §3.5 and §3.6 in [BD]

2.5.1 Solving nonhomogeneous equations

We have solved linear constant coefficient homogeneous equations. What about nonhomogeneous linear ODEs? For example, the equations for forced mechanical vibrations. That is, suppose we have an equation such as

$$y'' + 5y' + 6y = 2x + 1. \tag{2.6}$$

We will write $Ly = 2x + 1$ when the exact form of the operator is not important. We solve (2.6) in the following manner. First, we find the general solution y_c to the *associated homogeneous equation*

$$y'' + 5y' + 6y = 0. \tag{2.7}$$

We call y_c the *complementary solution*. Next, we find a single *particular solution* y_p to (2.6) in some way. Then

$$y = y_c + y_p$$

is the general solution to (2.6). We have $Ly_c = 0$ and $Ly_p = 2x + 1$. As L is a *linear operator* we verify that y is a solution, $Ly = L(y_c + y_p) = Ly_c + Ly_p = 0 + (2x + 1)$. Let us see why we obtain the *general* solution.

Let y_p and $\tilde{y}_p$ be two different particular solutions to (2.6). Write the difference as $w = y_p - \tilde{y}_p$. Then plug w into the left-hand side of the equation to get

$$w'' + 5w' + 6w = (y_p'' + 5y_p' + 6y_p) - (\tilde{y}_p'' + 5\tilde{y}_p' + 6\tilde{y}_p) = (2x + 1) - (2x + 1) = 0.$$

Using the operator notation the calculation becomes simpler. As L is a linear operator we write

$$Lw = L(y_p - \tilde{y}_p) = Ly_p - L\tilde{y}_p = (2x + 1) - (2x + 1) = 0.$$

So $w = y_p - \tilde{y}_p$ is a solution to (2.7), that is $Lw = 0$. Any two solutions of (2.6) differ by a solution to the homogeneous equation (2.7). The solution $y = y_c + y_p$ includes *all* solutions to (2.6), since y_c is the general solution to the associated homogeneous equation.

Theorem 2.5.1. *Let $Ly = f(x)$ be a linear ODE (not necessarily constant coefficient). Let y_c be the complementary solution (the general solution to the associated homogeneous equation $Ly = 0$) and let y_p be any particular solution to $Ly = f(x)$. Then the general solution to $Ly = f(x)$ is*

$$y = y_c + y_p.$$

The moral of the story is that we can find the particular solution in any old way. If we find a different particular solution (by a different method, or simply by guessing), then we still get the same general solution. The formula may look different, and the constants we have to choose to satisfy the initial conditions may be different, but it is the same solution.

2.5.2 Undetermined coefficients

The trick is to somehow, in a smart way, guess one particular solution to (2.6). Note that $2x + 1$ is a polynomial, and the left-hand side of the equation will be a polynomial if we let y be a polynomial of the same degree. Let us try

$$y_p = Ax + B.$$

We plug y_p into the left hand side to obtain

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= (Ax + B)'' + 5(Ax + B)' + 6(Ax + B) \\ &= 0 + 5A + 6Ax + 6B = 6Ax + (5A + 6B). \end{aligned}$$

So $6Ax+(5A+6B) = 2x+1$. Therefore, $A = 1/3$ and $B = -1/9$. That means $y_p = \frac{1}{3}x - \frac{1}{9} = \frac{3x-1}{9}$. Solving the complementary problem (exercise!) we get

$$y_c = C_1 e^{-2x} + C_2 e^{-3x}.$$

Hence the general solution to (2.6) is

$$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{3x-1}{9}.$$

Now suppose we are further given some initial conditions. For example, $y(0) = 0$ and $y'(0) = 1/3$. First find $y' = -2C_1 e^{-2x} - 3C_2 e^{-3x} + 1/3$. Then

$$0 = y(0) = C_1 + C_2 - \frac{1}{9}, \qquad \frac{1}{3} = y'(0) = -2C_1 - 3C_2 + \frac{1}{3}.$$

We solve to get $C_1 = 1/3$ and $C_2 = -2/9$. The particular solution we want is

$$y(x) = \frac{1}{3}e^{-2x} - \frac{2}{9}e^{-3x} + \frac{3x-1}{9} = \frac{3e^{-2x} - 2e^{-3x} + 3x - 1}{9}.$$

Exercise* 2.5.1*: *Check that y really solves the equation (2.6) and the given initial conditions.*

Note: A common mistake is to solve for constants using the initial conditions with y_c and only add the particular solution y_p after that. That will *not* work. You need to first compute $y = y_c + y_p$ and *only then* solve for the constants using the initial conditions.

A right-hand side consisting of exponentials, sines, and cosines can be handled similarly. For example,

$$y'' + 2y' + 2y = \cos(2x).$$

Let us find some y_p. We start by guessing the solution includes some multiple of $\cos(2x)$. We may have to also add a multiple of $\sin(2x)$ to our guess since derivatives of cosine are sines. We try

$$y_p = A\cos(2x) + B\sin(2x).$$

We plug y_p into the equation and we get

$$\underbrace{-4A\cos(2x) - 4B\sin(2x)}_{y_p''} + 2\underbrace{\big(-2A\sin(2x) + 2B\cos(2x)\big)}_{y_p'} + 2\underbrace{\big(A\cos(2x) + 2B\sin(2x)\big)}_{y_p} = \cos(2x),$$

or

$$(-4A + 4B + 2A)\cos(2x) + (-4B - 4A + 2B)\sin(2x) = \cos(2x).$$

The left-hand side must equal to right-hand side. Namely, $-4A + 4B + 2A = 1$ and $-4B - 4A + 2B = 0$. So $-2A + 4B = 1$ and $2A + B = 0$ and hence $A = {-1}/{10}$ and $B = 1/5$. So

$$y_p = A\cos(2x) + B\sin(2x) = \frac{-\cos(2x) + 2\sin(2x)}{10}.$$

Similarly, if the right-hand side contains exponentials we try exponentials. If

$$Ly = e^{3x},$$

we try $y = Ae^{3x}$ as our guess and try to solve for A.

When the right-hand side is a multiple of sines, cosines, exponentials, and polynomials, we can use the product rule for differentiation to come up with a guess. We need to guess a form for y_p such that Ly_p is of the same form, and has all the terms needed to for the right-hand side. For example,

$$Ly = (1 + 3x^2)\,e^{-x}\cos(\pi x).$$

For this equation, we guess

$$y_p = (A + Bx + Cx^2)\,e^{-x}\cos(\pi x) + (D + Ex + Fx^2)\,e^{-x}\sin(\pi x).$$

We plug in and then hopefully get equations that we can solve for A, B, C, D, E, and F. As you can see this can make for a very long and tedious calculation very quickly. C'est la vie!

There is one hiccup in all this. It could be that our guess actually solves the associated homogeneous equation. That is, suppose we have

$$y'' - 9y = e^{3x}.$$

We would love to guess $y = Ae^{3x}$, but if we plug this into the left-hand side of the equation we get

$$y'' - 9y = 9Ae^{3x} - 9Ae^{3x} = 0 \neq e^{3x}.$$

There is no way we can choose A to make the left-hand side be e^{3x}. The trick in this case is to multiply our guess by x to get rid of duplication with the complementary solution. That is first we compute y_c (solution to $Ly = 0$)

$$y_c = C_1e^{-3x} + C_2e^{3x},$$

and we note that the e^{3x} term is a duplicate with our desired guess. We modify our guess to $y = Axe^{3x}$ so that there is no duplication anymore. Let us try: $y' = Ae^{3x} + 3Axe^{3x}$ and $y'' = 6Ae^{3x} + 9Axe^{3x}$, so

$$y'' - 9y = 6Ae^{3x} + 9Axe^{3x} - 9Axe^{3x} = 6Ae^{3x}.$$

Thus $6Ae^{3x}$ is supposed to equal e^{3x}. Hence, $6A = 1$ and so $A = 1/6$. We can now write the general solution as

$$y = y_c + y_p = C_1 e^{-3x} + C_2 e^{3x} + \frac{1}{6} x e^{3x}.$$

It is possible that multiplying by x does not get rid of all duplication. For example,

$$y'' - 6y' + 9y = e^{3x}.$$

The complementary solution is $y_c = C_1 e^{3x} + C_2 x e^{3x}$. Guessing $y = Axe^{3x}$ would not get us anywhere. In this case we want to guess $y_p = Ax^2 e^{3x}$. Basically, we want to multiply our guess by x until all duplication is gone. *But no more!* Multiplying too many times will not work.

Finally, what if the right-hand side has several terms, such as

$$Ly = e^{2x} + \cos x.$$

In this case we find u that solves $Lu = e^{2x}$ and v that solves $Lv = \cos x$ (that is, do each term separately). Then note that if $y = u + v$, then $Ly = e^{2x} + \cos x$. This is because L is linear; we have $Ly = L(u + v) = Lu + Lv = e^{2x} + \cos x$.

2.5.3 Variation of parameters

The method of undetermined coefficients works for many basic problems that crop up. But it does not work all the time. It only works when the right-hand side of the equation $Ly = f(x)$ has finitely many linearly independent derivatives, so that we can write a guess that consists of them all. Some equations are a bit tougher. Consider

$$y'' + y = \tan x.$$

Each new derivative of $\tan x$ looks completely different and cannot be written as a linear combination of the previous derivatives. If we start differentiating $\tan x$, we get:

$$\sec^2 x, \quad 2\sec^2 x \, \tan x, \quad 4\sec^2 x \, \tan^2 x + 2\sec^4 x,$$
$$8\sec^2 x \, \tan^3 x + 16\sec^4 x \, \tan x, \quad 16\sec^2 x \, \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x, \quad \ldots$$

This equation calls for a different method. We present the method of *variation of parameters*, which handles any equation of the form $Ly = f(x)$, provided we can solve certain integrals. For simplicity, we restrict ourselves to second order constant coefficient equations, but the method works for higher order equations just as well (the computations become more tedious). The method also works for equations with nonconstant coefficients, provided we can solve the associated homogeneous equation.

Perhaps it is best to explain this method by example. Let us try to solve the equation

$$Ly = y'' + y = \tan x.$$

First we find the complementary solution (solution to $Ly_c = 0$). We get $y_c = C_1 y_1 + C_2 y_2$, where $y_1 = \cos x$ and $y_2 = \sin x$. To find a particular solution to the nonhomogeneous equation we try

$$y_p = y = u_1 y_1 + u_2 y_2,$$

where u_1 and u_2 are *functions* and not constants. We are trying to satisfy $Ly = \tan x$. That gives us one condition on the functions u_1 and u_2. Compute (note the product rule!)

$$y' = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2').$$

We can still impose one more condition at our discretion to simplify computations (we have two unknown functions, so we should be allowed two conditions). We require that $(u_1' y_1 + u_2' y_2) = 0$. This makes computing the second derivative easier.

$$\begin{aligned} y' &= u_1 y_1' + u_2 y_2', \\ y'' &= (u_1' y_1' + u_2' y_2') + (u_1 y_1'' + u_2 y_2''). \end{aligned}$$

Since y_1 and y_2 are solutions to $y'' + y = 0$, we find $y_1'' = -y_1$ and $y_2'' = -y_2$. (If the equation was a more general $y'' + p(x)y' + q(x)y = 0$, we would have $y_i'' = -p(x)y_i' - q(x)y_i$.) So

$$y'' = (u_1' y_1' + u_2' y_2') - (u_1 y_1 + u_2 y_2).$$

We have $(u_1 y_1 + u_2 y_2) = y$ and so

$$y'' = (u_1' y_1' + u_2' y_2') - y,$$

and hence

$$y'' + y = Ly = u_1' y_1' + u_2' y_2'.$$

For y to satisfy $Ly = f(x)$ we must have $f(x) = u_1' y_1' + u_2' y_2'$.

What we need to solve are the two equations (conditions) we imposed on u_1 and u_2:

$$\boxed{\begin{aligned} u_1' y_1 + u_2' y_2 &= 0, \\ u_1' y_1' + u_2' y_2' &= f(x). \end{aligned}}$$

We solve for u_1' and u_2' in terms of $f(x)$, y_1 and y_2. We always get these formulas for any $Ly = f(x)$, where $Ly = y'' + p(x)y' + q(x)y$. There is a general formula for the solution we could just plug into, but instead of memorizing that, it is better, and easier, to just repeat what we do below. In our case the two equations are

$$\begin{aligned} u_1' \cos(x) + u_2' \sin(x) &= 0, \\ -u_1' \sin(x) + u_2' \cos(x) &= \tan(x). \end{aligned}$$

Hence

$$\begin{aligned} u_1' \cos(x) \sin(x) + u_2' \sin^2(x) &= 0, \\ -u_1' \sin(x) \cos(x) + u_2' \cos^2(x) &= \tan(x) \cos(x) = \sin(x). \end{aligned}$$

And thus

$$\begin{aligned} u_2' \big(\sin^2(x) + \cos^2(x)\big) &= \sin(x), \\ u_2' &= \sin(x), \\ u_1' &= \frac{-\sin^2(x)}{\cos(x)} = -\tan(x) \sin(x). \end{aligned}$$

We integrate u_1' and u_2' to get u_1 and u_2.

$$\begin{aligned} u_1 &= \int u_1' \, dx = \int -\tan(x) \sin(x) \, dx = \frac{1}{2} \ln \left| \frac{\sin(x) - 1}{\sin(x) + 1} \right| + \sin(x), \\ u_2 &= \int u_2' \, dx = \int \sin(x) \, dx = -\cos(x). \end{aligned}$$

So our particular solution is

$$\begin{aligned} y_p = u_1 y_1 + u_2 y_2 = \frac{1}{2} \cos(x) \ln \left| \frac{\sin(x) - 1}{\sin(x) + 1} \right| + \cos(x) \sin(x) - \cos(x) \sin(x) = \\ = \frac{1}{2} \cos(x) \ln \left| \frac{\sin(x) - 1}{\sin(x) + 1} \right|. \end{aligned}$$

The general solution to $y'' + y = \tan x$ is, therefore,

$$y = C_1 \cos(x) + C_2 \sin(x) + \frac{1}{2} \cos(x) \ln \left| \frac{\sin(x) - 1}{\sin(x) + 1} \right|.$$

2.5.4 Undetermined coefficients with Python

If you'd like to check your answer to an undetermined coefficients problem, you can do it using sympy, as we show below for the differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 2y = \cos 2x$ and the guess $y = A \cos 2x + B \sin 2x$.

```
from resources306 import *
x,A,B = sp.symbols('x A B')
cos2x,sin2x = sp.cos(2*x),sp.sin(2*x)    # shorthand
y = A*cos2x + B*sin2x  # our guessed form of the solution
de_lhs = sp.diff(y,x,x) + 2*sp.diff(y,x) + 2*y
de_rhs = cos2x
de = de_lhs - de_rhs # shove everything to the LHS, equate this to 0
de
```

$$-4A\sin(2x) + 2A\cos(2x) + 2B\sin(2x) + 4B\cos(2x) - 4(A\cos(2x) + B\sin(2x)) - \cos(2x)$$

```
eqs = [ de.coeff(cos2x), de.coeff(sin2x)) ]
eqs
```

$$[2A + 4B - 1, \quad -4A + 2B]$$

```
sp.solve( eqs, [A,B] )
```

$$\left\{A : \frac{1}{10}, \quad B : \frac{1}{5}\right\}$$

2.5.5 Exercises

***Exercise* 2.5.2:** *Find a particular solution of* $y'' - y' - 6y = e^{2x}$.

***Exercise* 2.5.3:** *Find a particular solution of* $y'' - 4y' + 4y = e^{2x}$.

***Exercise* 2.5.4:** *Solve the initial value problem* $y'' + 9y = \cos(3x) + \sin(3x)$ *for* $y(0) = 2$, $y'(0) = 1$.

***Exercise* 2.5.5:** *Set up the form of the particular solution but do not solve for the coefficients for* $y^{(4)} - 2y''' + y'' = e^x$.

***Exercise* 2.5.6:** *Set up the form of the particular solution but do not solve for the coefficients for* $y^{(4)} - 2y''' + y'' = e^x + x + \sin x$.

***Exercise* 2.5.7:**

a) *Using variation of parameters find a particular solution of* $y'' - 2y' + y = e^x$.

b) *Find a particular solution using undetermined coefficients.*

c) *Are the two solutions you found the same? See also Exercise 2.5.10.*

***Exercise* 2.5.8:** *Find a particular solution of* $y'' - 2y' + y = \sin(x^2)$. *It is OK to leave the answer as a definite integral.*

***Exercise* 2.5.9:** *For an arbitrary constant* c *find a particular solution to* $y'' - y = e^{cx}$. *Hint: Make sure to handle every possible real* c.

***Exercise* 2.5.10:**

a) *Using variation of parameters find a particular solution of* $y'' - y = e^x$.

b) *Find a particular solution using undetermined coefficients.*

c) *Are the two solutions you found the same? What is going on?*

***Exercise* 2.5.101:** *Find a particular solution to* $y'' - y' + y = 2\sin(3x)$

***Exercise* 2.5.102:**

a) *Find a particular solution to* $y'' + 2y = e^x + x^3$.

b) *Find the general solution.*

***Exercise* 2.5.103:** *Solve* $y'' + 2y' + y = x^2$, $y(0) = 1$, $y'(0) = 2$.

***Exercise* 2.5.104:** *Use variation of parameters to find a particular solution of* $y'' - y = \frac{1}{e^x + e^{-x}}$.

***Exercise* 2.5.105:** *For an arbitrary constant* c *find the general solution to* $y'' - 2y = \sin(x + c)$.

***Exercise* 2.5.151:** *Apply the method of undetermined coefficients to find the general solution to the following DEs. Include the form of* y_p*, but* <u>*do not determine the coefficients*</u>

a) $y''' - 4y'' + 5y' = 3x + 5e^{2x}\sin x$

b) $y''' - 4y' = x^2 + 3e^{2x} - e^{3x}$

c) $y''' + 4y'' + 13y' = 3x^2 + 5 - 6e^{-2x}\cos 3x$

d) $y^{(4)} + 2y''' - 3y'' = 4x + 2e^{-3x} - 3e^{5x}$

e) $y^{(4)} - 9y'' = 5e^{-3x} - 2x^2 + 7$

f) $y^{(5)} - 2y^{(4)} - 8y''' = 4x + 1 + 2e^{-2x} - 3e^{4x}$

g) $y^{(5)} + 2y^{(4)} + 2y''' = 4x^3 - x^2 + 5e^{-x}\sin x$

***Exercise* 2.5.152:** *Apply the method of undetermined coefficients to find the general solution to the following DEs.* <u>*Determine the form and coefficients of* y_p</u>

a) $y'' - 2y' = 8x + 5e^{3x}$

b) $y''' + y'' - 2y' = 2x + e^{2x}$

c) $y'' + 6y' + 13y = \cos x$

d) $y''' + y'' - 6y' = x^2 + 2x + 4e^{3x}$

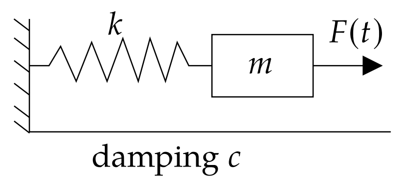
***Exercise* 2.5.153:** *Based upon the solutions in Exercise 2.5.152, solve the following IVPs:*

a) $y'' - 2y' = 8x + 5e^{3x}$; $y(0) = 0$, $y'(0) = -1$

b) $y''' + y'' - 2y' = 2x + e^{2x}$; $y(0) = 1$, $y'(0) = y''(0) = 0$

2.6 Forced oscillations and resonance

Note: 2 lectures, §3.6 in [EP], §3.8 in [BD]



Let us return back to the example of a mass on a spring. We examine the case of forced oscillations, which we did not yet handle. That is, we consider the equation

$$mx'' + cx' + kx = F(t),$$

for some nonzero $F(t)$. The setup is again: m is mass, c is friction, k is the spring constant, and $F(t)$ is an external force acting on the mass.

We are interested in periodic forcing, such as noncentered rotating parts, or perhaps loud sounds, or other sources of periodic force. Once we learn about Fourier series in chapter 4, we will see that we cover all periodic functions by simply considering $F(t) = F_0 \cos(\omega t)$ (or sine instead of cosine, the calculations are essentially the same).

2.6.1 Undamped forced motion and resonance

First let us consider undamped ($c = 0$) motion. We have the equation

$$mx'' + kx = F_0 \cos(\omega t).$$

This equation has the complementary solution (solution to the associated homogeneous equation)

$$x_c = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t),$$

where $\omega_0 = \sqrt{k/m}$ is the *natural frequency* (angular). It is the frequency at which the system "wants to oscillate" without external interference.

Suppose that $\omega_0 \neq \omega$. We try the solution $x_p = A\cos(\omega t)$ and solve for A. We do not need a sine in our trial solution as after plugging in we only have cosines. If you include a sine, it is fine; you will find that its coefficient is zero (I could not find a second rhyme).

We solve using the method of undetermined coefficients. We find that

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

We leave it as an exercise to do the algebra required.

The general solution is

$$\boxed{x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).}$$

Written another way

$$x = C \cos(\omega_0 t - \gamma) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t).$$

The solution is a superposition of two cosine waves at different frequencies.

Example 2.6.1: Take

$$0.5x'' + 8x = 10\cos(\pi t), \qquad x(0) = 0, \qquad x'(0) = 0.$$

Let us compute. First we read off the parameters: $\omega = \pi$, $\omega_0 = \sqrt{8/0.5} = 4$, $F_0 = 10$, $m = 0.5$. The general solution is

$$x = C_1 \cos(4t) + C_2 \sin(4t) + \frac{20}{16-\pi^2}\cos(\pi t).$$

Solve for C_1 and C_2 using the initial conditions: $C_1 = \frac{-20}{16-\pi^2}$ and $C_2 = 0$. Hence

$$x = \frac{20}{16-\pi^2}\bigl(\cos(\pi t) - \cos(4t)\bigr).$$

Notice the "beating" behavior in Figure 2.5. First use the trigonometric identity

$$2\sin\left(\frac{A-B}{2}\right)\sin\left(\frac{A+B}{2}\right) = \cos B - \cos A$$

to get

$$x = \frac{20}{16-\pi^2}\left(2\sin\left(\frac{4-\pi}{2}t\right)\sin\left(\frac{4+\pi}{2}t\right)\right).$$

The function x is a high frequency wave modulated by a low frequency wave.

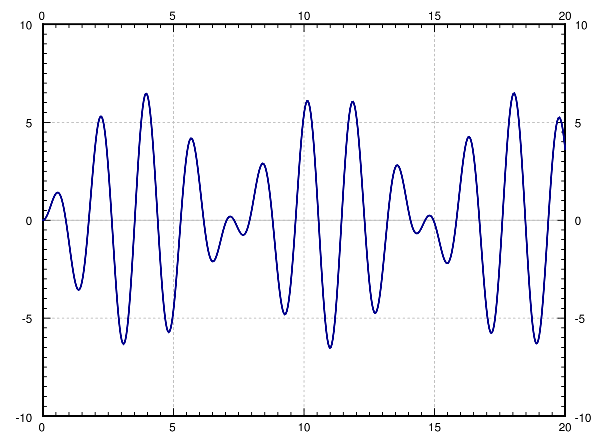


Figure 2.5: Graph of $\frac{20}{16-\pi^2}\bigl(\cos(\pi t) - \cos(4t)\bigr)$.

Now suppose $\omega_0 = \omega$. Obviously, we cannot try the solution $A\cos(\omega t)$ and then use the method of undetermined coefficients. We notice that $\cos(\omega t)$ solves the associated homogeneous equation. Therefore, we try $x_p = At\cos(\omega t) + Bt\sin(\omega t)$. This time we need the sine term, since the second derivative of $t\cos(\omega t)$ contains sines. We write the equation

$$x'' + \omega^2 x = \frac{F_0}{m}\cos(\omega t).$$

Plugging x_p into the left-hand side we get

$$2B\omega\cos(\omega t) - 2A\omega\sin(\omega t) = \frac{F_0}{m}\cos(\omega t).$$

Hence $A = 0$ and $B = \frac{F_0}{2m\omega}$. Our particular solution is $\frac{F_0}{2m\omega}\, t\sin(\omega t)$ and our general solution is

$$x = C_1\cos(\omega t) + C_2\sin(\omega t) + \frac{F_0}{2m\omega}\, t\sin(\omega t).$$

The important term is the last one (the particular solution we found). This term grows without bound as $t \to \infty$. In fact it oscillates between $\frac{F_0 t}{2m\omega}$ and $\frac{-F_0 t}{2m\omega}$. The first two terms only oscillate between $\pm\sqrt{C_1^2 + C_2^2}$, which becomes smaller and smaller in proportion to the oscillations of the last term as t gets larger. In Figure 2.6 on the next page we see the graph with $C_1 = C_2 = 0$, $F_0 = 2$, $m = 1$, $\omega = \pi$.

By forcing the system in just the right frequency we produce very wild oscillations. This kind of behavior is called *resonance* or perhaps *pure resonance*. Sometimes resonance is desired. For example, remember when as a kid you could start swinging by just moving back and forth on the swing seat in the "correct frequency"? You were trying to achieve resonance. The force of each one of your moves was small, but after a while it produced large swings.

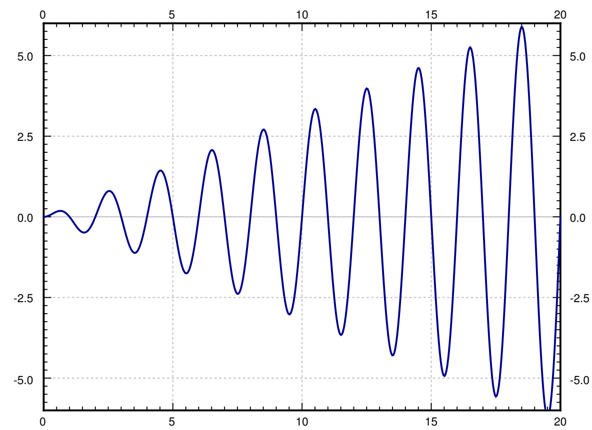


Figure 2.6: Graph of $\frac{1}{\pi} t \sin(\pi t)$.

On the other hand resonance can be destructive. In an earthquake some buildings collapse while others may be relatively undamaged. This is due to different buildings having different resonance frequencies. So figuring out the resonance frequency can be very important.

A common (but wrong) example of destructive force of resonance is the Tacoma Narrows bridge failure. It turns out there was a different phenomenon at play*.

2.6.2 Damped forced motion and practical resonance

In real life things are not as simple as they were above. There is, of course, some damping. Our equation becomes

$$mx'' + cx' + kx = F_0 \cos(\omega t), \tag{2.8}$$

for some $c > 0$. We solved the homogeneous problem before. We let

$$p = \frac{c}{2m}, \qquad \omega_0 = \sqrt{\frac{k}{m}}.$$

We replace equation (2.8) with

$$x'' + 2px' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t).$$

The roots of the characteristic equation of the associated homogeneous problem are $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$. The form of the general solution of the associated homogeneous equation depends on the sign of $p^2 - \omega_0^2$, or equivalently on the sign of $c^2 - 4km$, as before:

$$x_c = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t} & \text{if } c^2 > 4km, \\ C_1 e^{-pt} + C_2 t e^{-pt} & \text{if } c^2 = 4km, \\ e^{-pt}\big(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)\big) & \text{if } c^2 < 4km, \end{cases}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. In any case, we see that $x_c(t) \to 0$ as $t \to \infty$.

*K. Billah and R. Scanlan, *Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks*, American Journal of Physics, 59(2), 1991, 118–124, `http://www.ketchum.org/billah/Billah-Scanlan.pdf`

Let us find a particular solution. There can be no conflicts when trying to solve for the undetermined coefficients by trying $x_p = A\cos(\omega t) + B\sin(\omega t)$. Let us plug in and solve for A and B. We get (the tedious details are left to reader)

$$\big((\omega_0^2 - \omega^2)B - 2\omega p A\big)\sin(\omega t) + \big((\omega_0^2 - \omega^2)A + 2\omega p B\big)\cos(\omega t) = \frac{F_0}{m}\cos(\omega t).$$

We solve for A and B:

$$A = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2},$$
$$B = \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}.$$

We also compute $C = \sqrt{A^2 + B^2}$ to be

$$C = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}}.$$

Thus our particular solution is

$$x_p = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}\cos(\omega t) + \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}\sin(\omega t).$$

Or in the alternative notation we have amplitude C and phase shift γ where (if $\omega \neq \omega_0$)

$$\tan\gamma = \frac{B}{A} = \frac{2\omega p}{\omega_0^2 - \omega^2}.$$

Hence,

$$\boxed{x_p = \frac{F_0}{m\sqrt{(2\omega p)^2 + (\omega_0^2 - \omega^2)^2}}\cos(\omega t - \gamma).}$$

If $\omega = \omega_0$, then $A = 0$, $B = C = \frac{F_0}{2m\omega p}$, and $\gamma = \pi/2$.

For reasons we will explain in a moment, we call x_c the *transient solution* and denote it by x_{tr}. We call the x_p from above the *steady periodic solution* and denote it by x_{sp}. The general solution is

$$x = x_c + x_p = x_{tr} + x_{sp}.$$

The transient solution $x_c = x_{tr}$ goes to zero as $t \to \infty$, as all the terms involve an exponential with a negative exponent. So for large t, the effect of x_{tr} is negligible and we see essentially only x_{sp}. Hence the name *transient*. Notice that x_{sp} involves no arbitrary constants, and the initial conditions only affect x_{tr}. Thus, the effect of the initial conditions is negligible after some period of time. We might as well focus on the steady periodic solution and ignore the transient solution. See Figure 2.7 on the facing page for a graph given several different initial conditions.

The speed at which x_{tr} goes to zero depends on p (and hence c). The bigger p is (the bigger c is), the "faster" x_{tr} becomes negligible. So the smaller the damping, the longer the "transient region." This is consistent with the observation that when $c = 0$, the initial conditions affect the behavior for all time (i.e. an infinite "transient region").

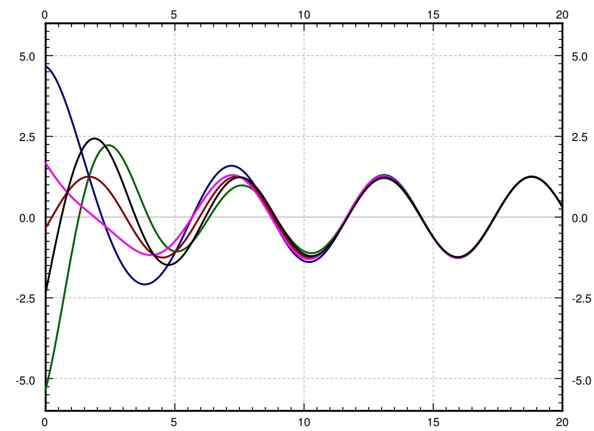


Figure 2.7: Solutions with different initial conditions for parameters $k = 1$, $m = 1$, $F_0 = 1$, $c = 0.7$, and $\omega = 1.1$.

Let us describe what we mean by resonance when damping is present. Since there were no conflicts when solving with undetermined coefficient, there is no term that goes to infinity. We look instead at the maximum value of the amplitude of the steady periodic solution. Let C be the amplitude of x_{sp}. If we plot C as a function of ω (with all other parameters fixed), we can find its maximum. We call the ω that achieves this maximum the *practical resonance frequency*. We call the maximal amplitude $C(\omega)$ the *practical resonance amplitude*. Thus when damping is present we talk of *practical resonance* rather than pure resonance. A sample plot for three different values of c is given in Figure 2.8. As you can see the practical resonance amplitude grows as damping gets smaller, and practical resonance can disappear altogether when damping is large.

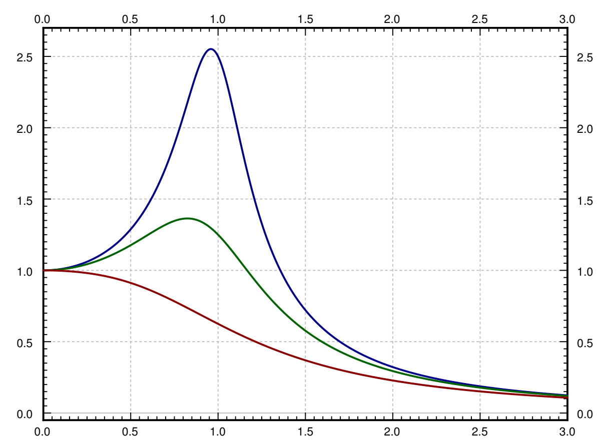


Figure 2.8: Graph of $C(\omega)$ showing practical resonance with parameters $k = 1$, $m = 1$, $F_0 = 1$. The top line is with $c = 0.4$, the middle line with $c = 0.8$, and the bottom line with $c = 1.6$.

To find the maximum we need to find the derivative $C'(\omega)$. Computation shows

$$C'(\omega) = \frac{-2\omega(2p^2 + \omega^2 - \omega_0^2)F_0}{m\big((2\omega p)^2 + (\omega_0^2 - \omega^2)^2\big)^{3/2}}.$$

This is zero either when $\omega = 0$ or when $2p^2 + \omega^2 - \omega_0^2 = 0$. In other words, $C'(\omega) = 0$ when

$$\boxed{\omega = \sqrt{\omega_0^2 - 2p^2} \quad \text{or} \quad \omega = 0.}$$

If $\omega_0^2 - 2p^2$ is positive, then $\sqrt{\omega_0^2 - 2p^2}$ is the practical resonance frequency (that is the point where $C(\omega)$ is maximal). This follows by the first derivative test for example as then $C'(\omega) > 0$ for small ω in this case. If on the other hand $\omega_0^2 - 2p^2$ is not positive, then $C(\omega)$ achieves its maximum at $\omega = 0$, and there is no practical resonance since we assume $\omega > 0$ in our system. In this case the amplitude gets larger as the forcing frequency gets smaller.

If practical resonance occurs, the frequency is smaller than ω_0. As the damping c (and hence p) becomes smaller, the practical resonance frequency goes to ω_0. So when damping is very small, ω_0 is a good estimate of the practical resonance frequency. This behavior agrees with the observation that when $c = 0$, then ω_0 is the resonance frequency.

Another interesting observation to make is that when $\omega \to \infty$, then $C \to 0$. This means that if the forcing frequency gets too high it does not manage to get the mass moving in the mass-spring system. This is quite reasonable intuitively. If we wiggle back and forth really fast while sitting on a swing, we will not get it moving at all, no matter how forceful. Fast vibrations just cancel each other out before the mass has any chance of responding by moving one way or the other.

The behavior is more complicated if the forcing function is not an exact cosine wave, but for example a square wave. A general periodic function will be the sum (superposition) of many cosine waves of different frequencies. The reader is encouraged to come back to this section once we have learned about the Fourier series.

2.6.3 Symbolic computation with Python

As an example of symbolic computation in Python, we repeat the above computation using sympy.

```
from resources306 import *
a,b,w,w0,p,t,G0  = sp.symbols('a b w w0 p t G0')
x = a*sp.cos(w*t) + b*sp.sin(w*t)
# Write the DE with everything on the LHS, implicitly set to 0
de = sp.diff(x,t,t) + 2*p*sp.diff(x,t) + w0**2*x -  G0*sp.cos(w*t)
display(de)
dec = de.subs(t,0)         # get the coefficient of cos(wt)
des = de.subs(t,sp.pi/2/w) # get the coefficient of sin(wt)
display(dec); display(des)
sol = sp.solve([dec,des],[a,b])
display(sol)
amp = sp.simplify( sp.sqrt(a**2 + b**2).subs(sol) )
display( amp )
```

```
amp = amp.subs({G0:1,w0:1})  # choose units of time and amplitude
for pval in [2,1,0.5,0.25,0.125,0.0625]:
        ampp = amp.subs(p,pval)
        expressionplot( ampp, w, 0, 3, label='damping = '+str(pval) )
plt.legend()
plt.xlabel('forcing frequency, w'); plt.ylabel('amplitude of response')
plt.savefig('myresonanceplot.png')
```

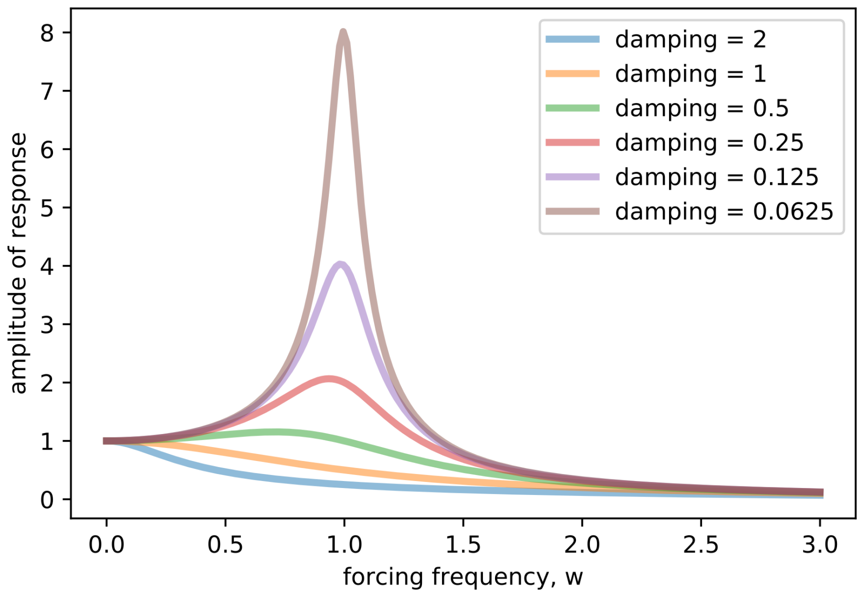
$$-G_0 \cos(tw) + 2p\left(-aw\sin(tw) + bw\cos(tw)\right) - w^2\left(a\cos(tw) + b\sin(tw)\right) + w_0^2\left(a\cos(tw) + b\sin(tw)\right)$$

$$-G_0 - aw^2 + aw_0^2 + 2bpw$$

$$-2apw - bw^2 + bw_0^2$$

$$\left\{a: -\frac{G_0\left(w^2 - w_0^2\right)}{4p^2w^2 + \left(w^2 - w_0^2\right)^2}, \quad b: \frac{2G_0pw}{4p^2w^2 + \left(w^2 - w_0^2\right)^2}\right\}$$

$$\sqrt{\frac{G_0^2}{4p^2w^2 + \left(w^2 - w_0^2\right)^2}}$$



2.6.4 Exercises

***Exercise* 2.6.1:** *Derive a formula for x_{sp} if the equation is $mx'' + cx' + kx = F_0 \sin(\omega t)$. Assume $c > 0$.*

***Exercise* 2.6.2:** *Derive a formula for x_{sp} if the equation is $mx'' + cx' + kx = F_0 \cos(\omega t) + F_1 \cos(3\omega t)$. Assume $c > 0$.*

***Exercise* 2.6.3:** *Take $mx'' + cx' + kx = F_0 \cos(\omega t)$. Fix $m > 0$, $k > 0$, and $F_0 > 0$. Consider the function $C(\omega)$. For what values of c (solve in terms of m, k, and F_0) will there be no practical resonance (that is, for what values of c is there no maximum of $C(\omega)$ for $\omega > 0$)?*

Exercise 2.6.4: *Take $mx'' + cx' + kx = F_0 \cos(\omega t)$. Fix $c > 0$, $k > 0$, and $F_0 > 0$. Consider the function $C(\omega)$. For what values of m (solve in terms of c, k, and F_0) will there be no practical resonance (that is, for what values of m is there no maximum of $C(\omega)$ for $\omega > 0$)?*

Exercise 2.6.5: *A water tower in an earthquake acts as a mass-spring system. Assume that the container on top is full and the water does not move around. The container then acts as the mass and the support acts as the spring, where the induced vibrations are horizontal. The container with water has a mass of $m = 10{,}000$ kg. It takes a force of 1000 newtons to displace the container 1 meter. For simplicity assume no friction. When the earthquake hits the water tower is at rest (it is not moving). The earthquake induces an external force $F(t) = mA\omega^2 \cos(\omega t)$.*

a) *What is the natural frequency of the water tower?*

b) *If ω is not the natural frequency, find a formula for the maximal amplitude of the resulting oscillations of the water container (the maximal deviation from the rest position). The motion will be a high frequency wave modulated by a low frequency wave, so simply find the constant in front of the sines.*

c) *Suppose $A = 1$ and an earthquake with frequency 0.5 cycles per second comes. What is the amplitude of the oscillations? Suppose that if the water tower moves more than 1.5 meter from the rest position, the tower collapses. Will the tower collapse?*

Exercise 2.6.101: *A mass of 4 kg on a spring with $k = 4$ N/m and a damping constant $c = 1$ Ns/m. Suppose that $F_0 = 2$ N. Using forcing function $F_0 \cos(\omega t)$, find the ω that causes practical resonance and find the amplitude.*

Exercise 2.6.102: *Derive a formula for x_{sp} for $mx'' + cx' + kx = F_0 \cos(\omega t) + A$, where A is some constant. Assume $c > 0$.*

Exercise 2.6.103: *Suppose there is no damping in a mass and spring system with $m = 5$, $k = 20$, and $F_0 = 5$. Suppose ω is chosen to be precisely the resonance frequency.*

a) *Find ω.*

b) *Find the amplitude of the oscillations at time $t = 100$, given the system is at rest at $t = 0$.*

Exercise 2.6.151: *Apply the method of undermined coefficients to solve the following initial value problems:*

a) $x'' + 4x = 15 \cos 3t$; $x(0) = 2$, $x'(0) = 1$

b) $x'' + 9x = 4 \sin 2t$; $x(0) = 3$, $x'(0) = -2$

c) $x'' + 16x = 2 \cos 3t + 5 \sin 3t$; $x(0) = -1$, $x'(0) = 1$

Chapter 3

Systems of ODEs

3.1 Introduction to systems of ODEs

Note: 1 to 1.5 lectures, §4.1 in [EP], §7.1 in [BD]

3.1.1 Systems

Often we do not have just one dependent variable and one equation. And as we will see, we may end up with systems of several equations and several dependent variables even if we start with a single equation.

If we have several dependent variables, suppose $y_1, y_2, \ldots, y_n$, then we can have a differential equation involving all of them and their derivatives with respect to one independent variable x. For example, $y_1'' = f(y_1', y_2', y_1, y_2, x)$. Usually, when we have two dependent variables we have two equations such as

$$\begin{aligned} y_1'' &= f_1(y_1', y_2', y_1, y_2, x), \\ y_2'' &= f_2(y_1', y_2', y_1, y_2, x), \end{aligned}$$

for some functions f_1 and f_2. We call the above a *system of differential equations*. More precisely, the above is a *second order system* of ODEs as second order derivatives appear. The system

$$\begin{aligned} x_1' &= g_1(x_1, x_2, x_3, t), \\ x_2' &= g_2(x_1, x_2, x_3, t), \\ x_3' &= g_3(x_1, x_2, x_3, t), \end{aligned}$$

is a *first order system*, where x_1, x_2, x_3 are the dependent variables, and t is the independent variable.

The terminology for systems is essentially the same as for single equations. For the system above, a *solution* is a set of three functions $x_1(t)$, $x_2(t)$, $x_3(t)$, such that

$$\begin{aligned} x_1'(t) &= g_1\big(x_1(t), x_2(t), x_3(t), t\big), \\ x_2'(t) &= g_2\big(x_1(t), x_2(t), x_3(t), t\big), \\ x_3'(t) &= g_3\big(x_1(t), x_2(t), x_3(t), t\big). \end{aligned}$$

We usually also have an *initial condition*. Just like for single equations we specify x_1, x_2, and x_3 for some fixed t. For example, $x_1(0) = a_1$, $x_2(0) = a_2$, $x_3(0) = a_3$. For some constants a_1, a_2, and a_3. For the second order system we would also specify the first derivatives at a point. And if we find a solution with constants in it, where by solving for the constants we find a solution for any initial condition, we call this solution the *general solution*. Best to look at a simple example.

Example 3.1.1: Sometimes a system is easy to solve by solving for one variable and then for the second variable. Take the first order system

$$\begin{aligned} y_1' &= y_1, \\ y_2' &= y_1 - y_2, \end{aligned}$$

with y_1, y_2 as the dependent variables and x as the independent variable. And consider initial conditions $y_1(0) = 1$, $y_2(0) = 2$.

We note that $y_1 = C_1 e^x$ is the general solution of the first equation. We then plug this y_1 into the second equation and get the equation $y_2' = C_1 e^x - y_2$, which is a linear first order equation that is easily solved for y_2. By the method of integrating factor we get

$$e^x y_2 = \frac{C_1}{2} e^{2x} + C_2,$$

or $y_2 = \frac{C_1}{2} e^x + C_2 e^{-x}$. The general solution to the system is, therefore,

$$y_1 = C_1 e^x, \qquad y_2 = \frac{C_1}{2} e^x + C_2 e^{-x}.$$

We solve for C_1 and C_2 given the initial conditions. We substitute $x = 0$ and find that $C_1 = 1$ and $C_2 = 3/2$. Thus the solution is $y_1 = e^x$, and $y_2 = (1/2)e^x + (3/2)e^{-x}$.

Generally, we will not be so lucky to be able to solve for each variable separately as in the example above, and we will have to solve for all variables at once. While we won't generally be able to solve for one variable and then the next, we will try to salvage as much as possible from this technique. It will turn out that in a certain sense we will still (try to) solve a bunch of single equations and put their solutions together. Let's not worry right now about how to solve systems yet.

We will mostly consider the *linear systems*. The example above is an example of a *linear first order system*. It is linear as none of the dependent variables or their derivatives appear in nonlinear functions or with powers higher than one (x, y, x' and y', constants, and functions of t can appear, but not xy or $(y')^2$ or x^3). Another, more complicated, example of a linear system is

$$\begin{aligned} y_1'' &= e^t y_1' + t^2 y_1 + 5y_2 + \sin(t), \\ y_2'' &= t y_1' - y_2' + 2y_1 + \cos(t). \end{aligned}$$

3.1.2 Applications

Let us consider some simple applications of systems and how to set up the equations.

Example 3.1.2: First, we consider salt and brine tanks, but this time water flows from one to the other and back. We again consider that the tanks are evenly mixed.

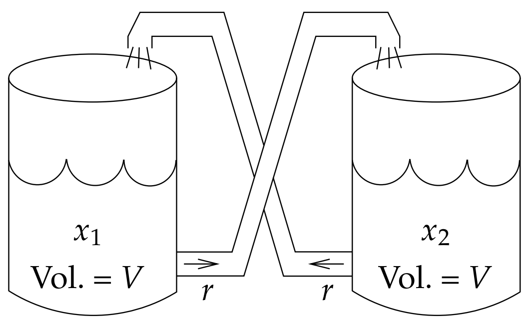


Figure 3.1: A closed system of two brine tanks.

Suppose we have two tanks, each containing volume V liters of salt brine. The amount of salt in the first tank is x_1 grams, and the amount of salt in the second tank is x_2 grams. The liquid is perfectly mixed and flows at the rate r liters per second out of each tank into the other. See Figure 3.1.

The rate of change of x_1, that is x_1', is the rate of salt coming in minus the rate going out. The rate coming in is the density of the salt in tank 2, that is $\frac{x_2}{V}$, times the rate r. The rate coming out is the density of the salt in tank 1, that is $\frac{x_1}{V}$, times the rate r. In other words it is

$$x_1' = \frac{x_2}{V} r - \frac{x_1}{V} r = \frac{r}{V} x_2 - \frac{r}{V} x_1 = \frac{r}{V}(x_2 - x_1).$$

Similarly we find the rate x_2', where the roles of x_1 and x_2 are reversed. All in all, the system of ODEs for this problem is

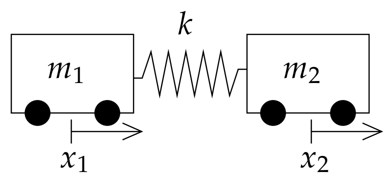
$$\begin{aligned} x_1' &= \frac{r}{V}(x_2 - x_1), \\ x_2' &= \frac{r}{V}(x_1 - x_2). \end{aligned}$$

In this system we cannot solve for x_1 or x_2 separately. We must solve for both x_1 and x_2 at once, which is intuitively clear since the amount of salt in one tank affects the amount in the other. We can't know x_1 before we know x_2, and vice versa.

We don't yet know how to find all the solutions, but intuitively we can at least find some solutions. Suppose we know that initially the tanks have the same amount of salt. That is, we have an initial condition such as $x_1(0) = x_2(0) = C$. Then clearly the amount of salt coming and out of each tank is the same, so the amounts are not changing. In other words, $x_1 = C$ and $x_2 = C$ (the constant functions) is a solution: $x_1' = x_2' = 0$, and $x_2 - x_1 = x_1 - x_2 = 0$, so the equations are satisfied.

Let us think about the setup a little bit more without solving it. Suppose the initial conditions are $x_1(0) = A$ and $x_2(0) = B$, for two different constants A and B. Since no salt is coming in or out of this closed system, the total amount of salt is constant. That is, $x_1 + x_2$ is constant, and so it equals $A + B$. Intuitively if A is bigger than B, then more salt will flow out of tank one than into it. Eventually, after a long time we would then expect the amount of salt in each tank to equalize. In other words, the solutions of both x_1 and x_2 should tend towards $\frac{A+B}{2}$. Once you know how to solve systems you will find out that this really is so.

Example 3.1.3: Let us look at a second order example. We return to the mass and spring setup, but this time we consider two masses.



Consider one spring with constant k and two masses m_1 and m_2. Think of the masses as carts that ride along a straight track with no friction. Let x_1 be the displacement of the first cart and x_2 be the displacement of the second cart. That is, we put the two carts somewhere with no tension on the spring, and we mark the position of the first and second cart and call those the zero positions. Then x_1 measures how far the first cart is from its zero position, and x_2 measures how far the second cart is from its zero position. The force exerted by the spring on the first cart is $k(x_2 - x_1)$, since $x_2 - x_1$ is how far the string is stretched (or compressed) from the rest position. The force exerted on the second cart is the opposite, thus the same thing with a negative sign. Newton's second law states that force equals mass times acceleration. So the system of equations is

$$
\begin{aligned}
m_1 x_1'' &= k(x_2 - x_1), \\
m_2 x_2'' &= -k(x_2 - x_1).
\end{aligned}
$$

Again, we cannot solve for the x_1 or x_2 variable separately. That we must solve for both x_1 and x_2 at once is intuitively clear, since where the first cart goes depends on exactly where the second cart goes and vice-versa.

3.1.3 Changing to first order

Before we talk about how to handle systems, let us note that in some sense we need only consider first order systems. Let us take an n^{th} order differential equation

$$
y^{(n)} = F(y^{(n-1)}, \ldots, y', y, x).
$$

We define new variables $u_1, u_2, \ldots, u_n$ and write the system

$$\begin{aligned} u_1' &= u_2, \\ u_2' &= u_3, \\ &\vdots \\ u_{n-1}' &= u_n, \\ u_n' &= F(u_n, u_{n-1}, \ldots, u_2, u_1, x). \end{aligned}$$

We solve this system for $u_1, u_2, \ldots, u_n$. Once we have solved for the u's, we can discard u_2 through u_n and let $y = u_1$. This y solves the original equation.

Example 3.1.4: Take $x''' = 2x'' + 8x' + x + t$. Letting $u_1 = x$, $u_2 = x'$, $u_3 = x''$, we find the system:

$$u_1' = u_2, \qquad u_2' = u_3, \qquad u_3' = 2u_3 + 8u_2 + u_1 + t.$$

A similar process can be followed for a system of higher order differential equations. For example, a system of k differential equations in k unknowns, all of order n, can be transformed into a first order system of $n \times k$ equations and $n \times k$ unknowns.

Example 3.1.5: Consider the system from the carts example,

$$m_1 x_1'' = k(x_2 - x_1), \qquad m_2 x_2'' = -k(x_2 - x_1).$$

Let $u_1 = x_1$, $u_2 = x_1'$, $u_3 = x_2$, $u_4 = x_2'$. The second order system becomes the first order system

$$u_1' = u_2, \qquad m_1 u_2' = k(u_3 - u_1), \qquad u_3' = u_4, \qquad m_2 u_4' = -k(u_3 - u_1).$$

Example 3.1.6: The idea works in reverse as well. Consider the system

$$x' = 2y - x, \qquad y' = x,$$

where the independent variable is t. We wish to solve for the initial conditions $x(0) = 1$, $y(0) = 0$.

If we differentiate the second equation, we get $y'' = x'$. We know what x' is in terms of x and y, and we know that $x = y'$. So,

$$y'' = x' = 2y - x = 2y - y'.$$

We now have the equation $y'' + y' - 2y = 0$. We know how to solve this equation and we find that $y = C_1 e^{-2t} + C_2 e^t$. Once we have y, we use the equation $y' = x$ to get x.

$$x = y' = -2C_1 e^{-2t} + C_2 e^t.$$

We solve for the initial conditions $1 = x(0) = -2C_1 + C_2$ and $0 = y(0) = C_1 + C_2$. Hence, $C_1 = -C_2$ and $1 = 3C_2$. So $C_1 = -1/3$ and $C_2 = 1/3$. Our solution is

$$x = \frac{2e^{-2t} + e^t}{3}, \qquad y = \frac{-e^{-2t} + e^t}{3}.$$

***Exercise* 3.1.1:** *Plug in and check that this really is the solution.*

It is useful to go back and forth between systems and higher order equations for other reasons. For example, software for solving ODE numerically (approximation) is generally for first order systems. So to use it, you have to take whatever ODE you want to solve and convert it to a first order system. In fact, it is not very hard to adapt computer code for the Euler or Runge–Kutta method for first order equations to handle first order systems. We essentially just treat the dependent variable not as a number but as a vector. In many mathematical computer languages there is almost no distinction in syntax.

3.1.4 Autonomous systems and vector fields

A system where the equations do not depend on the independent variable is called an *autonomous system*. For example the system $y' = 2y - x$, $y' = x$ is autonomous as t is the independent variable but does not appear in the equations.

For autonomous systems we can the so-called *direction field* or *vector field*, a plot similar to a slope field, but instead of giving a slope at each point, we give a direction (and a magnitude). The previous example, $x' = 2y - x$, $y' = x$, says that at the point (x, y) the direction in which we should travel to satisfy the equations should be the direction of the vector $(2y - x, x)$ with the speed equal to the magnitude of this vector. So we draw the vector $(2y - x, x)$ at the point (x, y) and we do this for many points on the xy-plane. For example, at the point $(1, 2)$ we draw the vector $\bigl(2(2) - 1, 1\bigr) = (3, 1)$, a vector pointing to the right and a little bit up, while at the point $(2, 1)$ we draw the vector $\bigl(2(1) - 2, 2\bigr) = (0, 2)$ a vector that points straight up. When drawing the vectors, we will scale down their size to fit many of them on the same direction field. We are mostly interested in their direction and relative size. See Figure 3.2 on the next page.

We can draw a path of the solution in the plane. Suppose the solution is given by $x = f(t)$, $y = g(t)$. We pick an interval of t (say $0 \leq t \leq 2$ for our example) and plot all the points $\bigl(f(t), g(t)\bigr)$ for t in the selected range. The resulting picture is called the *phase portrait* (or phase plane portrait). The particular curve obtained is called the *trajectory* or *solution curve*. See an example plot in Figure 3.3 on the facing page. In the figure the solution starts at $(1, 0)$ and travels along the vector field for a distance of 2 units of t. We solved this system precisely, so we compute $x(2)$ and $y(2)$ to find $x(2) \approx 2.475$ and $y(2) \approx 2.457$. This point corresponds to the top right end of the plotted solution curve in the figure.

Notice the similarity to the diagrams we drew for autonomous systems in one dimension. But note how much more complicated things become when we allow just one extra dimension.

We can draw phase portraits and trajectories in the xy-plane even if the system is not autonomous. In this case however we cannot draw the direction field, since the field changes as t changes. For each t we would get a different direction field.

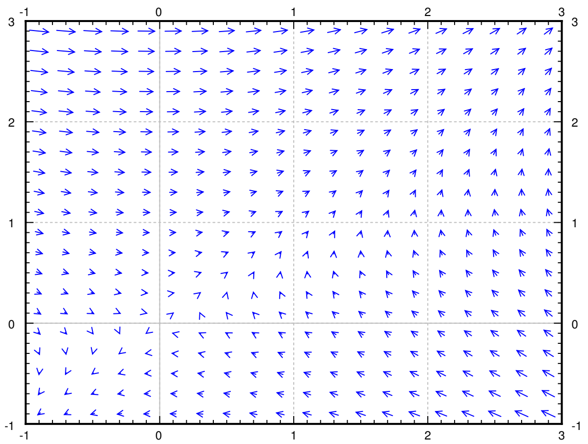


Figure 3.2: The direction field for $x' = 2y - x$, $y' = x$.

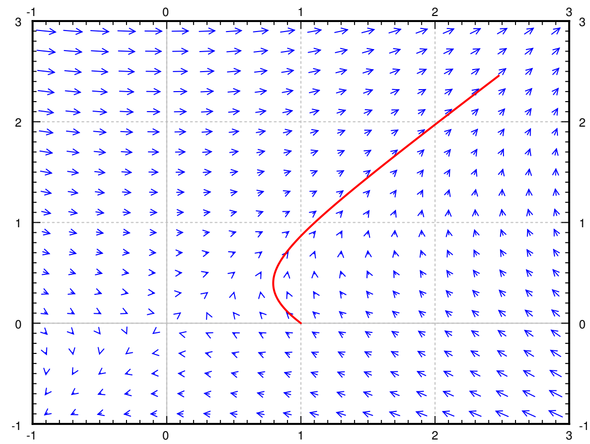


Figure 3.3: The direction field for $x' = 2y - x$, $y' = x$ with the trajectory of the solution starting at $(1, 0)$ for $0 \leq t \leq 2$.

3.1.5 Picard's theorem

Perhaps before going further, let us mention that Picard's theorem on existence and uniqueness still holds for systems of ODE. Let us restate this theorem in the setting of systems. A general first order system is of the form

$$\begin{aligned} x_1' &= F_1(x_1, x_2, \ldots, x_n, t), \\ x_2' &= F_2(x_1, x_2, \ldots, x_n, t), \\ &\vdots \\ x_n' &= F_n(x_1, x_2, \ldots, x_n, t). \end{aligned} \tag{3.1}$$

Theorem 3.1.1 (Picard's theorem on existence and uniqueness for systems)**.** *If for every $j = 1, 2, \ldots, n$ and every $k = 1, 2, \ldots, n$ each F_j is continuous and the derivative $\frac{\partial F_j}{\partial x_k}$ exists and is continuous near some $(x_1^0, x_2^0, \ldots, x_n^0, t^0)$, then a solution to (3.1) subject to the initial condition $x_1(t^0) = x_1^0$, $x_2(t^0) = x_2^0$, $\ldots$, $x_n(t^0) = x_n^0$ exists (at least for some small interval of t's) and is unique.*

That is, a unique solution exists for any initial condition given that the system is reasonable (F_j and its partial derivatives in the x variables are continuous). As for single equations we may not have a solution for all time t, but at least for some short period of time.

As we can change any nth order ODE into a first order system, then we notice that this theorem provides also the existence and uniqueness of solutions for higher order equations that we have until now not stated explicitly.

3.1.6 Exercises

Exercise* 3.1.2*: *Find the general solution of* $x_1' = x_2 - x_1 + t$, $x_2' = x_2$.

Exercise* 3.1.3*: *Find the general solution of* $x_1' = 3x_1 - x_2 + e^t$, $x_2' = x_1$.

Exercise* 3.1.4*: *Write* $ay'' + by' + cy = f(x)$ *as a first order system of ODEs.*

Exercise* 3.1.5*: *Write* $x'' + y^2y' - x^3 = \sin(t)$, $y'' + (x' + y')^2 - x = 0$ *as a first order system of ODEs.*

Exercise* 3.1.6*: *Suppose two masses on carts on frictionless surface are at displacements* x_1 *and* x_2 *as in* *Example* 3.1.3 *on page* 132*. Suppose that a rocket applies force* F *in the positive direction on cart* x_1*. Set up the system of equations.*

Exercise* 3.1.7*: *Suppose the tanks are as in* *Example* 3.1.2 *on page* 131*, starting both at volume* V*, but now the rate of flow from tank* 1 *to tank* 2 *is* r_1*, and rate of flow from tank* 2 *to tank one is* r_2*. In particular, the volumes will now be changing. Set up the system of equations.*

Exercise* 3.1.101*: *Find the general solution to* $y_1' = 3y_1$, $y_2' = y_1 + y_2$, $y_3' = y_1 + y_3$.

Exercise* 3.1.102*: *Solve* $y' = 2x$, $x' = x + y$, $x(0) = 1$, $y(0) = 3$.

Exercise* 3.1.103*: *Write* $x''' = x + t$ *as a first order system.*

Exercise* 3.1.104*: *Write* $y_1'' + y_1 + y_2 = t$, $y_2'' + y_1 - y_2 = t^2$ *as a first order system.*

Exercise* 3.1.105*: *Suppose two masses on carts on frictionless surface are at displacements* x_1 *and* x_2 *as in* *Example* 3.1.3 *on page* 132*. Suppose initial displacement is* $x_1(0) = x_2(0) = 0$*, and initial velocity is* $x_1'(0) = x_2'(0) = a$ *for some number* a*. Use your intuition to solve the system, explain your reasoning.*

Exercise* 3.1.106*: *Suppose the tanks are as in* *Example* 3.1.2 *on page* 131 *except that clean water flows in at the rate* s *liters per second into tank* 1*, and brine flows out of tank* 2 *and into the sewer also at the rate of* s *liters per second.*

a) *Draw the picture.*

b) *Set up the system of equations.*

c) *Intuitively, what happens as* t *goes to infinity, explain.*

3.2 Matrices and linear systems

Note: 1.5 lectures, first part of §5.1 in [EP], §7.2 and §7.3 in [BD]

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3.2.1 Matrices and vectors

Before we start talking about linear systems of ODEs, we need to talk about matrices, so let us review these briefly. A *matrix* is an $m \times n$ array of numbers (m rows and n columns). For example, we denote a 3×5 matrix as follows

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}.$$

The numbers a_{ij} are called *elements* or *entries*.

By a *vector* we usually mean a *column vector*, that is an $m \times 1$ matrix. If we mean a *row vector*, we will explicitly say so (a row vector is a $1 \times n$ matrix). We usually denote matrices by upper case letters and vectors by lower case letters with an arrow such as $\vec{x}$ or $\vec{b}$. By $\vec{0}$ we mean the vector of all zeros.

We define some operations on matrices. We want 1×1 matrices to really act like numbers, so our operations have to be compatible with this viewpoint.

First, we can multiply a matrix by a *scalar* (a number). We simply multiply each entry in the matrix by the scalar. For example,

$$2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

Matrix addition is also easy. We add matrices element by element. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 7 & 10 \end{bmatrix}.$$

If the sizes do not match, then addition is not defined.

If we denote by 0 the matrix with all zero entries, by c, d scalars, and by A, B, C matrices, we have the following familiar rules:

$$\begin{aligned} A + 0 = A &= 0 + A, \\ A + B &= B + A, \\ (A + B) + C &= A + (B + C), \\ c(A + B) &= cA + cB, \\ (c + d)A &= cA + dA. \end{aligned}$$

Another useful operation for matrices is the so-called *transpose*. This operation just swaps rows and columns of a matrix. The transpose of A is denoted by A^T. Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

3.2.2 Matrix multiplication

Let us now define matrix multiplication. First we define the so-called *dot product* (or *inner product*) of two vectors. Usually this will be a row vector multiplied with a column vector of the same size. For the dot product we multiply each pair of entries from the first and the second vector and we sum these products. The result is a single number. For example,

$$\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 .$$

And similarly for larger (or smaller) vectors.

Armed with the dot product we define the *product of matrices*. First let us denote by $\operatorname{row}_i(A)$ the i^{th} row of A and by $\operatorname{column}_j(A)$ the j^{th} column of A. For an $m \times n$ matrix A and an $n \times p$ matrix B we can define the product AB. We let AB be an $m \times p$ matrix whose ij^{th} entry is the dot product

$$\operatorname{row}_i(A) \cdot \operatorname{column}_j(B).$$

Do note how the sizes match up: $m \times n$ multiplied by $n \times p$ is $m \times p$. Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 \cdot 1 + 2 \cdot 1 + 3 \cdot 1 & 1 \cdot 0 + 2 \cdot 1 + 3 \cdot 0 & 1 \cdot (-1) + 2 \cdot 1 + 3 \cdot 0 \\ 4 \cdot 1 + 5 \cdot 1 + 6 \cdot 1 & 4 \cdot 0 + 5 \cdot 1 + 6 \cdot 0 & 4 \cdot (-1) + 5 \cdot 1 + 6 \cdot 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 1 \\ 15 & 5 & 1 \end{bmatrix}$$

For multiplication we want an analogue of a 1. This analogue is the so-called *identity matrix*. The identity matrix is a square matrix with 1s on the diagonal and zeros everywhere else. It is usually denoted by I. For each size we have a different identity matrix and so sometimes we may denote the size as a subscript. For example, the I_3 would be the 3×3 identity matrix

$$I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} .$$

We have the following rules for matrix multiplication. Suppose that A, B, C are matrices of the correct sizes so that the following make sense. Let α denote a scalar (number).

$$\begin{aligned} A(BC) &= (AB)C, \\ A(B+C) &= AB + AC, \\ (B+C)A &= BA + CA, \\ \alpha(AB) &= (\alpha A)B = A(\alpha B), \\ IA &= A = AI. \end{aligned}$$

A few warnings are in order.

(i) $AB \neq BA$ in general (it may be true by fluke sometimes). That is, matrices do not commute. For example, take $A = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$.

(ii) $AB = AC$ does not necessarily imply $B = C$, even if A is not 0.

(iii) $AB = 0$ does not necessarily mean that $A = 0$ or $B = 0$. Try, for example, $A = B = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$.

For the last two items to hold we would need to "divide" by a matrix. This is where the *matrix inverse* comes in. Suppose that A and B are $n \times n$ matrices such that

$$AB = I = BA.$$

Then we call B the inverse of A and we denote B by A^{-1}. If the inverse of A exists, then we call A *invertible*. If A is not invertible, we sometimes say A is *singular*.

If A is invertible, then $AB = AC$ does imply that $B = C$ (in particular the inverse of A is unique). We just multiply both sides by A^{-1} (on the left) to get $A^{-1}AB = A^{-1}AC$ or $IB = IC$ or $B = C$. It is also not hard to see that $(A^{-1})^{-1} = A$.

3.2.3 The determinant

For square matrices we define a useful quantity called the *determinant*. We define the determinant of a 1×1 matrix as the value of its only entry. For a 2×2 matrix we define

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \overset{\text{def}}{=} ad - bc.$$

Before trying to define the determinant for larger matrices, let us note the meaning of the determinant. Consider an $n \times n$ matrix as a mapping of the n-dimensional euclidean space $\mathbb{R}^n$ to itself, where $\vec{x}$ gets sent to $A\vec{x}$. In particular, a 2×2 matrix A is a mapping of the plane to itself. The determinant of A is the factor by which the area of objects changes. If we take the unit square (square of side 1) in the plane, then A takes the square to a parallelogram of area $|\det(A)|$. The sign of $\det(A)$ denotes changing of orientation (negative if the axes get flipped). For example, let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Then $\det(A) = 1 + 1 = 2$. Let us see where the (unit) square with vertices $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$ gets sent. Clearly $(0,0)$ gets sent to $(0,0)$.

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

The image of the square is another square with vertices $(0,0)$, $(1,-1)$, $(1,1)$, and $(2,0)$. The image square has a side of length $\sqrt{2}$ and is therefore of area 2.

If you think back to high school geometry, you may have seen a formula for computing the area of a parallelogram with vertices $(0,0)$, (a,c), (b,d) and $(a+b,c+d)$. And it is precisely

$$\left| \det \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) \right|.$$

The vertical lines above mean absolute value. The matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ carries the unit square to the given parallelogram.

Let us look at the determinant for larger matrices. We define A_{ij} as the matrix A with the i^{th} row and the j^{th} column deleted. To compute the determinant of a matrix, pick one row, say the i^{th} row and compute:

$$\boxed{\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).}$$

For the first row we get

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \cdots \begin{cases} +a_{1n}\det(A_{1n}) & \text{if } n \text{ is odd,} \\ -a_{1n}\det(A_{1n}) & \text{if } n \text{ even.} \end{cases}$$

We alternately add and subtract the determinants of the submatrices A_{ij} multiplied by a_{ij} for a fixed i and all j. For a 3×3 matrix, picking the first row, we get $\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$. For example,

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \right) &= 1 \cdot \det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} \right) - 2 \cdot \det\left(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} \right) + 3 \cdot \det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} \right) \\ &= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0. \end{aligned}$$

The numbers $(-1)^{i+j}\det(A_{ij})$ are called *cofactors* of the matrix and this way of computing the determinant is called the *cofactor expansion*. No matter which row you pick, you always get the same number. It is also possible to compute the determinant by expanding along columns (picking a column instead of a row above). It is true that $\det(A) = \det(A^T)$.

A common notation for the determinant is a pair of vertical lines:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right).$$

I personally find this notation confusing as vertical lines usually mean a positive quantity, while determinants can be negative. Also think about how to write the absolute value of a determinant. I will not use this notation in this book.

Think of the determinants telling you the scaling of a mapping. If B doubles the sizes of geometric objects and A triples them, then AB (which applies B to an object and then A) should make size go up by a factor of 6. This is true in general:

$$\det(AB) = \det(A)\det(B).$$

This property is one of the most useful, and it is employed often to actually compute determinants. A particularly interesting consequence is to note what it means for existence of inverses. Take A and B to be inverses of each other, that is $AB = I$. Then

$$\det(A)\det(B) = \det(AB) = \det(I) = 1.$$

Neither $\det(A)$ nor $\det(B)$ can be zero. Let us state this as a theorem as it will be very important in the context of this course.

Theorem 3.2.1. *An $n \times n$ matrix A is invertible if and only if* $\det(A) \neq 0$.

In fact, $\det(A^{-1})\det(A) = 1$ says that $\det(A^{-1}) = \frac{1}{\det(A)}$. So we even know what the determinant of A^{-1} is before we know how to compute A^{-1}.

There is a simple formula for the inverse of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Notice the determinant of the matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in the denominator of the fraction. The formula only works if the determinant is nonzero, otherwise we are dividing by zero.

3.2.4 Solving linear systems

One application of matrices we will need is to solve systems of linear equations. This is best shown by example. Suppose that we have the following system of linear equations

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2, \\ x_1 + x_2 + 3x_3 &= 5, \\ x_1 + 4x_2 + x_3 &= 10. \end{aligned}$$

Without changing the solution, we could swap equations in this system, we could multiply any of the equations by a nonzero number, and we could add a multiple of one equation to another equation. It turns out these operations always suffice to find a solution.

It is easier to write the system as a matrix equation. The system above can be written as

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}.$$

To solve the system we put the coefficient matrix (the matrix on the left-hand side of the equation) together with the vector on the right and side and get the so-called *augmented matrix*

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array}\right].$$

We apply the following three elementary operations.

(i) Swap two rows.

(ii) Multiply a row by a nonzero number.

(iii) Add a multiple of one row to another row.

We keep doing these operations until we get into a state where it is easy to read off the answer, or until we get into a contradiction indicating no solution, for example if we come up with an equation such as $0 = 1$.

Let us work through the example. First multiply the first row by $1/2$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array}\right].$$

Now subtract the first row from the second and third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 9 \end{array}\right]$$

Multiply the last row by $1/3$ and the second row by $1/2$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{array}\right]$$

Swap rows 2 and 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Subtract the last row from the first, then subtract the second row from the first.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

If we think about what equations this augmented matrix represents, we see that $x_1 = -4$, $x_2 = 3$, and $x_3 = 2$. We try this solution in the original system and, voilà, it works!

***Exercise* 3.2.1:** *Check that the solution above really solves the given equations.*

We write this equation in matrix notation as

$$A\vec{x} = \vec{b},$$

where A is the matrix $\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix}$ and $\vec{b}$ is the vector $\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$. The solution can also be computed via the inverse,

$$\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{b}.$$

It is possible that the solution is not unique, or that no solution exists. It is easy to tell if a solution does not exist. If during the row reduction you come up with a row where all the entries except the last one are zero (the last entry in a row corresponds to the right-hand side of the equation), then the system is *inconsistent* and has no solution. For example, for a system of 3 equations and 3 unknowns, if you find a row such as $[\,0 \quad 0 \quad 0 \mid 1\,]$ in the augmented matrix, you know the system is inconsistent. That row corresponds to $0 = 1$.

You generally try to use row operations until the following conditions are satisfied. The first (from the left) nonzero entry in each row is called the *leading entry*.

(i) The leading entry in any row is strictly to the right of the leading entry of the row above.

(ii) Any zero rows are below all the nonzero rows.

(iii) All leading entries are 1.

(iv) All the entries above and below a leading entry are zero.

Such a matrix is said to be in *reduced row echelon form*. The variables corresponding to columns with no leading entries are said to be *free variables*. Free variables mean that we can pick those variables to be anything we want and then solve for the rest of the unknowns.

Example 3.2.1: The following augmented matrix is in reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Suppose the variables are x_1, x_2, and x_3. Then x_2 is the free variable, $x_1 = 3 - 2x_2$, and $x_3 = 1$.

On the other hand if during the row reduction process you come up with the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 13 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right],$$

there is no need to go further. The last row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 3$, which is preposterous. Hence, no solution exists.

3.2.5 Computing the inverse

If the matrix A is square and there exists a unique solution $\vec{x}$ to $A\vec{x} = \vec{b}$ for any $\vec{b}$ (there are no free variables), then A is invertible. Multiplying both sides by A^{-1}, you can see that $\vec{x} = A^{-1}\vec{b}$. So it is useful to compute the inverse if you want to solve the equation for many different right-hand sides $\vec{b}$.

We have a formula for the 2×2 inverse, but it is also not hard to compute inverses of larger matrices. While we will not have too much occasion to compute inverses for larger matrices than 2×2 by hand, let us touch on how to do it. Finding the inverse of A is actually just solving a bunch of linear equations. If we can solve $A\vec{x}_k = \vec{e}_k$ where $\vec{e}_k$ is the vector with all zeros except a 1 at the k^{th} position, then the inverse is the matrix with the columns $\vec{x}_k$ for $k = 1, 2, \ldots, n$ (exercise: why?). Therefore, to find the inverse we write a larger $n \times 2n$ augmented matrix $[\,A \mid I\,]$, where I is the identity matrix. We then perform row reduction. The reduced row echelon form of $[\,A \mid I\,]$ will be of the form $[\,I \mid A^{-1}\,]$ if and only if A is invertible. We then just read off the inverse A^{-1}.

3.2.6 Exercises

***Exercise* 3.2.2:** *Solve* $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ *by using matrix inverse.*

***Exercise* 3.2.3:** *Compute determinant of* $\begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix}$.

***Exercise* 3.2.4:** *Compute determinant of* $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 0 & 5 & 0 \\ 6 & 0 & 7 & 0 \\ 8 & 0 & 10 & 1 \end{bmatrix}$. *Hint*: *Expand along the proper row or column to make the calculations simpler.*

***Exercise* 3.2.5:** *Compute inverse of* $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

***Exercise* 3.2.6:** *For which h is* $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & h \end{bmatrix}$ *not invertible*? *Is there only one such h*? *Are there several*? *Infinitely many*?

***Exercise* 3.2.7:** *For which h is* $\begin{bmatrix} h & 1 & 1 \\ 0 & h & 0 \\ 1 & 1 & h \end{bmatrix}$ *not invertible*? *Find all such h.*

***Exercise* 3.2.8:** *Solve* $\begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

***Exercise* 3.2.9:** *Solve* $\begin{bmatrix} 5 & 3 & 7 \\ 8 & 4 & 4 \\ 6 & 3 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

***Exercise* 3.2.10:** *Solve* $\begin{bmatrix} 3 & 2 & 3 & 0 \\ 3 & 3 & 3 & 3 \\ 0 & 2 & 4 & 2 \\ 2 & 3 & 4 & 3 \end{bmatrix} \vec{x} = \begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$.

***Exercise* 3.2.11:** *Find* 3 *nonzero* 2×2 *matrices A, B, and C such that AB = AC but B ≠ C.*

***Exercise* 3.2.51:** *Solve* $\begin{bmatrix} 2 & -3 \\ -1 & 5 \end{bmatrix} \vec{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ *by using the matrix inverse.*

***Exercise* 3.2.52:** *For a general* 2×2 *matrix* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, *show* $AI = IA = A$.

***Exercise* 3.2.53:** *For each of the following pairs of matrices* A, B *calculate the products* AB *and* BA, *if they exist.*

a) $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 \\ 2 & -4 \end{bmatrix}$

b) $A = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 5 \\ 2 & 4 & 3 \end{bmatrix}$

c) $A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & -1 & 4 \\ 2 & 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}$

***Exercise* 3.2.54:** *For each of the following matrices, compute* A^{-1}, *then show that* $AA^{-1} = A^{-1}A = I$.

a) $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

b) $A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$

***Exercise* 3.2.101:** *Compute determinant of* $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -5 \\ 1 & -1 & 0 \end{bmatrix}$

***Exercise* 3.2.102:** *Find* t *such that* $\begin{bmatrix} 1 & t \\ -1 & 2 \end{bmatrix}$ *is not invertible.*

***Exercise* 3.2.103:** *Solve* $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$.

***Exercise* 3.2.104:** *Suppose* a, b, c *are nonzero numbers. Let* $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $N = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.

a) *Compute* M^{-1}.

b) *Compute* N^{-1}.

3.3 Linear systems of ODEs

Note: less than 1 lecture, second part of §5.1 in [EP], §7.4 in [BD]

First let us talk about matrix- or vector-valued functions. Such a function is just a matrix or vector whose entries depend on some variable. If t is the independent variable, we write a *vector-valued function* $\vec{x}(t)$ as

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}.$$

Similarly a *matrix-valued function* $A(t)$ is

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}.$$

The derivative $A'(t)$ or $\frac{dA}{dt}$ is just the matrix-valued function whose ij^{th} entry is $a'_{ij}(t)$.

Rules of differentiation of matrix-valued functions are similar to rules for normal functions. Let $A(t)$ and $B(t)$ be matrix-valued functions. Let c a scalar and let C be a constant matrix. Then

$$\begin{aligned} \big(A(t) + B(t)\big)' &= A'(t) + B'(t), \\ \big(A(t)B(t)\big)' &= A'(t)B(t) + A(t)B'(t), \\ \big(cA(t)\big)' &= cA'(t), \\ \big(CA(t)\big)' &= CA'(t), \\ \big(A(t)\,C\big)' &= A'(t)\,C. \end{aligned}$$

Note the order of the multiplication in the last two expressions.

A *first order linear system of ODEs* is a system that can be written as the vector equation

$$\vec{x}'(t) = P(t)\vec{x}(t) + \vec{f}(t),$$

where $P(t)$ is a matrix-valued function, and $\vec{x}(t)$ and $\vec{f}(t)$ are vector-valued functions. We will often suppress the dependence on t and only write $\vec{x}' = P\vec{x} + \vec{f}$. A solution of the system is a vector-valued function $\vec{x}$ satisfying the vector equation.

For example, the equations

$$\begin{aligned} x_1' &= 2tx_1 + e^t x_2 + t^2, \\ x_2' &= \frac{x_1}{t} - x_2 + e^t, \end{aligned}$$

can be written as

$$\vec{x}' = \begin{bmatrix} 2t & e^t \\ 1/t & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} t^2 \\ e^t \end{bmatrix}.$$

We will mostly concentrate on equations that are not just linear, but are in fact *constant coefficient* equations. That is, the matrix P will be constant; it will not depend on t.

When $\vec{f} = \vec{0}$ (the zero vector), then we say the system is *homogeneous*. For homogeneous linear systems we have the principle of superposition, just like for single homogeneous equations.

Theorem 3.3.1 (Superposition)**.** *Let $\vec{x}' = P\vec{x}$ be a linear homogeneous system of ODEs. Suppose that $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are n solutions of the equation and $c_1, c_2, \ldots, c_n$ are any constants, then*

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n, \tag{3.2}$$

is also a solution. Furthermore, if this is a system of n equations (P is $n \times n$), and $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent, then every solution $\vec{x}$ can be written as (3.2).

Linear independence for vector-valued functions is the same idea as for normal functions. The vector-valued functions $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent when

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n = \vec{0}$$

has only the solution $c_1 = c_2 = \cdots = c_n = 0$, where the equation must hold for all t.

Example 3.3.1: $\vec{x}_1 = \begin{bmatrix} t^2 \\ t \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ 1+t \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} -t^2 \\ 1 \end{bmatrix}$ are linearly dependent because $\vec{x}_1 + \vec{x}_3 = \vec{x}_2$, and this holds for all t. So $c_1 = 1$, $c_2 = -1$, and $c_3 = 1$ above will work.

On the other hand if we change the example just slightly $\vec{x}_1 = \begin{bmatrix} t^2 \\ t \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 0 \\ t \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} -t^2 \\ 1 \end{bmatrix}$, then the functions are linearly independent. First write $c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \vec{0}$ and note that it has to hold for all t. We get that

$$c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \begin{bmatrix} c_1t^2 - c_3t^2 \\ c_1t + c_2t + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

In other words $c_1t^2 - c_3t^2 = 0$ and $c_1t + c_2t + c_3 = 0$. If we set $t = 0$, then the second equation becomes $c_3 = 0$. But then the first equation becomes $c_1t^2 = 0$ for all t and so $c_1 = 0$. Thus the second equation is just $c_2t = 0$, which means $c_2 = 0$. So $c_1 = c_2 = c_3 = 0$ is the only solution and $\vec{x}_1$, $\vec{x}_2$, and $\vec{x}_3$ are linearly independent.

The linear combination $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$ could always be written as

$$X(t)\,\vec{c},$$

where $X(t)$ is the matrix with columns $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$, and $\vec{c}$ is the column vector with entries $c_1, c_2, \ldots, c_n$. Assuming that $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent, the matrix-valued function $X(t)$ is called a *fundamental matrix*, or a *fundamental matrix solution*.

To solve nonhomogeneous first order linear systems, we use the same technique as we applied to solve single linear nonhomogeneous equations.

Theorem 3.3.2. *Let $\vec{x}' = P\vec{x} + \vec{f}$ be a linear system of ODEs. Suppose $\vec{x}_p$ is one particular solution. Then every solution can be written as*

$$\vec{x} = \vec{x}_c + \vec{x}_p,$$

where $\vec{x}_c$ is a solution to the associated homogeneous equation ($\vec{x}' = P\vec{x}$).

The procedure for systems is the same as for single equations. We find a particular solution to the nonhomogeneous equation, then we find the general solution to the associated homogeneous equation, and finally we add the two together.

Alright, suppose you have found the general solution of $\vec{x}' = P\vec{x} + \vec{f}$. Next suppose you are given an initial condition of the form

$$\vec{x}(t_0) = \vec{b}$$

for some fixed t_0 and a constant vector $\vec{b}$. Let $X(t)$ be a fundamental matrix solution of the associated homogeneous equation (i.e. columns of $X(t)$ are solutions). The general solution can be written as

$$\vec{x}(t) = X(t)\,\vec{c} + \vec{x}_p(t).$$

We are seeking a vector $\vec{c}$ such that

$$\vec{b} = \vec{x}(t_0) = X(t_0)\,\vec{c} + \vec{x}_p(t_0).$$

In other words, we are solving for $\vec{c}$ the nonhomogeneous system of linear equations

$$X(t_0)\,\vec{c} = \vec{b} - \vec{x}_p(t_0).$$

Example 3.3.2: In § 3.1 we solved the system

$$x_1' = x_1,$$
$$x_2' = x_1 - x_2,$$

with initial conditions $x_1(0) = 1$, $x_2(0) = 2$. Let us consider this problem in the language of this section.

The system is homogeneous, so $\vec{f}(t) = \vec{0}$. We write the system and the initial conditions as

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

We found the general solution is $x_1 = c_1 e^t$ and $x_2 = \frac{c_1}{2} e^t + c_2 e^{-t}$. Letting $c_1 = 1$ and $c_2 = 0$, we obtain the solution $\left[\begin{smallmatrix} e^t \\ (1/2)e^t \end{smallmatrix}\right]$. Letting $c_1 = 0$ and $c_2 = 1$, we obtain $\left[\begin{smallmatrix} 0 \\ e^{-t} \end{smallmatrix}\right]$. These two solutions are linearly independent, as can be seen by setting $t = 0$, and noting that the resulting constant vectors are linearly independent. In matrix notation, a fundamental matrix solution is, therefore,

$$X(t) = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}e^t & e^{-t} \end{bmatrix}.$$

To solve the initial value problem we solve for $\vec{c}$ in the equation

$$X(0)\,\vec{c} = \vec{b},$$

or in other words,

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

A single elementary row operation shows $\vec{c} = \left[\begin{smallmatrix} 1 \\ 3/2 \end{smallmatrix}\right]$. Our solution is

$$\vec{x}(t) = X(t)\,\vec{c} = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}e^t & e^{-t} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} e^t \\ \frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{bmatrix}.$$

This new solution agrees with our previous solution from § 3.1.

3.3.1 Exercises

***Exercise* 3.3.1:** *Write the system $x_1' = 2x_1 - 3tx_2 + \sin t$, $x_2' = e^t x_1 + 3x_2 + \cos t$ in the form $\vec{x}' = P(t)\vec{x} + \vec{f}(t)$.*

***Exercise* 3.3.2:**

a) *Verify that the system $\vec{x}' = \left[\begin{smallmatrix} 1 & 3 \\ 3 & 1 \end{smallmatrix}\right] \vec{x}$ has the two solutions $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] e^{4t}$ and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] e^{-2t}$.*

b) *Write down the general solution.*

c) *Write down the general solution in the form $x_1 =?$, $x_2 =?$ (i.e. write down a formula for each element of the solution).*

***Exercise* 3.3.3:** *Verify that $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] e^t$ and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] e^t$ are linearly independent. Hint: Just plug in $t = 0$.*

***Exercise* 3.3.4:** *Verify that $\left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right] e^t$ and $\left[\begin{smallmatrix} 1 \\ -1 \\ 1 \end{smallmatrix}\right] e^t$ and $\left[\begin{smallmatrix} 1 \\ -1 \\ 1 \end{smallmatrix}\right] e^{2t}$ are linearly independent. Hint: You must be a bit more tricky than in the previous exercise.*

***Exercise* 3.3.5:** *Verify that $\left[\begin{smallmatrix} t \\ t^2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} t^3 \\ t^4 \end{smallmatrix}\right]$ are linearly independent.*

***Exercise* 3.3.6:** *Take the system $x_1' + x_2' = x_1$, $x_1' - x_2' = x_2$.*

a) *Write it in the form $A\vec{x}' = B\vec{x}$ for matrices A and B.*

b) *Compute A^{-1} and use that to write the system in the form $\vec{x}' = P\vec{x}$.*

***Exercise* 3.3.51:** *Write the following n^{th}- order DEs as a system of n 1^{st}- order DEs, then write the system of the form $\vec{x}\,'(t) = P(t)\vec{x}(t) + \vec{f}(t)$*

a) $x'' + e^t x' - (\sin t)x = \cos t$

b) $x''' - (\tan^{-1} t)x'' + t^2x' - 3x = t^4$

c) $x''' - t^3x' + e^{t^2}x = \sin^2 t$

d) $tx''' + te^tx'' - t^2x' + x = e^{t^4},\ t > 0$

e) $x^{(4)} + 5x''' - (6\cos t)x'' + t^2x = (\tan^{-1} t)$

f) $t^2x^{(4)} - 2x'' + tx = \ln t,\ t > 0$

Exercise* 3.3.101*: *Are* $\begin{bmatrix} e^{2t} \\ e^t \end{bmatrix}$ *and* $\begin{bmatrix} e^t \\ e^{2t} \end{bmatrix}$ *linearly independent? Justify.*

Exercise* 3.3.102*: *Are* $\begin{bmatrix} \cosh(t) \\ 1 \end{bmatrix}$, $\begin{bmatrix} e^t \\ 1 \end{bmatrix}$, *and* $\begin{bmatrix} e^{-t} \\ 1 \end{bmatrix}$ *linearly independent? Justify.*

Exercise* 3.3.103*: *Write* $x' = 3x - y + e^t$, $y' = tx$ *in matrix notation.*

Exercise* 3.3.104*:

a) *Write* $x_1' = 2tx_2$, $x_2' = 2tx_2$ *in matrix notation.*

b) *Solve and write the solution in matrix notation.*

3.4 Eigenvalue method

Note: 2 lectures, §5.2 in [EP], part of §7.3, §7.5, and §7.6 in [BD]

In this section we will learn how to solve linear homogeneous constant coefficient systems of ODEs by the eigenvalue method. Suppose we have such a system

$$\vec{x}' = P\vec{x},$$

where P is a constant square matrix. We wish to adapt the method for the single constant coefficient equation by trying the function $e^{\lambda t}$. However, $\vec{x}$ is a vector. So we try $\vec{x} = \vec{v}e^{\lambda t}$, where $\vec{v}$ is an arbitrary constant vector. We plug this $\vec{x}$ into the equation to get

$$\underbrace{\lambda\vec{v}e^{\lambda t}}_{\vec{x}'} = \underbrace{P\vec{v}e^{\lambda t}}_{P\vec{x}}.$$

We divide by $e^{\lambda t}$ and notice that we are looking for a scalar λ and a vector $\vec{v}$ that satisfy the equation

$$\lambda\vec{v} = P\vec{v}.$$

To solve this equation we need a little bit more linear algebra, which we now review.

3.4.1 Eigenvalues and eigenvectors of a matrix

Let A be a constant square matrix. Suppose there is a scalar λ and a nonzero vector $\vec{v}$ such that

$$A\vec{v} = \lambda\vec{v}.$$

We call λ an *eigenvalue* of A and we call $\vec{v}$ a corresponding *eigenvector*.

Example 3.4.1: The matrix $\left[\begin{smallmatrix} 2 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ has an eigenvalue $\lambda = 2$ with a corresponding eigenvector $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ as

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let us see how to compute eigenvalues for any matrix. Rewrite the equation for an eigenvalue as

$$(A - \lambda I)\vec{v} = \vec{0}.$$

This equation has a nonzero solution $\vec{v}$ only if $A - \lambda I$ is not invertible. Were it invertible, we could write $(A - \lambda I)^{-1}(A - \lambda I)\vec{v} = (A - \lambda I)^{-1}\vec{0}$, which implies $\vec{v} = \vec{0}$. Therefore, A has the eigenvalue λ if and only if λ solves the equation

$$\det(A - \lambda I) = 0.$$

Consequently, we will be able to find an eigenvalue of A without finding a corresponding eigenvector. An eigenvector will have to be found later, once λ is known.

Example 3.4.2: Find all eigenvalues of $\left[\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right]$.

We write

$$\det\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) =$$
$$= (2-\lambda)\big((2-\lambda)^2 - 1\big) = -(\lambda-1)(\lambda-2)(\lambda-3).$$

So the eigenvalues are $\lambda = 1$, $\lambda = 2$, and $\lambda = 3$.

For an $n \times n$ matrix, the polynomial we get by computing $\det(A - \lambda I)$ is of degree n, and hence in general, we have n eigenvalues. Some may be repeated, some may be complex.

To find an eigenvector corresponding to an eigenvalue λ, we write

$$(A - \lambda I)\vec{v} = \vec{0},$$

and solve for a nontrivial (nonzero) vector $\vec{v}$. If λ is an eigenvalue, there will be at least one free variable, and so for each distinct eigenvalue λ, we can always find an eigenvector.

Example 3.4.3: Find an eigenvector of $\left[\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right]$ corresponding to the eigenvalue $\lambda = 3$.

We write

$$(A - \lambda I)\vec{v} = \left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \vec{0}.$$

It is easy to solve this system of linear equations. We write down the augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right],$$

and perform row operations (exercise: which ones?) until we get:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The entries of $\vec{v}$ have to satisfy the equations $v_1 - v_2 = 0$, $v_3 = 0$, and v_2 is a free variable. We can pick v_2 to be arbitrary (but nonzero), let $v_1 = v_2$, and of course $v_3 = 0$. For example, if we pick $v_2 = 1$, then $\vec{v} = \left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right]$. Let us verify that $\vec{v}$ really is an eigenvector corresponding to $\lambda = 3$:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Yay! It worked.

Exercise* 3.4.1** (easy):*** *Are eigenvectors unique? Can you find a different eigenvector for $\lambda = 3$ in the example above? How are the two eigenvectors related?*

Exercise* 3.4.2*: *When the matrix is 2×2 you do not need to do row operations when computing an eigenvector, you can read it off from $A - \lambda I$ (if you have computed the eigenvalues correctly). Can you see why? Explain. Try it for the matrix $\left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$.*

3.4.2 The eigenvalue method with distinct real eigenvalues

OK. We have the system of equations

$$\vec{x}' = P\vec{x}.$$

We find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix P, and corresponding eigenvectors $\vec{v}_1$, $\vec{v}_2, \ldots, \vec{v}_n$. Now we notice that the functions $\vec{v}_1 e^{\lambda_1 t}, \vec{v}_2 e^{\lambda_2 t}, \ldots, \vec{v}_n e^{\lambda_n t}$ are solutions of the system of equations and hence $\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \cdots + c_n \vec{v}_n e^{\lambda_n t}$ is a solution.

Theorem 3.4.1. *Take $\vec{x}' = P\vec{x}$. If P is an $n \times n$ constant matrix that has n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then there exist n linearly independent corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and the general solution to $\vec{x}' = P\vec{x}$ can be written as*

$$\boxed{\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \cdots + c_n \vec{v}_n e^{\lambda_n t}.}$$

The corresponding fundamental matrix solution is

$$X(t) = \begin{bmatrix} \vec{v}_1 e^{\lambda_1 t} & \vec{v}_2 e^{\lambda_2 t} & \cdots & \vec{v}_n e^{\lambda_n t} \end{bmatrix}.$$

That is, $X(t)$ is the matrix whose j^{th} column is $\vec{v}_j e^{\lambda_j t}$.

Example 3.4.4: Consider the system

$$\vec{x}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}.$$

Find the general solution.

Earlier, we found the eigenvalues are $1, 2, 3$. We found the eigenvector $\left[\begin{smallmatrix} 1 \\ 1 \\ 0 \end{smallmatrix}\right]$ for the eigenvalue 3. Similarly we find the eigenvector $\left[\begin{smallmatrix} 1 \\ -1 \\ 0 \end{smallmatrix}\right]$ for the eigenvalue 1, and $\left[\begin{smallmatrix} 0 \\ 1 \\ -1 \end{smallmatrix}\right]$ for the eigenvalue 2 (exercise: check). Hence our general solution is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t} = \begin{bmatrix} c_1 e^t + c_3 e^{3t} \\ -c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ -c_2 e^{2t} \end{bmatrix}.$$

In terms of a fundamental matrix solution,

$$\vec{x} = X(t)\,\vec{c} = \begin{bmatrix} e^t & 0 & e^{3t} \\ -e^t & e^{2t} & e^{3t} \\ 0 & -e^{2t} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

***Exercise* 3.4.3:** *Check that this $\vec{x}$ really solves the system.*

Note: If we write a single homogeneous linear constant coefficient n^{th} order equation as a first order system (as we did in § 3.1), then the eigenvalue equation

$$\det(P - \lambda I) = 0$$

is essentially the same as the characteristic equation we got in § 2.2 and § 2.3.

3.4.3 Complex eigenvalues

A matrix may very well have complex eigenvalues even if all the entries are real. Take, for example,

$$\vec{x}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}.$$

Let us compute the eigenvalues of the matrix $P = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 1 \end{smallmatrix}\right]$.

$$\det(P - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 + 1 = \lambda^2 - 2\lambda + 2 = 0.$$

Thus $\lambda = 1 \pm i$. Corresponding eigenvectors are also complex. Start with $\lambda = 1 - i$.

$$\begin{aligned} \big(P - (1-i)I\big)\vec{v} &= \vec{0}, \\ \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \vec{v} &= \vec{0}. \end{aligned}$$

The equations $iv_1 + v_2 = 0$ and $-v_1 + iv_2 = 0$ are multiples of each other. So we only need to consider one of them. After picking $v_2 = 1$, for example, we have an eigenvector $\vec{v} = \left[\begin{smallmatrix} i \\ 1 \end{smallmatrix}\right]$. In similar fashion we find that $\left[\begin{smallmatrix} -i \\ 1 \end{smallmatrix}\right]$ is an eigenvector corresponding to the eigenvalue $1 + i$.

We could write the solution as

$$\vec{x} = c_1 \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1-i)t} + c_2 \begin{bmatrix} -i \\ 1 \end{bmatrix} e^{(1+i)t} = \begin{bmatrix} c_1 i e^{(1-i)t} - c_2 i e^{(1+i)t} \\ c_1 e^{(1-i)t} + c_2 e^{(1+i)t} \end{bmatrix}.$$

We would then need to look for complex values c_1 and c_2 to solve any initial conditions. It is perhaps not completely clear that we get a real solution. After solving for c_1 and c_2, we could use Euler's formula and do the whole song and dance we did before, but we will not. We will apply the formula in a smarter way first to find independent real solutions.

We claim that we did not have to look for a second eigenvector (nor for the second eigenvalue). All complex eigenvalues come in pairs (because the matrix P is real).

First a small detour. The real part of a complex number z can be computed as $\frac{z+\bar{z}}{2}$, where the bar above z means $\overline{a+ib} = a - ib$. This operation is called the *complex conjugate*. If a is a real number, then $\bar{a} = a$. Similarly we bar whole vectors or matrices by taking the complex conjugate of every entry. Suppose a matrix P is real. Then $\overline{P} = P$, and so $\overline{P\vec{x}} = \overline{P}\,\overline{\vec{x}} = P\overline{\vec{x}}$. Also the complex conjugate of 0 is still 0, therefore,

$$\vec{0} = \overline{\vec{0}} = \overline{(P - \lambda I)\vec{v}} = (P - \bar{\lambda} I)\overline{\vec{v}}.$$

In other words, if $\lambda = a + ib$ is an eigenvalue, then so is $\bar{\lambda} = a - ib$. And if $\vec{v}$ is an eigenvector corresponding to the eigenvalue λ, then $\overline{\vec{v}}$ is an eigenvector corresponding to the eigenvalue $\bar{\lambda}$.

Suppose $a + ib$ is a complex eigenvalue of P, and $\vec{v}$ is a corresponding eigenvector. Then

$$\vec{x}_1 = \vec{v} e^{(a+ib)t}$$

is a solution (complex-valued) of $\vec{x}' = P\vec{x}$. Euler's formula shows that $\overline{e^{a+ib}} = e^{a-ib}$, and so

$$\vec{x}_2 = \overline{\vec{x}_1} = \overline{\vec{v}} e^{(a-ib)t}$$

is also a solution. As $\vec{x}_1$ and $\vec{x}_2$ are solutions, the function

$$\vec{x}_3 = \operatorname{Re} \vec{x}_1 = \operatorname{Re} \vec{v} e^{(a+ib)t} = \frac{\vec{x}_1 + \overline{\vec{x}_1}}{2} = \frac{\vec{x}_1 + \vec{x}_2}{2} = \frac{1}{2}\vec{x}_1 + \frac{1}{2}\vec{x}_2$$

is also a solution. And $\vec{x}_3$ is real-valued! Similarly as $\operatorname{Im} z = \frac{z - \bar{z}}{2i}$ is the imaginary part, we find that

$$\vec{x}_4 = \operatorname{Im} \vec{x}_1 = \frac{\vec{x}_1 - \overline{\vec{x}_1}}{2i} = \frac{\vec{x}_1 - \vec{x}_2}{2i}.$$

is also a real-valued solution. It turns out that $\vec{x}_3$ and $\vec{x}_4$ are linearly independent. We will use Euler's formula to separate out the real and imaginary part.

Returning to our problem,

$$\vec{x}_1 = \begin{bmatrix} i \\ 1 \end{bmatrix} e^{(1-i)t} = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(e^t \cos t - i e^t \sin t \right) = \begin{bmatrix} i e^t \cos t + e^t \sin t \\ e^t \cos t - i e^t \sin t \end{bmatrix} = \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + i \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix}.$$

Then

$$\operatorname{Re} \vec{x}_1 = \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix}, \qquad \text{and} \qquad \operatorname{Im} \vec{x}_1 = \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix},$$

are the two real-valued linearly independent solutions we seek.

***Exercise* 3.4.4:** *Check that these really are solutions.*

The general solution is

$$\vec{x} = c_1 \begin{bmatrix} e^t \sin t \\ e^t \cos t \end{bmatrix} + c_2 \begin{bmatrix} e^t \cos t \\ -e^t \sin t \end{bmatrix} = \begin{bmatrix} c_1 e^t \sin t + c_2 e^t \cos t \\ c_1 e^t \cos t - c_2 e^t \sin t \end{bmatrix}.$$

This solution is real-valued for real c_1 and c_2. We now solve for any initial conditions we may have.

Let us summarize as a theorem.

Theorem 3.4.2. *Let P be a real-valued constant matrix. If P has a complex eigenvalue $a + ib$ and a corresponding eigenvector $\vec{v}$, then P also has a complex eigenvalue $a - ib$ with a corresponding eigenvector $\overline{\vec{v}}$. Furthermore, $\vec{x}' = P\vec{x}$ has two linearly independent real-valued solutions*

$$\vec{x}_1 = \operatorname{Re} \vec{v}e^{(a+ib)t}, \qquad \textit{and} \qquad \vec{x}_2 = \operatorname{Im} \vec{v}e^{(a+ib)t}.$$

For each pair of complex eigenvalues $a + ib$ and $a - ib$, we get two real-valued linearly independent solutions. We then go on to the next eigenvalue, which is either a real eigenvalue or another complex eigenvalue pair. If we have n distinct eigenvalues (real or complex), then we end up with n linearly independent solutions. If we had only two equations ($n = 2$) as in the example above, then once we found two solutions we are finished, and our general solution is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1\big(\operatorname{Re} \vec{v}e^{(a+ib)t}\big) + c_2\big(\operatorname{Im} \vec{v}e^{(a+ib)t}\big).$$

We can now find a real-valued general solution to any homogeneous system where the matrix has distinct eigenvalues. When we have repeated eigenvalues, matters get a bit more complicated and we will look at that situation in § 3.7.

3.4.4 Eigenvalues and eigenvectors with Python

Both numpy and sympy provide functions to compute eigenvalues and eigenvectors. They present the results in rather different ways. *numpy.linalg.eig()* gives a list of the eigenvalues ("j" is used for the imaginary unit), followed by an array whose columns are the corresponding eigenvectors. *sympy.Matrix.eigenvects()* returns a list of tuples, each of which has the eigenvalue, its algebraic multiplicity, and a list of corresponding eigenvectors. Numpy uses floating point arithmetic, while sympy does exact rational arithmetic if the input is integer or rational.

```
from resources306 import *
```

```
A = [[7,7,0],[-7,7,0],[0,0,13]]
```

```
np.linalg.eig(A)
```

```
(array([ 7.+7.j,  7.-7.j, 13.+0.j]),
 array([[0.        -0.70710678j, 0.        +0.70710678j,
         0.        +0.j        ],
        [0.70710678+0.j        , 0.70710678-0.j        ,
         0.        +0.j        ],
        [0.        +0.j        , 0.        -0.j        ,
         1.        +0.j        ]]))
```

```
sp.Matrix(A).eigenvects()
```

$$\left[\left(13, \quad 1, \quad \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right] \right), \quad \left(7 - 7i, \quad 1, \quad \left[\begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} \right] \right), \quad \left(7 + 7i, \quad 1, \quad \left[\begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} \right] \right) \right]$$

```
B = [[5,5],[0,5]]
```

```
np.linalg.eig(B)
```

```
(array([5., 5.]), array([[ 1.00000000e+00, -1.00000000e+00],
        [ 0.00000000e+00,  2.22044605e-16]]))
```

```
sp.Matrix(B).eigenvects()
```

$$\left[\left(5, \quad 2, \quad \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right] \right) \right]$$

```
C = [[3,0],[0,3]]
```

```
np.linalg.eig(C)
```

```
(array([3., 3.]), array([[1., 0.],
        [0., 1.]]))
```

```
sp.Matrix(C).eigenvects()
```

$$\left[\left(3, \quad 2, \quad \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] \right) \right]$$

3.4.5 Exercises

***Exercise* 3.4.5** (easy)*: Let A be a 3×3 matrix with an eigenvalue of 3 and a corresponding eigenvector $\vec{v} = \left[\begin{smallmatrix} 1 \\ -1 \\ 3 \end{smallmatrix}\right]$. Find $A\vec{v}$.*

***Exercise* 3.4.6:**

a) *Find the general solution of $x_1' = 2x_1$, $x_2' = 3x_2$ using the eigenvalue method (first write the system in the form $\vec{x}' = A\vec{x}$).*

b) *Solve the system by solving each equation separately and verify you get the same general solution.*

***Exercise* 3.4.7:** *Find the general solution of $x_1' = 3x_1 + x_2$, $x_2' = 2x_1 + 4x_2$ using the eigenvalue method.*

***Exercise* 3.4.8:** *Find the general solution of $x_1' = x_1 - 2x_2$, $x_2' = 2x_1 + x_2$ using the eigenvalue method. Do not use complex exponentials in your solution.*

***Exercise* 3.4.9:**

a) *Compute eigenvalues and eigenvectors of $A = \left[\begin{smallmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{smallmatrix}\right]$.*

b) *Find the general solution of $\vec{x}' = A\vec{x}$.*

***Exercise* 3.4.10:** *Compute eigenvalues and eigenvectors of $\left[\begin{smallmatrix} -2 & -1 & -1 \\ 3 & 2 & 1 \\ -3 & -1 & 0 \end{smallmatrix}\right]$.*

***Exercise* 3.4.11:** *Let a, b, c, d, e, f be numbers. Find the eigenvalues of $\left[\begin{smallmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{smallmatrix}\right]$.*

***Exercise* 3.4.101:**

a) *Compute eigenvalues and eigenvectors of $A = \left[\begin{smallmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & 2 \end{smallmatrix}\right]$.*

b) *Solve the system $\vec{x}\,' = A\vec{x}$.*

***Exercise* 3.4.102:**

a) *Compute eigenvalues and eigenvectors of $A = \left[\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}\right]$.*

b) *Solve the system $\vec{x}\,' = A\vec{x}$.*

***Exercise* 3.4.103:** *Solve $x_1' = x_2$, $x_2' = x_1$ using the eigenvalue method.*

***Exercise* 3.4.104:** *Solve $x_1' = x_2$, $x_2' = -x_1$ using the eigenvalue method.*

***Exercise* 3.4.151:** *Solve each of the following systems by the eigenvalue method. If ICs are given, find the particular solution to the system. If no ICs are given, find the general solution. Write all solutions in vector form.*

a) $x_1' = 3x_1 - x_2,\ x_2' = 7x_1 - 5x_2$

b) $x_1' = 4x_1 + x_2,\ x_2' = 6x_1 - x_2$

c) $x_1' = x_1 - x_2,\ x_2' = 5x_1 - 3x_2$

d) $x_1' = -2x_1 + 5x_2,\ x_2' = -6x_1 + 9x_2;\ x_1(0) = 1,\ x_2(0) = 3$

e) $x_1' = 3x_1 + 4x_2,\ x_2' = -5x_1 + 7x_2$

f) $x_1' = -x_1 - x_2,\ x_2' = 5x_1 + x_2;\ x_1(0) = -2,\ x_2(0) = 1$

g) $x_1' = -4x_1 + x_2,\ x_2' = 2x_1 - 3x_2;\ x_1(0) = 2,\ x_2(0) = -3$

h) $x_1' = -x_1 - 2x_2,\ x_2' = 9x_1 + 5x_2$

i) $x_1' = x_1 + 5x_2,\ x_2' = -2x_1 - x_2$

j) $x_1' = 5x_1 + 7x_2,\ x_2' = -2x_1 - 4x_2;\ x_1(0) = -2,\ x_2(0) = -3$

3.5 Two-dimensional systems and their vector fields

Note: 1 lecture, part of §6.2 in [EP], parts of §7.5 and §7.6 in [BD]

Let us take a moment to talk about constant coefficient linear homogeneous systems in the plane. Much intuition can be obtained by studying this simple case. Suppose we use coordinates (x, y) for the plane as usual, and suppose $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix. Consider the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = P \begin{bmatrix} x \\ y \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{3.3}$$

The system is autonomous (compare this section to § 1.6) and so we can draw a vector field (see the end of § 3.1). We will be able to visually tell what the vector field looks like and how the solutions behave, once we find the eigenvalues and eigenvectors of the matrix P. For this section, we assume that P has two eigenvalues and two corresponding eigenvectors.

Case 1. Suppose that the eigenvalues of P are real and positive. We find two corresponding eigenvectors and plot them in the plane. For example, take the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. The eigenvalues are 1 and 2 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. See Figure 3.4.

Suppose the point (x, y) is on the line determined by an eigenvector $\vec{v}$ for an eigenvalue λ. That is, $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha\vec{v}$ for some scalar α. Then

$$\begin{bmatrix} x \\ y \end{bmatrix}' = P \begin{bmatrix} x \\ y \end{bmatrix} = P(\alpha\vec{v}) = \alpha(P\vec{v}) = \alpha\lambda\vec{v}.$$

The derivative is a multiple of $\vec{v}$ and hence points along the line determined by $\vec{v}$. As $\lambda > 0$, the derivative points in the direction of $\vec{v}$ when α is positive and in the opposite direction when α is negative. Let us draw the lines determined by the eigenvectors, and let us draw arrows on the lines to indicate the directions. See Figure 3.5 on the facing page.

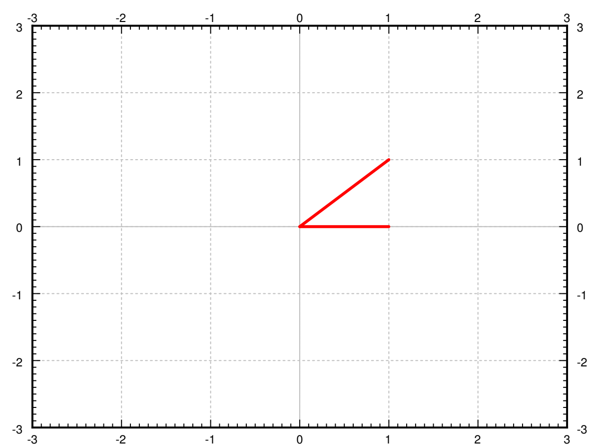


Figure 3.4: Eigenvectors of P.

We fill in the rest of the arrows for the vector field and we also draw a few solutions. See Figure 3.6 on the next page. The picture looks like a source with arrows coming out from the origin. Hence we call this type of picture a *source* or sometimes an *unstable node*.

Case 2. Suppose both eigenvalues are negative. For example, take the negation of the matrix in case 1, $\begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$. The eigenvalues are -1 and -2 and corresponding eigenvectors are the same, $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The calculation and the picture are almost the same. The only difference is that the eigenvalues are negative and hence all arrows are reversed. We get the picture in Figure 3.7 on the facing page. We call this kind of picture a *sink* or a *stable node*.

Case 3. Suppose one eigenvalue is positive and one is negative. For example the matrix $\begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}$. The eigenvalues are 1 and -2 and corresponding eigenvectors are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$.

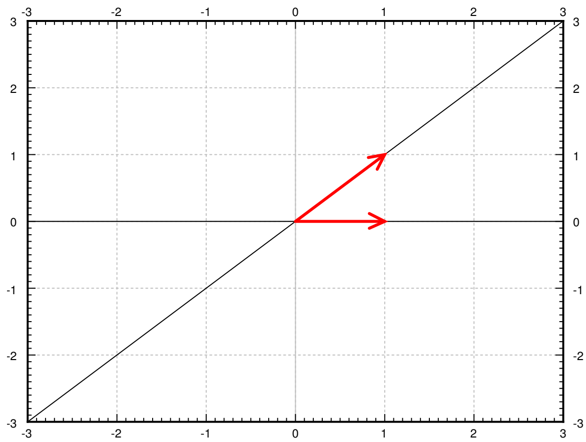


Figure 3.5: Eigenvectors of P with directions.

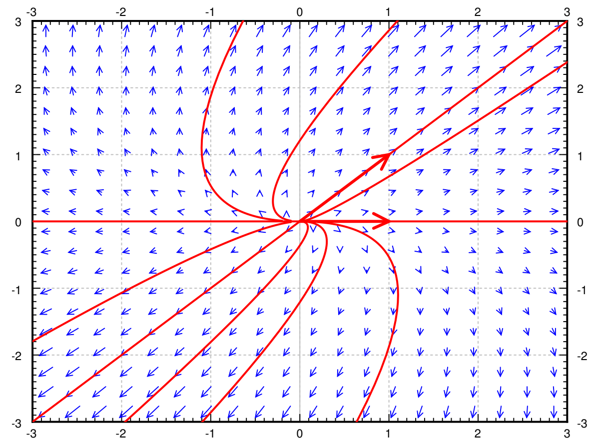


Figure 3.6: Example source vector field with eigenvectors and solutions.

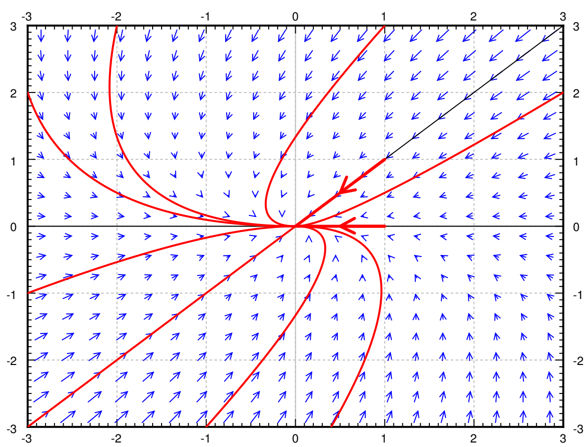


Figure 3.7: Example sink vector field with eigenvectors and solutions.

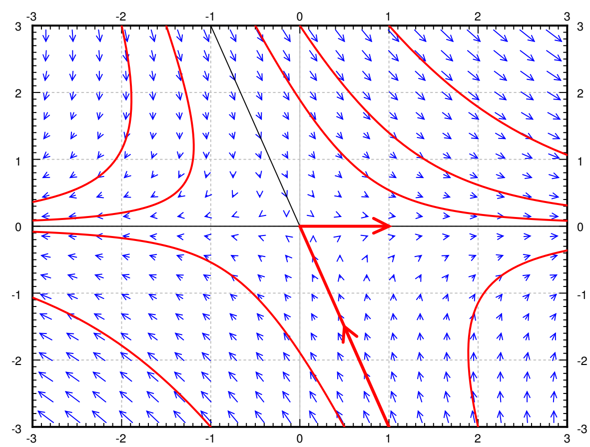


Figure 3.8: Example saddle vector field with eigenvectors and solutions.

We reverse the arrows on one line (corresponding to the negative eigenvalue) and we obtain the picture in Figure 3.8. We call this picture a *saddle point*.

For the next three cases we will assume the eigenvalues are complex. In this case the eigenvectors are also complex and we cannot just plot them in the plane.

Case 4. Suppose the eigenvalues are purely imaginary. That is, suppose the eigenvalues are $\pm ib$. For example, let $P = \left[\begin{smallmatrix} 0 & 1 \\ -4 & 0 \end{smallmatrix}\right]$. The eigenvalues turn out to be $\pm 2i$ and eigenvectors are $\left[\begin{smallmatrix} 1 \\ 2i \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -2i \end{smallmatrix}\right]$. Consider the eigenvalue $2i$ and its eigenvector $\left[\begin{smallmatrix} 1 \\ 2i \end{smallmatrix}\right]$. The real and imaginary parts of $\vec{v}e^{2it}$ are

$$\operatorname{Re} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} = \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}, \qquad \operatorname{Im} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{2it} = \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}.$$

We can take any linear combination of them to get other solutions, which one we take depends on the initial conditions. Now note that the real part is a parametric equation for an ellipse. Same with the imaginary part and in fact any linear combination of the two. This is what happens in general when the eigenvalues are purely imaginary. So when the eigenvalues are purely imaginary, we get *ellipses* for the solutions. This type of picture is sometimes called a *center*. See Figure 3.9.

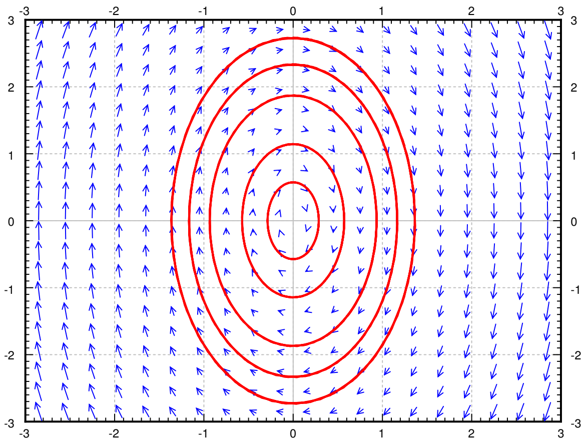


Figure 3.9: Example center vector field.

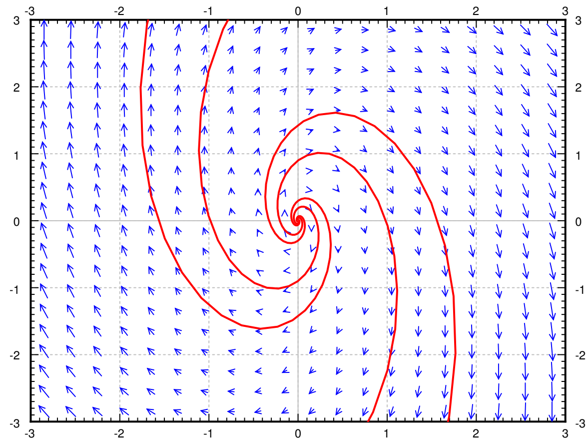


Figure 3.10: Example spiral source vector field.

Case 5. Now suppose the complex eigenvalues have a positive real part. That is, suppose the eigenvalues are $a \pm ib$ for some $a > 0$. For example, let $P = \left[\begin{smallmatrix} 1 & 1 \\ -4 & 1 \end{smallmatrix}\right]$. The eigenvalues turn out to be $1 \pm 2i$ and eigenvectors are $\left[\begin{smallmatrix} 1 \\ 2i \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -2i \end{smallmatrix}\right]$. We take $1 + 2i$ and its eigenvector $\left[\begin{smallmatrix} 1 \\ 2i \end{smallmatrix}\right]$ and find the real and imaginary parts of $\vec{v}e^{(1+2i)t}$ are

$$\operatorname{Re} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(1+2i)t} = e^t \begin{bmatrix} \cos(2t) \\ -2\sin(2t) \end{bmatrix}, \qquad \operatorname{Im} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(1+2i)t} = e^t \begin{bmatrix} \sin(2t) \\ 2\cos(2t) \end{bmatrix}.$$

Note the e^t in front of the solutions. The solutions grow in magnitude while spinning around the origin. Hence we get a *spiral source*. See Figure 3.10.

Case 6. Finally suppose the complex eigenvalues have a negative real part. That is, suppose the eigenvalues are $-a \pm ib$ for some $a > 0$. For example, let $P = \left[\begin{smallmatrix} -1 & -1 \\ 4 & -1 \end{smallmatrix}\right]$. The eigenvalues turn out to be $-1 \pm 2i$ and eigenvectors are $\left[\begin{smallmatrix} 1 \\ -2i \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ 2i \end{smallmatrix}\right]$. We take $-1 - 2i$ and its eigenvector $\left[\begin{smallmatrix} 1 \\ 2i \end{smallmatrix}\right]$ and find the real and imaginary parts of $\vec{v}e^{(-1-2i)t}$ are

$$\operatorname{Re} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(-1-2i)t} = e^{-t} \begin{bmatrix} \cos(2t) \\ 2\sin(2t) \end{bmatrix}, \qquad \operatorname{Im} \begin{bmatrix} 1 \\ 2i \end{bmatrix} e^{(-1-2i)t} = e^{-t} \begin{bmatrix} -\sin(2t) \\ 2\cos(2t) \end{bmatrix}.$$

Note the e^{-t} in front of the solutions. The solutions shrink in magnitude while spinning around the origin. Hence we get a *spiral sink*. See Figure 3.11 on the facing page.

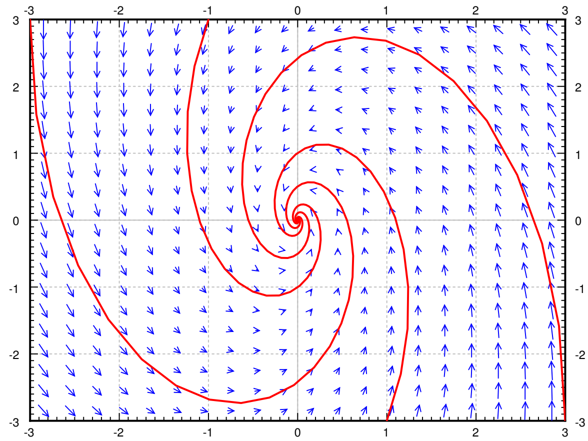


Figure 3.11: Example spiral sink vector field.

We summarize the behavior of linear homogeneous two-dimensional systems given by a nonsingular matrix in Table 3.1. Systems where one of the eigenvalues is zero (the matrix is singular) come up in practice from time to time, see Example 3.1.2 on page 131, and the pictures are somewhat different (simpler in a way). See the exercises.

Eigenvalues	Behavior
real and both positive	source / unstable node
real and both negative	sink / stable node
real and opposite signs	saddle
purely imaginary	center point / ellipses
complex with positive real part	spiral source
complex with negative real part	spiral sink

Table 3.1: Summary of behavior of linear homogeneous two-dimensional systems.

3.5.1 Exercises

***Exercise* 3.5.1:** *Take the equation $mx'' + cx' + kx = 0$, with $m > 0$, $c \geq 0$, $k > 0$ for the mass-spring system.*

a) *Convert this to a system of first order equations.*

b) *Classify for what m, c, k do you get which behavior.*

c) *Can you explain from physical intuition why you do not get all the different kinds of behavior here?*

Exercise* 3.5.2*: *What happens in the case when* $P = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$? *In this case the eigenvalue is repeated and there is only one independent eigenvector. What picture does this look like*?

Exercise* 3.5.3*: *What happens in the case when* $P = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$? *Does this look like any of the pictures we have drawn*?

Exercise* 3.5.4*: *Which behaviors are possible if* P *is diagonal, that is* $P = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$? *You can assume that* a *and* b *are not zero.*

Exercise* 3.5.5*: *Take the system from Example* 3.1.2 *on page* 131, $x_1' = \frac{r}{V}(x_2 - x_1)$, $x_2' = \frac{r}{V}(x_1 - x_2)$. *As we said, one of the eigenvalues is zero. What is the other eigenvalue, how does the picture look like and what happens when* t *goes to infinity.*

Exercise* 3.5.101*: *Describe the behavior of the following systems without solving*:

a) $x' = x + y, \quad y' = x - y.$

b) $x_1' = x_1 + x_2, \quad x_2' = 2x_2.$

c) $x_1' = -2x_2, \quad x_2' = 2x_1.$

d) $x' = x + 3y, \quad y' = -2x - 4y.$

e) $x' = x - 4y, \quad y' = -4x + y.$

Exercise* 3.5.102*: *Suppose that* $\vec{x}\,' = A\vec{x}$ *where* A *is a* 2 *by* 2 *matrix with eigenvalues* $2 \pm i$. *Describe the behavior.*

Exercise* 3.5.103*: *Take* $\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]' = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]$. *Draw the vector field and describe the behavior. Is it one of the behaviors that we have seen before*?

3.6 Second order systems and applications

Note: more than 2 lectures, §5.4 in [EP], not in [BD]

3.6.1 Undamped mass-spring systems

While we did say that we will usually only look at first order systems, it is sometimes more convenient to study the system in the way it arises naturally. For example, suppose we have 3 masses connected by springs between two walls. We could pick any higher number, and the math would be essentially the same, but for simplicity we pick 3 right now. Let us also assume no friction, that is, the system is undamped. The masses are m_1, m_2, and m_3 and the spring constants are k_1, k_2, k_3, and k_4. Let x_1 be the displacement from rest position of the first mass, and x_2 and x_3 the displacement of the second and third mass. We make, as usual, positive values go right (as x_1 grows, the first mass is moving right). See Figure 3.12.

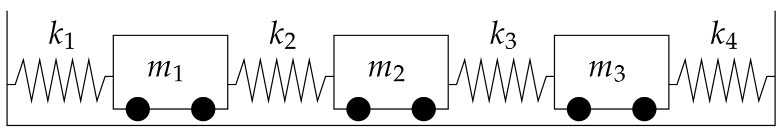


Figure 3.12: System of masses and springs.

This simple system turns up in unexpected places. For example, our world really consists of many small particles of matter interacting together. When we try the system above with many more masses, we obtain a good approximation to how an elastic material behaves. By somehow taking a limit of the number of masses going to infinity, we obtain the continuous one-dimensional wave equation (that we study in § 4.7). But we digress.

Let us set up the equations for the three mass system. By Hooke's law, the force acting on the mass equals the spring compression times the spring constant. By Newton's second law, force is mass times acceleration. So if we sum the forces acting on each mass, put the right sign in front of each term, depending on the direction in which it is acting, and set this equal to mass times the acceleration, we end up with the desired system of equations.

$$\begin{aligned}
m_1 x_1'' &= -k_1 x_1 + k_2(x_2 - x_1) & &= -(k_1 + k_2)x_1 + k_2 x_2, \\
m_2 x_2'' &= -k_2(x_2 - x_1) + k_3(x_3 - x_2) & &= k_2 x_1 - (k_2 + k_3)x_2 + k_3 x_3, \\
m_3 x_3'' &= -k_3(x_3 - x_2) - k_4 x_3 & &= k_3 x_2 - (k_3 + k_4)x_3.
\end{aligned}$$

We define the matrices

$$M = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \qquad \text{and} \qquad K = \begin{bmatrix} -(k_1+k_2) & k_2 & 0 \\ k_2 & -(k_2+k_3) & k_3 \\ 0 & k_3 & -(k_3+k_4) \end{bmatrix}.$$

We write the equation simply as
$$M\vec{x}'' = K\vec{x}.$$

At this point we could introduce 3 new variables and write out a system of 6 first order equations. We claim this simple setup is easier to handle as a second order system. We call $\vec{x}$ the *displacement vector*, M the *mass matrix*, and K the *stiffness matrix*.

***Exercise* 3.6.1:** *Repeat this setup for* 4 *masses (find the matrices M and K). Do it for* 5 *masses. Can you find a prescription to do it for n masses?*

As with a single equation we want to "divide by M." This means computing the inverse of M. The masses are all nonzero and M is a diagonal matrix, so computing the inverse is easy:
$$M^{-1} = \begin{bmatrix} \frac{1}{m_1} & 0 & 0 \\ 0 & \frac{1}{m_2} & 0 \\ 0 & 0 & \frac{1}{m_3} \end{bmatrix}.$$
This fact follows readily by how we multiply diagonal matrices. As an exercise, you should verify that $MM^{-1} = M^{-1}M = I$.

Let $A = M^{-1}K$. We look at the system $\vec{x}'' = M^{-1}K\vec{x}$, or
$$\vec{x}'' = A\vec{x}.$$

Many real world systems can be modeled by this equation. For simplicity, we will only talk about the given masses-and-springs problem. We try a solution of the form
$$\vec{x} = \vec{v}e^{\alpha t}.$$

We compute that for this guess, $\vec{x}'' = \alpha^2\vec{v}e^{\alpha t}$. We plug our guess into the equation and get
$$\alpha^2\vec{v}e^{\alpha t} = A\vec{v}e^{\alpha t}.$$

We divide by $e^{\alpha t}$ to arrive at $\alpha^2\vec{v} = A\vec{v}$. Hence if α^2 is an eigenvalue of A and $\vec{v}$ is a corresponding eigenvector, we have found a solution.

In our example, and in other common applications, A has only real negative eigenvalues (and possibly a zero eigenvalue). So we study only this case. When an eigenvalue λ is negative, it means that $\alpha^2 = \lambda$ is negative. Hence there is some real number ω such that $-\omega^2 = \lambda$. Then $\alpha = \pm i\omega$. The solution we guessed was
$$\vec{x} = \vec{v}\,\big(\cos(\omega t) + i\sin(\omega t)\big).$$

By taking the real and imaginary parts (note that $\vec{v}$ is real), we find that $\vec{v}\cos(\omega t)$ and $\vec{v}\sin(\omega t)$ are linearly independent solutions.

If an eigenvalue is zero, it turns out that both $\vec{v}$ and $\vec{v}t$ are solutions, where $\vec{v}$ is an eigenvector corresponding to the eigenvalue 0.

***Exercise* 3.6.2:** *Show that if A has a zero eigenvalue and $\vec{v}$ is a corresponding eigenvector, then $\vec{x} = \vec{v}(a + bt)$ is a solution of $\vec{x}'' = A\vec{x}$ for arbitrary constants a and b.*

Theorem 3.6.1. *Let A be a real $n \times n$ matrix with n distinct real negative (or zero) eigenvalues we denote by $-\omega_1^2 > -\omega_2^2 > \cdots > -\omega_n^2$, and corresponding eigenvectors by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. If A is invertible (that is, if $\omega_1 > 0$), then*

$$\boxed{\vec{x}(t) = \sum_{i=1}^{n} \vec{v}_i \big(a_i \cos(\omega_i t) + b_i \sin(\omega_i t)\big),}$$

is the general solution of

$$\vec{x}'' = A\vec{x},$$

for some arbitrary constants a_i and b_i. If A has a zero eigenvalue, that is $\omega_1 = 0$, and all other eigenvalues are distinct and negative, then the general solution can be written as

$$\boxed{\vec{x}(t) = \vec{v}_1(a_1 + b_1 t) + \sum_{i=2}^{n} \vec{v}_i \big(a_i \cos(\omega_i t) + b_i \sin(\omega_i t)\big).}$$

We use this solution and the setup from the introduction of this section even when some of the masses and springs are missing. For example, when there are only 2 masses and only 2 springs, simply take only the equations for the two masses and set all the spring constants for the springs that are missing to zero.

3.6.2 Examples

Example 3.6.1: Consider the setup in Figure 3.13, with $m_1 = 2\,\text{kg}$, $m_2 = 1\,\text{kg}$, $k_1 = 4\,{}^{\text{N}}\!/\!_{\text{m}}$, and $k_2 = 2\,{}^{\text{N}}\!/\!_{\text{m}}$.

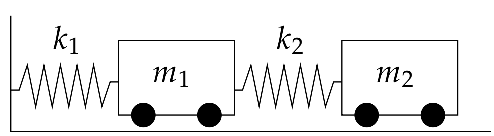


Figure 3.13: System of masses and springs.

The equations we write down are

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -(4+2) & 2 \\ 2 & -2 \end{bmatrix} \vec{x},$$

or

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

We find the eigenvalues of A to be $\lambda = -1, -4$ (exercise). We find corresponding eigenvectors to be $\left[\begin{smallmatrix}1\\2\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]$ respectively (exercise).

We check the theorem and note that $\omega_1 = 1$ and $\omega_2 = 2$. Hence the general solution is

$$\vec{x} = \begin{bmatrix}1\\2\end{bmatrix}\big(a_1\cos(t) + b_1\sin(t)\big) + \begin{bmatrix}1\\-1\end{bmatrix}\big(a_2\cos(2t) + b_2\sin(2t)\big).$$

The two terms in the solution represent the two so-called *natural* or *normal modes of oscillation*. And the two (angular) frequencies are the *natural frequencies*. The first natural frequency is 1, and second natural frequency is 2. The two modes are plotted in Figure 3.14.

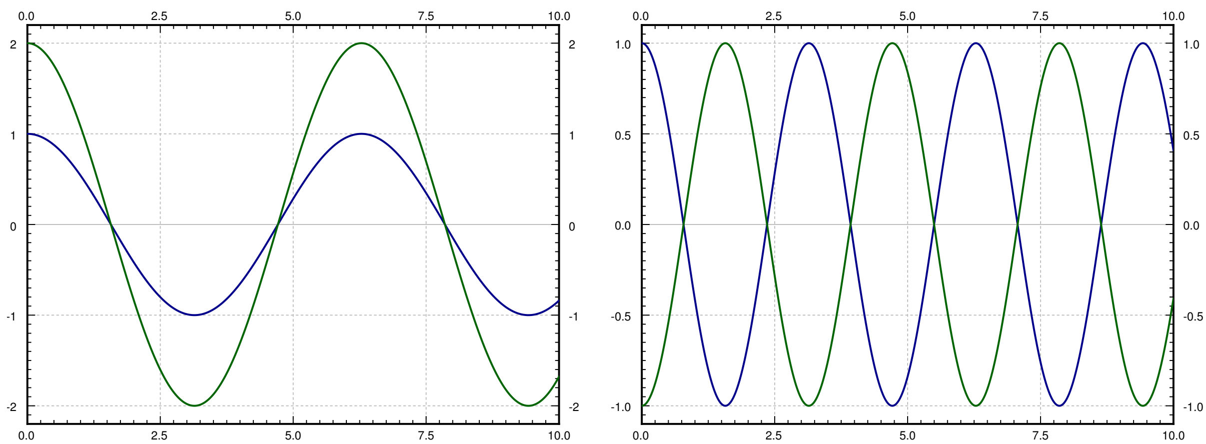


Figure 3.14: The two modes of the mass-spring system. In the left plot the masses are moving in unison and in the right plot are masses moving in the opposite direction.

Let us write the solution as

$$\vec{x} = \begin{bmatrix}1\\2\end{bmatrix} c_1\cos(t-\alpha_1) + \begin{bmatrix}1\\-1\end{bmatrix} c_2\cos(2t-\alpha_2).$$

The first term,

$$\begin{bmatrix}1\\2\end{bmatrix} c_1\cos(t-\alpha_1) = \begin{bmatrix}c_1\cos(t-\alpha_1)\\2c_1\cos(t-\alpha_1)\end{bmatrix},$$

corresponds to the mode where the masses move synchronously in the same direction.

The second term,

$$\begin{bmatrix}1\\-1\end{bmatrix} c_2\cos(2t-\alpha_2) = \begin{bmatrix}c_2\cos(2t-\alpha_2)\\-c_2\cos(2t-\alpha_2)\end{bmatrix},$$

corresponds to the mode where the masses move synchronously but in opposite directions.

The general solution is a combination of the two modes. That is, the initial conditions determine the amplitude and phase shift of each mode. As an example, suppose we have initial conditions

$$\vec{x}(0) = \begin{bmatrix}1\\-1\end{bmatrix}, \qquad \vec{x}'(0) = \begin{bmatrix}0\\6\end{bmatrix}.$$

We use the a_j, b_j constants to solve for initial conditions. First

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} a_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} a_2 = \begin{bmatrix} a_1 + a_2 \\ 2a_1 - a_2 \end{bmatrix}.$$

We solve (exercise) to find $a_1 = 0$, $a_2 = 1$. To find the b_1 and b_2, we differentiate first:

$$\vec{x}' = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \big(-a_1 \sin(t) + b_1 \cos(t)\big) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \big(-2a_2 \sin(2t) + 2b_2 \cos(2t)\big).$$

Now we solve:

$$\begin{bmatrix} 0 \\ 6 \end{bmatrix} = \vec{x}'(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} b_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} 2b_2 = \begin{bmatrix} b_1 + 2b_2 \\ 2b_1 - 2b_2 \end{bmatrix}.$$

Again solve (exercise) to find $b_1 = 2$, $b_2 = -1$. So our solution is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} 2\sin(t) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \big(\cos(2t) - \sin(2t)\big) = \begin{bmatrix} 2\sin(t) + \cos(2t) - \sin(2t) \\ 4\sin(t) - \cos(2t) + \sin(2t) \end{bmatrix}.$$

The graphs of the two displacements, x_1 and x_2 of the two carts is in Figure 3.15.

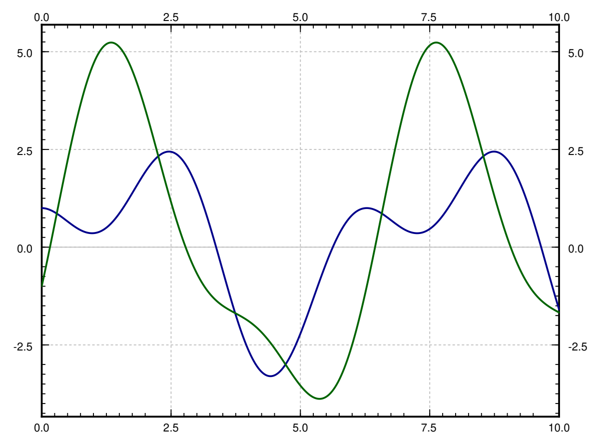


Figure 3.15: Superposition of the two modes given the initial conditions.

Example 3.6.2: We have two toy rail cars. Car 1 of mass 2 kg is traveling at 3 m/s towards the second rail car of mass 1 kg. There is a bumper on the second rail car that engages at the moment the cars hit (it connects to two cars) and does not let go. The bumper acts like a spring of spring constant $k = 2$ N/m. The second car is 10 meters from a wall. See Figure 3.16 on the next page.

We want to ask several questions. At what time after the cars link does impact with the wall happen? What is the speed of car 2 when it hits the wall?

OK, let us first set the system up. Let $t = 0$ be the time when the two cars link up. Let x_1 be the displacement of the first car from the position at $t = 0$, and let x_2 be the displacement

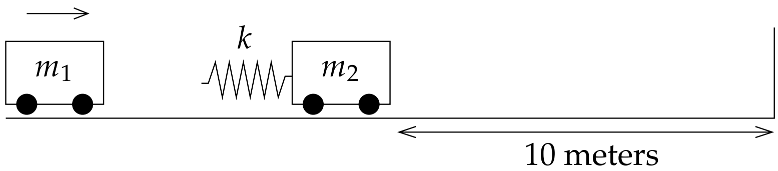


Figure 3.16: The crash of two rail cars.

of the second car from its original location. Then the time when $x_2(t) = 10$ is exactly the time when impact with wall occurs. For this t, $x_2'(t)$ is the speed at impact. This system acts just like the system of the previous example but without k_1. Hence the equation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} \vec{x},$$

or

$$\vec{x}'' = \begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

We compute the eigenvalues of A. It is not hard to see that the eigenvalues are 0 and -3 (exercise). Furthermore, eigenvectors are $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]$ respectively (exercise). Then $\omega_1 = 0$, $\omega_2 = \sqrt{3}$, and by the second part of the theorem the general solution is

$$\begin{aligned} \vec{x} &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} (a_1 + b_1 t) + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \Big(a_2 \cos(\sqrt{3}\, t) + b_2 \sin(\sqrt{3}\, t) \Big) \\ &= \begin{bmatrix} a_1 + b_1 t + a_2 \cos(\sqrt{3}\, t) + b_2 \sin(\sqrt{3}\, t) \\ a_1 + b_1 t - 2a_2 \cos(\sqrt{3}\, t) - 2b_2 \sin(\sqrt{3}\, t) \end{bmatrix}. \end{aligned}$$

We now apply the initial conditions. First the cars start at position 0 so $x_1(0) = 0$ and $x_2(0) = 0$. The first car is traveling at $3\,{}^{\mathrm{m}}\!/\!{}_{\mathrm{s}}$, so $x_1'(0) = 3$ and the second car starts at rest, so $x_2'(0) = 0$. The first conditions says

$$\vec{0} = \vec{x}(0) = \begin{bmatrix} a_1 + a_2 \\ a_1 - 2a_2 \end{bmatrix}.$$

It is not hard to see that $a_1 = a_2 = 0$. We set $a_1 = 0$ and $a_2 = 0$ in $\vec{x}(t)$ and differentiate to get

$$\vec{x}'(t) = \begin{bmatrix} b_1 + \sqrt{3}\, b_2 \cos(\sqrt{3}\, t) \\ b_1 - 2\sqrt{3}\, b_2 \cos(\sqrt{3}\, t) \end{bmatrix}.$$

So

$$\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \vec{x}'(0) = \begin{bmatrix} b_1 + \sqrt{3}\, b_2 \\ b_1 - 2\sqrt{3}\, b_2 \end{bmatrix}.$$

Solving these two equations we find $b_1 = 2$ and $b_2 = \frac{1}{\sqrt{3}}$. Hence the position of our cars is (until the impact with the wall)

$$\vec{x} = \begin{bmatrix} 2t + \frac{1}{\sqrt{3}} \sin(\sqrt{3}\, t) \\ 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3}\, t) \end{bmatrix}.$$

Note how the presence of the zero eigenvalue resulted in a term containing t. This means that the cars will be traveling in the positive direction as time grows, which is what we expect.

What we are really interested in is the second expression, the one for x_2. We have $x_2(t) = 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3}\, t)$. See Figure 3.17 for the plot of x_2 versus time.

Just from the graph we can see that time of impact will be a little more than 5 seconds from time zero. For this we have to solve the equation $10 = x_2(t) = 2t - \frac{2}{\sqrt{3}} \sin(\sqrt{3}\, t)$. Using a computer (or even a graphing calculator) we find that $t_{\text{impact}} \approx 5.22$ seconds.

The speed of the second car is $x_2' = 2 - 2\cos(\sqrt{3}\, t)$. At the time of impact (5.22 seconds from $t = 0$) we get $x_2'(t_{\text{impact}}) \approx 3.85$. The maximum speed is the maximum of $2 - 2\cos(\sqrt{3}\, t)$, which is 4. We are traveling at almost the maximum speed when we hit the wall.

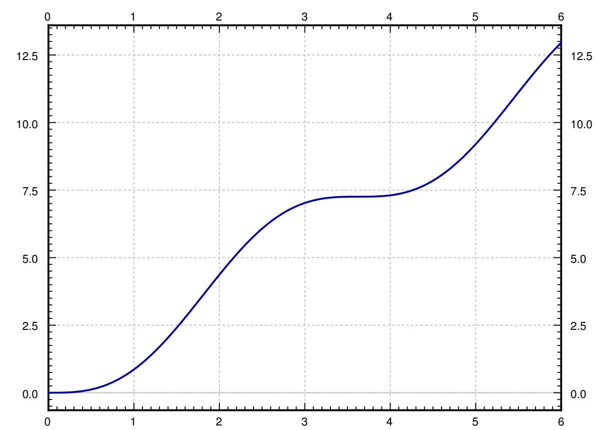


Figure 3.17: Position of the second car in time (ignoring the wall).

Suppose that Bob is a tiny person sitting on car 2. Bob has a Martini in his hand and would like not to spill it. Let us suppose Bob would not spill his Martini when the first car links up with car 2, but if car 2 hits the wall at any speed greater than zero, Bob will spill his drink. Suppose Bob can move car 2 a few meters towards or away from the wall (he cannot go all the way to the wall, nor can he get out of the way of the first car). Is there a "safe" distance for him to be at? A distance such that the impact with the wall is at zero speed?

The answer is yes. Looking at Figure 3.17, we note the "plateau" between $t = 3$ and $t = 4$. There is a point where the speed is zero. To find it we solve $x_2'(t) = 0$. This is when $\cos(\sqrt{3}\, t) = 1$ or in other words when $t = \frac{2\pi}{\sqrt{3}}, \frac{4\pi}{\sqrt{3}}, \ldots$ and so on. We plug in the first value to obtain $x_2\left(\frac{2\pi}{\sqrt{3}}\right) = \frac{4\pi}{\sqrt{3}} \approx 7.26$. So a "safe" distance is about 7 and a quarter meters from the wall.

Alternatively Bob could move away from the wall towards the incoming car 2, where another safe distance is $x_2\left(\frac{4\pi}{\sqrt{3}}\right) = \frac{8\pi}{\sqrt{3}} \approx 14.51$ and so on. We can use all the different t such that $x_2'(t) = 0$. Of course $t = 0$ is also a solution, corresponding to $x_2 = 0$, but that means standing right at the wall.

3.6.3 Forced oscillations

Finally we move to forced oscillations. Suppose that now our system is

$$\vec{x}'' = A\vec{x} + \vec{F}\cos(\omega t). \tag{3.4}$$

That is, we are adding periodic forcing to the system in the direction of the vector $\vec{F}$.

As before, this system just requires us to find one particular solution $\vec{x}_p$, add it to the general solution of the associated homogeneous system $\vec{x}_c$, and we will have the general solution to (3.4). Let us suppose that ω is not one of the natural frequencies of $\vec{x}'' = A\vec{x}$, then we can guess

$$\vec{x}_p = \vec{c}\cos(\omega t),$$

where $\vec{c}$ is an unknown constant vector. Note that we do not need to use sine since there are only second derivatives. We solve for $\vec{c}$ to find $\vec{x}_p$. This is really just the method of *undetermined coefficients* for systems. Let us differentiate $\vec{x}_p$ twice to get

$$\vec{x}_p'' = -\omega^2\vec{c}\cos(\omega t).$$

Plug $\vec{x}_p$ and $\vec{x}_p''$ into equation (3.4):

$$\overbrace{-\omega^2\vec{c}\cos(\omega t)}^{\vec{x}_p''} = \overbrace{A\vec{c}\cos(\omega t)}^{A\vec{x}_p} + \vec{F}\cos(\omega t).$$

We cancel out the cosine and rearrange the equation to obtain

$$(A + \omega^2 I)\vec{c} = -\vec{F}.$$

So

$$\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F}).$$

Of course this is possible only if $(A + \omega^2 I) = \big(A - (-\omega^2)I\big)$ is invertible. That matrix is invertible if and only if $-\omega^2$ is not an eigenvalue of A. That is true if and only if ω is not a natural frequency of the system.

We simplified things a little bit. If we wish to have the forcing term to be in the units of force, say Newtons, then we must write

$$M\vec{x}'' = K\vec{x} + \vec{G}\cos(\omega t).$$

If we then write things in terms of $A = M^{-1}K$, we have

$$\vec{x}'' = M^{-1}K\vec{x} + M^{-1}\vec{G}\cos(\omega t) \qquad \text{or} \qquad \vec{x}'' = A\vec{x} + \vec{F}\cos(\omega t),$$

where $\vec{F} = M^{-1}\vec{G}$.

Example 3.6.3: Let us take the example in Figure 3.13 on page 167 with the same parameters as before: $m_1 = 2$, $m_2 = 1$, $k_1 = 4$, and $k_2 = 2$. Now suppose that there is a force $2\cos(3t)$ acting on the second cart.

The equation is

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -4 & 2 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(3t) \qquad \text{or} \qquad \vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(3t).$$

We solved the associated homogeneous equation before and found the complementary solution to be

$$\vec{x}_c = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \bigl(a_1 \cos(t) + b_1 \sin(t)\bigr) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bigl(a_2 \cos(2t) + b_2 \sin(2t)\bigr).$$

The natural frequencies are 1 and 2. As 3 is not a natural frequency, we try $\vec{c}\cos(3t)$. We invert $(A + 3^2 I)$:

$$\left(\begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} + 3^2 I \right)^{-1} = \begin{bmatrix} 6 & 1 \\ 2 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{20} & \frac{3}{20} \end{bmatrix}.$$

Hence,

$$\vec{c} = (A + \omega^2 I)^{-1}(-\vec{F}) = \begin{bmatrix} \frac{7}{40} & \frac{-1}{40} \\ \frac{-1}{20} & \frac{3}{20} \end{bmatrix} \begin{bmatrix} 0 \\ -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{20} \\ \frac{-3}{10} \end{bmatrix}.$$

Combining with the general solution of the associated homogeneous problem, we get that the general solution to $\vec{x}'' = A\vec{x} + \vec{F}\cos(\omega t)$ is

$$\vec{x} = \vec{x}_c + \vec{x}_p = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \bigl(a_1 \cos(t) + b_1 \sin(t)\bigr) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \bigl(a_2 \cos(2t) + b_2 \sin(2t)\bigr) + \begin{bmatrix} \frac{1}{20} \\ \frac{-3}{10} \end{bmatrix} \cos(3t).$$

We then solve for the constants a_1, a_2, b_1, and b_2 using any initial conditions we are given.

Note that given force $\vec{f}$, we write the equation as $M\vec{x}'' = K\vec{x} + \vec{f}$ to get the units right. Then we write $\vec{x}'' = M^{-1}K\vec{x} + M^{-1}\vec{f}$. The term $\vec{g} = M^{-1}\vec{f}$ in $\vec{x}'' = A\vec{x} + \vec{g}$ is in units of force per unit mass.

If ω is a natural frequency of the system, *resonance* may occur, because we will have to try a particular solution of the form

$$\vec{x}_p = \vec{c}\, t \sin(\omega t) + \vec{d}\, \cos(\omega t).$$

That is assuming that the eigenvalues of the coefficient matrix are distinct. Next, note that the amplitude of this solution grows without bound as t grows.

3.6.4 Exercises

***Exercise* 3.6.3:** *Find a particular solution to*

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(2t).$$

Exercise* 3.6.4** (challenging):*** *Let us take the example in Figure 3.13 on page 167 with the same parameters as before: $m_1 = 2$, $k_1 = 4$, and $k_2 = 2$, except for m_2, which is unknown. Suppose that there is a force $\cos(5t)$ acting on the first mass. Find an m_2 such that there exists a particular solution where the first mass does not move.*

Note: This idea is called dynamic damping*. In practice there will be a small amount of damping and so any transient solution will disappear and after long enough time, the first mass will always come to a stop.*

Exercise* 3.6.5*: *Let us take the Example 3.6.2 on page 169, but that at time of impact, car 2 is moving to the left at the speed of 3 m/s.*

a) Find the behavior of the system after linkup.

b) Will the second car hit the wall, or will it be moving away from the wall as time goes on?

c) At what speed would the first car have to be traveling for the system to essentially stay in place after linkup?

Exercise* 3.6.6*: *Let us take the example in Figure 3.13 on page 167 with parameters $m_1 = m_2 = 1$, $k_1 = k_2 = 1$. Does there exist a set of initial conditions for which the first cart moves but the second cart does not? If so, find those conditions. If not, argue why not.*

Exercise* 3.6.101*: *Find the general solution to* $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{x}'' = \begin{bmatrix} -3 & 0 & 0 \\ 2 & -4 & 0 \\ 0 & 6 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} \cos(2t) \\ 0 \\ 0 \end{bmatrix}$.

Exercise* 3.6.102*: *Suppose there are three carts of equal mass m and connected by two springs of constant k (and no connections to walls). Set up the system and find its general solution.*

Exercise* 3.6.103*: *Suppose a cart of mass 2 kg is attached by a spring of constant $k = 1$ to a cart of mass 3 kg, which is attached to the wall by a spring also of constant $k = 1$. Suppose that the initial position of the first cart is 1 meter in the positive direction from the rest position, and the second mass starts at the rest position. The masses are not moving and are let go. Find the position of the second mass as a function of time.*

3.7 Multiple eigenvalues

Note: 1 or 1.5 lectures, §5.5 in [EP], §7.8 in [BD]

It may happen that a matrix A has some "repeated" eigenvalues. That is, the characteristic equation $\det(A - \lambda I) = 0$ may have repeated roots. This is actually unlikely to happen for a random matrix. If we take a small perturbation of A (we change the entries of A slightly), we get a matrix with distinct eigenvalues. As any system we want to solve in practice is an approximation to reality anyway, it is not absolutely indispensable to know how to solve these corner cases. On the other hand, these cases do come up in applications from time to time. Furthermore, if we have distinct but very close eigenvalues, the behavior is similar to that of repeated eigenvalues, and so understanding that case will give us insight into what is going on.

3.7.1 Geometric multiplicity

Take the diagonal matrix

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

A has an eigenvalue 3 of multiplicity 2. We call the multiplicity of the eigenvalue in the characteristic equation the *algebraic multiplicity*. In this case, there also exist 2 linearly independent eigenvectors, $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ corresponding to the eigenvalue 3. This means that the so-called *geometric multiplicity* of this eigenvalue is also 2.

In all the theorems where we required a matrix to have n distinct eigenvalues, we only really needed to have n linearly independent eigenvectors. For example, $\vec{x}' = A\vec{x}$ has the general solution

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}.$$

Let us restate the theorem about real eigenvalues. In the following theorem we will repeat eigenvalues according to (algebraic) multiplicity. So for the matrix A above, we would say that it has eigenvalues 3 and 3.

Theorem 3.7.1. *Suppose the $n \times n$ matrix P has n real eigenvalues (not necessarily distinct), λ_1, $\lambda_2, \ldots, \lambda_n$, and there are n linearly independent corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Then the general solution to $\vec{x}' = P\vec{x}$ can be written as*

$$\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \cdots + c_n \vec{v}_n e^{\lambda_n t}.$$

The *geometric multiplicity* of an eigenvalue of algebraic multiplicity n is equal to the number of corresponding linearly independent eigenvectors. The geometric multiplicity is always less than or equal to the algebraic multiplicity. The theorem handles the case when these two multiplicities are equal for all eigenvalues. If for an eigenvalue the geometric multiplicity is equal to the algebraic multiplicity, then we say the eigenvalue is *complete*.

In other words, the hypothesis of the theorem could be stated as saying that if all the eigenvalues of P are complete, then there are n linearly independent eigenvectors and thus we have the given general solution.

If the geometric multiplicity of an eigenvalue is 2 or greater, then the set of linearly independent eigenvectors is not unique up to multiples as it was before. For example, for the diagonal matrix $A = \left[\begin{smallmatrix} 3 & 0 \\ 0 & 3 \end{smallmatrix}\right]$ we could also pick eigenvectors $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$, or in fact any pair of two linearly independent vectors. The number of linearly independent eigenvectors corresponding to λ is the number of free variables we obtain when solving $A\vec{v} = \lambda\vec{v}$. We pick specific values for those free variables to obtain eigenvectors. If you pick different values, you may get different eigenvectors.

3.7.2 Defective eigenvalues

If an $n \times n$ matrix has less than n linearly independent eigenvectors, it is said to be *deficient*. Then there is at least one eigenvalue with an algebraic multiplicity that is higher than its geometric multiplicity. We call this eigenvalue *defective* and the difference between the two multiplicities we call the *defect*.

Example 3.7.1: The matrix

$$\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

has an eigenvalue 3 of algebraic multiplicity 2. Let us try to compute eigenvectors.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}.$$

We must have that $v_2 = 0$. Hence any eigenvector is of the form $\left[\begin{smallmatrix} v_1 \\ 0 \end{smallmatrix}\right]$. Any two such vectors are linearly dependent, and hence the geometric multiplicity of the eigenvalue is 1. Therefore, the defect is 1, and we can no longer apply the eigenvalue method directly to a system of ODEs with such a coefficient matrix.

Roughly, the key observation is that if λ is an eigenvalue of A of algebraic multiplicity m, then we can find certain m linearly independent vectors solving $(A - \lambda I)^k \vec{v} = \vec{0}$ for various powers k. We will call these *generalized eigenvectors*.

Let us continue with the example $A = \left[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right]$ and the equation $\vec{x}' = A\vec{x}$. We found an eigenvalue $\lambda = 3$ of (algebraic) multiplicity 2 and defect 1. We found one eigenvector $\vec{v} = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. We have one solution

$$\vec{x}_1 = \vec{v}e^{3t} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t}.$$

We are now stuck, we get no other solutions from standard eigenvectors. But we need two linearly independent solutions to find the general solution of the equation.

Let us try (in the spirit of repeated roots of the characteristic equation for a single equation) another solution of the form

$$\vec{x}_2 = (\vec{v}_2 + \vec{v}_1 t)\, e^{3t}.$$

We differentiate to get

$$\vec{x}_2' = \vec{v}_1 e^{3t} + 3(\vec{v}_2 + \vec{v}_1 t)\, e^{3t} = (3\vec{v}_2 + \vec{v}_1)\, e^{3t} + 3\vec{v}_1 t e^{3t}.$$

As we are assuming that $\vec{x}_2$ is a solution, $\vec{x}_2'$ must equal $A\vec{x}_2$. So let's compute $A\vec{x}_2$:

$$A\vec{x}_2 = A(\vec{v}_2 + \vec{v}_1 t)\, e^{3t} = A\vec{v}_2 e^{3t} + A\vec{v}_1 t e^{3t}.$$

By looking at the coefficients of e^{3t} and te^{3t} we see $3\vec{v}_2 + \vec{v}_1 = A\vec{v}_2$ and $3\vec{v}_1 = A\vec{v}_1$. This means that

$$(A - 3I)\vec{v}_2 = \vec{v}_1, \qquad \text{and} \qquad (A - 3I)\vec{v}_1 = \vec{0}.$$

Therefore, $\vec{x}_2$ is a solution if these two equations are satisfied. The second equation is satisfied if $\vec{v}_1$ is an eigenvector, and we found the eigenvector above, so let $\vec{v}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. So, if we can find a $\vec{v}_2$ that solves $(A - 3I)\vec{v}_2 = \vec{v}_1$, then we are done. This is just a bunch of linear equations to solve and we are by now very good at that. Let us solve $(A - 3I)\vec{v}_2 = \vec{v}_1$. Write

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

By inspection we see that letting $a = 0$ (a could be anything in fact) and $b = 1$ does the job. Hence we can take $\vec{v}_2 = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$. Our general solution to $\vec{x}' = A\vec{x}$ is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t \right) e^{3t} = \begin{bmatrix} c_1 e^{3t} + c_2 t e^{3t} \\ c_2 e^{3t} \end{bmatrix}.$$

Let us check that we really do have the solution. First $x_1' = c_1 3 e^{3t} + c_2 e^{3t} + 3c_2 t e^{3t} = 3x_1 + x_2$. Good. Now $x_2' = 3c_2 e^{3t} = 3x_2$. Good.

In the example, if we plug $(A - 3I)\vec{v}_2 = \vec{v}_1$ into $(A - 3I)\vec{v}_1 = \vec{0}$ we find

$$(A - 3I)(A - 3I)\vec{v}_2 = \vec{0}, \qquad \text{or} \qquad (A - 3I)^2\vec{v}_2 = \vec{0}.$$

Furthermore, if $(A - 3I)\vec{w} \neq \vec{0}$, then $(A - 3I)\vec{w}$ is an eigenvector, a multiple of $\vec{v}_1$. In this 2×2 case $(A - 3I)^2$ is just the zero matrix (exercise). So any vector $\vec{w}$ solves $(A - 3I)^2\vec{w} = \vec{0}$ and we just need a $\vec{w}$ such that $(A - 3I)\vec{w} \neq \vec{0}$. Then we could use $\vec{w}$ for $\vec{v}_2$, and $(A - 3I)\vec{w}$ for $\vec{v}_1$.

Note that the system $\vec{x}' = A\vec{x}$ has a simpler solution since A is a so-called *upper triangular matrix*, that is every entry below the diagonal is zero. In particular, the equation for x_2 does not depend on x_1. Mind you, not every defective matrix is triangular.

***Exercise* 3.7.1:** *Solve $\vec{x}' = \left[\begin{smallmatrix} 3 & 1 \\ 0 & 3 \end{smallmatrix}\right] \vec{x}$ by first solving for x_2 and then for x_1 independently. Check that you got the same solution as we did above.*

Let us describe the general algorithm. Suppose that λ is an eigenvalue of multiplicity 2, defect 1. First find an eigenvector $\vec{v}_1$ of λ. That is, $\vec{v}_1$ solves $(A - \lambda I)\vec{v}_1 = \vec{0}$. Then, find a vector $\vec{v}_2$ such that

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1.$$

This gives us two linearly independent solutions

$$\begin{aligned} \vec{x}_1 &= \vec{v}_1 e^{\lambda t}, \\ \vec{x}_2 &= \left(\vec{v}_2 + \vec{v}_1 t\right) e^{\lambda t}. \end{aligned}$$

Example 3.7.2: Consider the system

$$\vec{x}' = \begin{bmatrix} 2 & -5 & 0 \\ 0 & 2 & 0 \\ -1 & 4 & 1 \end{bmatrix} \vec{x}.$$

Compute the eigenvalues,

$$0 = \det(A - \lambda I) = \det\left(\begin{bmatrix} 2-\lambda & -5 & 0 \\ 0 & 2-\lambda & 0 \\ -1 & 4 & 1-\lambda \end{bmatrix}\right) = (2-\lambda)^2(1-\lambda).$$

The eigenvalues are 1 and 2, where 2 has multiplicity 2. We leave it to the reader to find that $\left[\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right]$ is an eigenvector for the eigenvalue $\lambda = 1$.

Let's focus on $\lambda = 2$. We compute eigenvectors:

$$\vec{0} = (A - 2I)\vec{v} = \begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ -1 & 4 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

The first equation says that $v_2 = 0$, so the last equation is $-v_1 - v_3 = 0$. Let v_3 be the free variable to find that $v_1 = -v_3$. Perhaps let $v_3 = -1$ to find an eigenvector $\left[\begin{smallmatrix} 1 \\ 0 \\ -1 \end{smallmatrix}\right]$. Problem is that setting v_3 to anything else just gets multiples of this vector and so we have a defect of 1. Let $\vec{v}_1$ be the eigenvector and let's look for a generalized eigenvector $\vec{v}_2$:

$$(A - 2I)\vec{v}_2 = \vec{v}_1,$$

or

$$\begin{bmatrix} 0 & -5 & 0 \\ 0 & 0 & 0 \\ -1 & 4 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix},$$

where we used a, b, c as components of $\vec{v}_2$ for simplicity. The first equation says $-5b = 1$ so $b = {-1}/{5}$. The second equation says nothing. The last equation is $-a + 4b - c = -1$, or

$a + 4/5 + c = 1$, or $a + c = 1/5$. We let c be the free variable and we choose $c = 0$. We find $\vec{v}_2 = \begin{bmatrix} 1/5 \\ -1/5 \\ 0 \end{bmatrix}$.

The general solution is therefore,

$$\vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 1/5 \\ -1/5 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} t \right) e^{2t}.$$

This machinery can also be generalized to higher multiplicities and higher defects. We will not go over this method in detail, but let us just sketch the ideas. Suppose that A has an eigenvalue λ of multiplicity m. We find vectors such that

$$(A - \lambda I)^k \vec{v} = \vec{0}, \qquad \text{but} \qquad (A - \lambda I)^{k-1} \vec{v} \neq \vec{0}.$$

Such vectors are called *generalized eigenvectors* (then $\vec{v}_1 = (A - \lambda I)^{k-1}\vec{v}$ is an eigenvector). For the eigenvector $\vec{v}_1$ there is a chain of generalized eigenvectors $\vec{v}_2$ through $\vec{v}_k$ such that:

$$\begin{aligned} (A - \lambda I)\vec{v}_1 &= \vec{0}, \\ (A - \lambda I)\vec{v}_2 &= \vec{v}_1, \\ &\vdots \\ (A - \lambda I)\vec{v}_k &= \vec{v}_{k-1}. \end{aligned}$$

Really once you find the $\vec{v}_k$ such that $(A - \lambda I)^k \vec{v}_k = \vec{0}$ but $(A - \lambda I)^{k-1}\vec{v}_k \neq \vec{0}$, you find the entire chain since you can compute the rest, $\vec{v}_{k-1} = (A - \lambda I)\vec{v}_k$, $\vec{v}_{k-2} = (A - \lambda I)\vec{v}_{k-1}$, etc. We form the linearly independent solutions

$$\begin{aligned} \vec{x}_1 &= \vec{v}_1 e^{\lambda t}, \\ \vec{x}_2 &= (\vec{v}_2 + \vec{v}_1 t)\, e^{\lambda t}, \\ &\vdots \\ \vec{x}_k &= \left(\vec{v}_k + \vec{v}_{k-1} t + \vec{v}_{k-2} \frac{t^2}{2} + \cdots + \vec{v}_2 \frac{t^{k-2}}{(k-2)!} + \vec{v}_1 \frac{t^{k-1}}{(k-1)!} \right) e^{\lambda t}. \end{aligned}$$

Recall that $k! = 1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k$ is the factorial. If you have an eigenvalue of geometric multiplicity ℓ, you will have to find ℓ such chains (some of them might be short: just the single eigenvector equation). We go until we form m linearly independent solutions where m is the algebraic multiplicity. We don't quite know which specific eigenvectors go with which chain, so start by finding $\vec{v}_k$ first for the longest possible chain and go from there.

For example, if λ is an eigenvalue of A of algebraic multiplicity 3 and defect 2, then solve

$$(A - \lambda I)\vec{v}_1 = \vec{0}, \qquad (A - \lambda I)\vec{v}_2 = \vec{v}_1, \qquad (A - \lambda I)\vec{v}_3 = \vec{v}_2.$$

That is, find $\vec{v}_3$ such that $(A-\lambda I)^3\vec{v}_3 = \vec{0}$, but $(A-\lambda I)^2\vec{v}_3 \neq \vec{0}$. Then you are done as $\vec{v}_2 = (A-\lambda I)\vec{v}_3$ and $\vec{v}_1 = (A-\lambda I)\vec{v}_2$. The 3 linearly independent solutions are

$$\vec{x}_1 = \vec{v}_1 e^{\lambda t}, \qquad \vec{x}_2 = (\vec{v}_2 + \vec{v}_1 t)\, e^{\lambda t}, \qquad \vec{x}_3 = \left(\vec{v}_3 + \vec{v}_2 t + \vec{v}_1 \frac{t^2}{2}\right) e^{\lambda t}.$$

If on the other hand A has an eigenvalue λ of algebraic multiplicity 3 and defect 1, then solve

$$(A-\lambda I)\vec{v}_1 = \vec{0}, \qquad (A-\lambda I)\vec{v}_2 = \vec{0}, \qquad (A-\lambda I)\vec{v}_3 = \vec{v}_2.$$

Here $\vec{v}_1$ and $\vec{v}_2$ are actual honest eigenvectors, and $\vec{v}_3$ is a generalized eigenvector. So there are two chains. To solve, first find a $\vec{v}_3$ such that $(A-\lambda I)^2\vec{v}_3 = \vec{0}$, but $(A-\lambda I)\vec{v}_3 \neq \vec{0}$. Then $\vec{v}_2 = (A-\lambda I)\vec{v}_3$ is going to be an eigenvector. Then solve for an eigenvector $\vec{v}_1$ that is linearly independent from $\vec{v}_2$. You get 3 linearly independent solutions

$$\vec{x}_1 = \vec{v}_1 e^{\lambda t}, \qquad \vec{x}_2 = \vec{v}_2 e^{\lambda t}, \qquad \vec{x}_3 = (\vec{v}_3 + \vec{v}_2 t)\, e^{\lambda t}.$$

3.7.3 Exercises

***Exercise* 3.7.2:** *Let* $A = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}$. *Find the general solution of* $\vec{x}\,' = A\vec{x}$.

***Exercise* 3.7.3:** *Let* $A = \begin{bmatrix} 5 & -4 & 4 \\ 0 & 3 & 0 \\ -2 & 4 & -1 \end{bmatrix}$.

a) What are the eigenvalues?

b) What is/are the defect(s) of the eigenvalue(s)?

c) Find the general solution of $\vec{x}\,' = A\vec{x}$.

***Exercise* 3.7.4:** *Let* $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

a) What are the eigenvalues?

b) What is/are the defect(s) of the eigenvalue(s)?

c) Find the general solution of $\vec{x}\,' = A\vec{x}$ *in two different ways and verify you get the same answer.*

***Exercise* 3.7.5:** *Let* $A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -2 & -2 \\ -4 & 4 & 7 \end{bmatrix}$.

a) What are the eigenvalues?

b) What is/are the defect(s) of the eigenvalue(s)?

c) Find the general solution of $\vec{x}\,' = A\vec{x}$.

Exercise 3.7.6: *Let* $A = \begin{bmatrix} 0 & 4 & -2 \\ -1 & -4 & 1 \\ 0 & 0 & -2 \end{bmatrix}$.

a) *What are the eigenvalues?*

b) *What is/are the defect(s) of the eigenvalue(s)?*

c) *Find the general solution of* $\vec{x}' = A\vec{x}$.

Exercise 3.7.7: *Let* $A = \begin{bmatrix} 2 & 1 & -1 \\ -1 & 0 & 2 \\ -1 & -2 & 4 \end{bmatrix}$.

a) *What are the eigenvalues?*

b) *What is/are the defect(s) of the eigenvalue(s)?*

c) *Find the general solution of* $\vec{x}' = A\vec{x}$.

Exercise 3.7.8: *Suppose that* A *is a* 2×2 *matrix with a repeated eigenvalue* λ. *Suppose that there are two linearly independent eigenvectors. Show that* $A = \lambda I$.

Exercise 3.7.101: *Let* $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

a) *What are the eigenvalues?*

b) *What is/are the defect(s) of the eigenvalue(s)?*

c) *Find the general solution of* $\vec{x}\,' = A\vec{x}$.

Exercise 3.7.102: *Let* $A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 1 & 0 \\ -1 & 1 & 2 \end{bmatrix}$.

a) *What are the eigenvalues?*

b) *What is/are the defect(s) of the eigenvalue(s)?*

c) *Find the general solution of* $\vec{x}\,' = A\vec{x}$.

Exercise 3.7.103: *Let* $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & -1 & 9 \\ 0 & -1 & 5 \end{bmatrix}$.

a) *What are the eigenvalues?*

b) *What is/are the defect(s) of the eigenvalue(s)?*

c) *Find the general solution of* $\vec{x}\,' = A\vec{x}$.

Exercise 3.7.104: *Let* $A = \begin{bmatrix} a & a \\ b & c \end{bmatrix}$, *where* a, b, *and* c *are unknowns. Suppose that* 5 *is a doubled eigenvalue of defect* 1, *and suppose that* $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *is a corresponding eigenvector. Find* A *and show that there is only one such matrix* A.

Exercise* 3.7.151*: *Find the general solution in vector form to each of the following systems. If ICs are given, find the particular solution. Classify the critical point for each system, then sketch the qualitatively- different solution curves for the general solution.*

a) $x_1' = 9x_1 + 4x_2,\ x_2' = -4x_1 + x_2$

b) $x_1' = -5x_1 + x_2,\ x_2' = -x_1 - 3x_2$

c) $x_1' = -x_1 - 2x_2,\ x_2' = 2x_1 - 5x_2;\ x_1(0) = -1,\ x_2(0) = 3$

d) $x_1' = 6x_1 + x_2,\ x_2' = -x_1 + 8x_2;\ x_1(0) = -2,\ x_2(0) = 5$

e) $x_1' = 7x_1 + x_2,\ x_2' = -4x_1 + 3x_2$

f) $x_1' = -8x_1 - x_2,\ x_2' = 4x_1 - 4x_2$

3.8 Matrix exponentials

Note: 2 lectures, §5.6 in [EP], §7.7 in [BD]

3.8.1 Definition

There is another way of finding a fundamental matrix solution of a system. Consider the constant coefficient equation

$$\vec{x}\,' = P\vec{x}.$$

If this would be just one equation (when P is a number or a 1×1 matrix), then the solution would be

$$\vec{x} = e^{Pt}.$$

That doesn't make sense if P is a larger matrix, but essentially the same computation that led to the above works for matrices when we define e^{Pt} properly. First let us write down the Taylor series for e^{at} for some number a:

$$e^{at} = 1 + at + \frac{(at)^2}{2} + \frac{(at)^3}{6} + \frac{(at)^4}{24} + \cdots = \sum_{k=0}^{\infty} \frac{(at)^k}{k!}.$$

Recall $k! = 1 \cdot 2 \cdot 3 \cdots k$ is the factorial, and $0! = 1$. We differentiate this series term by term

$$\frac{d}{dt}\left(e^{at}\right) = 0 + a + a^2 t + \frac{a^3 t^2}{2} + \frac{a^4 t^3}{6} + \cdots = a\left(1 + at + \frac{(at)^2}{2} + \frac{(at)^3}{6} + \cdots\right) = ae^{at}.$$

Maybe we can try the same trick with matrices. For an $n \times n$ matrix A we define the *matrix exponential* as

$$\boxed{e^{A} \overset{\text{def}}{=} I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots + \frac{1}{k!}A^k + \cdots}$$

Let us not worry about convergence. The series really does always converge. We usually write Pt as tP by convention when P is a matrix. With this small change and by the exact same calculation as above we have that

$$\frac{d}{dt}\left(e^{tP}\right) = Pe^{tP}.$$

Now P and hence e^{tP} is an $n \times n$ matrix. What we are looking for is a vector. In the 1×1 case we would at this point multiply by an arbitrary constant to get the general solution. In the matrix case we multiply by a column vector $\vec{c}$.

Theorem 3.8.1. *Let P be an $n \times n$ matrix. Then the general solution to $\vec{x}\,' = P\vec{x}$ is*

$$\vec{x} = e^{tP}\vec{c},$$

where $\vec{c}$ is an arbitrary constant vector. In fact, $\vec{x}(0) = \vec{c}$.

Let us check:

$$\frac{d}{dt}\vec{x} = \frac{d}{dt}\left(e^{tP}\vec{c}\right) = Pe^{tP}\vec{c} = P\vec{x}.$$

Hence e^{tP} is a fundamental matrix solution of the homogeneous system. So if we can compute the matrix exponential, we have another method of solving constant coefficient homogeneous systems. It also makes it easy to solve for initial conditions. To solve $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \vec{b}$, we take the solution

$$\vec{x} = e^{tA}\vec{b}.$$

This equation follows because $e^{0A} = I$, so $\vec{x}(0) = e^{0A}\vec{b} = \vec{b}$.

We mention a drawback of matrix exponentials. In general $e^{A+B} \neq e^A e^B$. The trouble is that matrices do not commute, that is, in general $AB \neq BA$. If you try to prove $e^{A+B} \neq e^A e^B$ using the Taylor series, you will see why the lack of commutativity becomes a problem. However, it is still true that if $AB = BA$, that is, if A and B commute, then $e^{A+B} = e^A e^B$. We will find this fact useful. Let us restate this as a theorem to make a point.

Theorem 3.8.2. *If $AB = BA$, then $e^{A+B} = e^A e^B$. Otherwise, $e^{A+B} \neq e^A e^B$ in general.*

3.8.2 Simple cases

In some instances it may work to just plug into the series definition. Suppose the matrix is diagonal. For example, $D = \left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right]$. Then

$$D^k = \begin{bmatrix} a^k & 0 \\ 0 & b^k \end{bmatrix},$$

and

$$\begin{aligned} e^D &= I + D + \frac{1}{2}D^2 + \frac{1}{6}D^3 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \frac{1}{2}\begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} a^3 & 0 \\ 0 & b^3 \end{bmatrix} + \cdots = \begin{bmatrix} e^a & 0 \\ 0 & e^b \end{bmatrix}. \end{aligned}$$

So by this rationale

$$e^I = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix} \qquad \text{and} \qquad e^{aI} = \begin{bmatrix} e^a & 0 \\ 0 & e^a \end{bmatrix}.$$

This makes exponentials of certain other matrices easy to compute. For example, the matrix $A = \left[\begin{smallmatrix} 5 & 4 \\ -1 & 1 \end{smallmatrix}\right]$ can be written as $3I + B$ where $B = \left[\begin{smallmatrix} 2 & 4 \\ -1 & -2 \end{smallmatrix}\right]$. Notice that $B^2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. So $B^k = 0$ for all $k \geq 2$. Therefore, $e^B = I + B$. Suppose we actually want to compute e^{tA}. The matrices $3tI$ and tB commute (exercise: check this) and $e^{tB} = I + tB$, since $(tB)^2 = t^2B^2 = 0$. We write

$$\begin{aligned} e^{tA} = e^{3tI+tB} = e^{3tI}e^{tB} &= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix}(I + tB) = \\ &= \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} 1+2t & 4t \\ -t & 1-2t \end{bmatrix} = \begin{bmatrix} (1+2t)\,e^{3t} & 4te^{3t} \\ -te^{3t} & (1-2t)\,e^{3t} \end{bmatrix}. \end{aligned}$$

We found a fundamental matrix solution for the system $\vec{x}' = A\vec{x}$. Note that this matrix has a repeated eigenvalue with a defect; there is only one eigenvector for the eigenvalue 3. So we found a perhaps easier way to handle this case. In fact, if a matrix A is 2×2 and has an eigenvalue λ of multiplicity 2, then either $A = \lambda I$, or $A = \lambda I + B$ where $B^2 = 0$. This is a good exercise.

***Exercise* 3.8.1:** *Suppose that A is 2×2 and λ is the only eigenvalue. Show that $(A - \lambda I)^2 = 0$, and therefore that we can write $A = \lambda I + B$, where $B^2 = 0$ (and possibly $B = 0$). Hint: First write down what does it mean for the eigenvalue to be of multiplicity 2. You will get an equation for the entries. Now compute the square of B.*

Matrices B such that $B^k = 0$ for some k are called *nilpotent*. Computation of the matrix exponential for nilpotent matrices is easy by just writing down the first k terms of the Taylor series.

3.8.3 General matrices

In general, the exponential is not as easy to compute as above. We usually cannot write a matrix as a sum of commuting matrices where the exponential is simple for each one. But fear not, it is still not too difficult provided we can find enough eigenvectors. First we need the following interesting result about matrix exponentials. For two square matrices A and B, with B invertible, we have

$$e^{BAB^{-1}} = Be^{A}B^{-1}.$$

This can be seen by writing down the Taylor series. First

$$(BAB^{-1})^2 = BAB^{-1}BAB^{-1} = BAIAB^{-1} = BA^2B^{-1}.$$

And by the same reasoning $(BAB^{-1})^k = BA^kB^{-1}$. Now write the Taylor series for $e^{BAB^{-1}}$:

$$\begin{aligned}
e^{BAB^{-1}} &= I + BAB^{-1} + \frac{1}{2}(BAB^{-1})^2 + \frac{1}{6}(BAB^{-1})^3 + \cdots \\
&= BB^{-1} + BAB^{-1} + \frac{1}{2}BA^2B^{-1} + \frac{1}{6}BA^3B^{-1} + \cdots \\
&= B\big(I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots\big)B^{-1} \\
&= Be^{A}B^{-1}.
\end{aligned}$$

Given a square matrix A, we can usually write $A = EDE^{-1}$, where D is diagonal and E invertible. This procedure is called *diagonalization*. If we can do that, the computation of the exponential becomes easy as e^{D} is just taking the exponential of the entries on the diagonal. Adding t into the mix, we can then compute the exponential

$$e^{tA} = Ee^{tD}E^{-1}.$$

To diagonalize A we need n linearly independent eigenvectors of A. Otherwise, this method of computing the exponential does not work and we need to be trickier, but we will not get into such details. Let E be the matrix with the eigenvectors as columns. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues and let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be the eigenvectors, then $E = [\, \vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_n \,]$. Make a diagonal matrix D with the eigenvalues on the diagonal:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

We compute

$$\begin{aligned} AE &= A[\, \vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_n \,] \\ &= [\, A\vec{v}_1 \quad A\vec{v}_2 \quad \cdots \quad A\vec{v}_n \,] \\ &= [\, \lambda_1\vec{v}_1 \quad \lambda_2\vec{v}_2 \quad \cdots \quad \lambda_n\vec{v}_n \,] \\ &= [\, \vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_n \,]D \\ &= ED. \end{aligned}$$

The columns of E are linearly independent as these are linearly independent eigenvectors of A. Hence E is invertible. Since $AE = ED$, we multiply on the right by E^{-1} and we get

$$A = EDE^{-1}.$$

This means that $e^A = Ee^DE^{-1}$. Multiplying the matrix by t we obtain

$$\boxed{e^{tA} = Ee^{tD}E^{-1} = E \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} E^{-1}.} \tag{3.5}$$

The formula (3.5), therefore, gives the formula for computing a fundamental matrix solution e^{tA} for the system $\vec{x}' = A\vec{x}$, in the case where we have n linearly independent eigenvectors.

This computation still works when the eigenvalues and eigenvectors are complex, though then you have to compute with complex numbers. It is clear from the definition that if A is real, then e^{tA} is real. So you will only need complex numbers in the computation and not for the result. You may need to apply Euler's formula to simplify the result. If simplified properly, the final matrix will not have any complex numbers in it.

Example 3.8.1: Compute a fundamental matrix solution using the matrix exponential for the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Then compute the particular solution for the initial conditions $x(0) = 4$ and $y(0) = 2$.

Let A be the coefficient matrix $\left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$. We first compute (exercise) that the eigenvalues are 3 and -1 and corresponding eigenvectors are $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$. Hence the diagonalization of A is

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}}_{E^{-1}}.$$

We write

$$\begin{aligned}
e^{tA} = Ee^{tD}E^{-1} &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \frac{-1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \frac{-1}{2} \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \\
&= \frac{-1}{2} \begin{bmatrix} -e^{3t} - e^{-t} & -e^{3t} + e^{-t} \\ -e^{3t} + e^{-t} & -e^{3t} - e^{-t} \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{2} \\ \frac{e^{3t}-e^{-t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix}.
\end{aligned}$$

The initial conditions are $x(0) = 4$ and $y(0) = 2$. Hence, by the property that $e^{0A} = I$ we find that the particular solution we are looking for is $e^{tA}\vec{b}$ where $\vec{b}$ is $\left[\begin{smallmatrix} 4 \\ 2 \end{smallmatrix}\right]$. Then the particular solution we are looking for is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{3t}-e^{-t}}{2} \\ \frac{e^{3t}-e^{-t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 2e^{3t} + 2e^{-t} + e^{3t} - e^{-t} \\ 2e^{3t} - 2e^{-t} + e^{3t} + e^{-t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} + e^{-t} \\ 3e^{3t} - e^{-t} \end{bmatrix}.$$

3.8.4 Fundamental matrix solutions

We note that if you can compute a fundamental matrix solution in a different way, you can use this to find the matrix exponential e^{tA}. A fundamental matrix solution of a system of ODEs is not unique. The exponential is the fundamental matrix solution with the property that for $t = 0$ we get the identity matrix. So we must find the right fundamental matrix solution. Let X be any fundamental matrix solution to $\vec{x}' = A\vec{x}$. Then we claim

$$e^{tA} = X(t)\,[X(0)]^{-1}.$$

Clearly, if we plug $t = 0$ into $X(t)\,[X(0)]^{-1}$ we get the identity. We can multiply a fundamental matrix solution on the right by any constant invertible matrix and we still get a fundamental matrix solution. All we are doing is changing what are the arbitrary constants in the general solution $\vec{x}(t) = X(t)\,\vec{c}$.

3.8.5 Approximations

If you think about it, the computation of any fundamental matrix solution X using the eigenvalue method is just as difficult as the computation of e^{tA}. So perhaps we did not gain much by this new tool. However, the Taylor series expansion actually gives us a way to approximate solutions, which the eigenvalue method did not.

The simplest thing we can do is to just compute the series up to a certain number of terms. There are better ways to approximate the exponential*. In many cases however, few terms of the Taylor series give a reasonable approximation for the exponential and may suffice for the application. For example, let us compute the first 4 terms of the series for the matrix $A = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$.

$$e^{tA} \approx I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{6}A^3 = I + t\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + t^2\begin{bmatrix} \frac{5}{2} & 2 \\ 2 & \frac{5}{2} \end{bmatrix} + t^3\begin{bmatrix} \frac{13}{6} & \frac{7}{3} \\ \frac{7}{3} & \frac{13}{6} \end{bmatrix} =$$
$$= \begin{bmatrix} 1 + t + \frac{5}{2}\,t^2 + \frac{13}{6}\,t^3 & 2\,t + 2\,t^2 + \frac{7}{3}\,t^3 \\ 2\,t + 2\,t^2 + \frac{7}{3}\,t^3 & 1 + t + \frac{5}{2}\,t^2 + \frac{13}{6}\,t^3 \end{bmatrix}.$$

Just like the scalar version of the Taylor series approximation, the approximation will be better for small t and worse for larger t. For larger t, we will generally have to compute more terms. Let us see how we stack up against the real solution with $t = 0.1$. The approximate solution is approximately (rounded to 8 decimal places)

$$e^{0.1\,A} \approx I + 0.1\,A + \frac{0.1^2}{2}A^2 + \frac{0.1^3}{6}A^3 = \begin{bmatrix} 1.12716667 & 0.22233333 \\ 0.22233333 & 1.12716667 \end{bmatrix}.$$

And plugging $t = 0.1$ into the real solution (rounded to 8 decimal places) we get

$$e^{0.1\,A} = \begin{bmatrix} 1.12734811 & 0.22251069 \\ 0.22251069 & 1.12734811 \end{bmatrix}.$$

Not bad at all! Although if we take the same approximation for $t = 1$ we get

$$I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 = \begin{bmatrix} 6.66666667 & 6.33333333 \\ 6.33333333 & 6.66666667 \end{bmatrix},$$

while the real value is (again rounded to 8 decimal places)

$$e^{A} = \begin{bmatrix} 10.22670818 & 9.85882874 \\ 9.85882874 & 10.22670818 \end{bmatrix}.$$

So the approximation is not very good once we get up to $t = 1$. To get a good approximation at $t = 1$ (say up to 2 decimal places) we would need to go up to the 11^{th} power (exercise).

*C. Moler and C.F. Van Loan, *Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later*, SIAM Review 45 (1), 2003, 3–49

3.8.6 Exercises

***Exercise* 3.8.2:** *Using the matrix exponential, find a fundamental matrix solution for the system* $x' = 3x + y$, $y' = x + 3y$.

***Exercise* 3.8.3:** *Find* e^{tA} *for the matrix* $A = \left[\begin{smallmatrix} 2 & 3 \\ 0 & 2 \end{smallmatrix}\right]$.

***Exercise* 3.8.4:** *Find a fundamental matrix solution for the system* $x_1' = 7x_1 + 4x_2 + 12x_3$, $x_2' = x_1 + 2x_2 + x_3$, $x_3' = -3x_1 - 2x_2 - 5x_3$. *Then find the solution that satisfies* $\vec{x}(0) = \left[\begin{smallmatrix} 0 \\ 1 \\ -2 \end{smallmatrix}\right]$.

***Exercise* 3.8.5:** *Compute the matrix exponential* e^A *for* $A = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$.

***Exercise* 3.8.6** (challenging)**:** *Suppose* $AB = BA$. *Show that under this assumption,* $e^{A+B} = e^A e^B$.

***Exercise* 3.8.7:** *Use Exercise 3.8.6 to show that* $(e^A)^{-1} = e^{-A}$. *In particular this means that* e^A *is invertible even if* A *is not.*

***Exercise* 3.8.8:** *Let* A *be a* 2×2 *matrix with eigenvalues* -1, 1, *and corresponding eigenvectors* $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$.

a) *Find matrix* A *with these properties.*

b) *Find a fundamental matrix solution to* $\vec{x}' = A\vec{x}$.

c) *Solve the system in with initial conditions* $\vec{x}(0) = \left[\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right]$.

***Exercise* 3.8.9:** *Suppose that* A *is an* $n \times n$ *matrix with a repeated eigenvalue* λ *of multiplicity* n. *Suppose that there are* n *linearly independent eigenvectors. Show that the matrix is diagonal, in particular* $A = \lambda I$. *Hint: Use diagonalization and the fact that the identity matrix commutes with every other matrix.*

***Exercise* 3.8.10:** *Let* $A = \left[\begin{smallmatrix} -1 & -1 \\ 1 & -3 \end{smallmatrix}\right]$.

a) *Find* e^{tA}.

b) *Solve* $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \left[\begin{smallmatrix} 1 \\ -2 \end{smallmatrix}\right]$.

***Exercise* 3.8.11:** *Let* $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$. *Approximate* e^{tA} *by expanding the power series up to the third order.*

***Exercise* 3.8.12:** *For any positive integer* n, *find a formula (or a recipe) for* A^n *for the following matrices:*

a) $\begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix}$

b) $\begin{bmatrix} 5 & 2 \\ 4 & 7 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

d) $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

***Exercise* 3.8.101:** *Compute* e^{tA} *where* $A = \left[\begin{smallmatrix} 1 & -2 \\ -2 & 1 \end{smallmatrix}\right]$.

***Exercise* 3.8.102:** *Compute* e^{tA} *where* $A = \left[\begin{smallmatrix} 1 & -3 & 2 \\ -2 & 1 & 2 \\ -1 & -3 & 4 \end{smallmatrix}\right]$.

Exercise* 3.8.103*:

a) *Compute* e^{tA} *where* $A = \left[\begin{smallmatrix} 3 & -1 \\ 1 & 1 \end{smallmatrix}\right]$.

b) *Solve* $\vec{x}\,' = A\vec{x}$ *for* $\vec{x}(0) = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$.

Exercise* 3.8.104*: *Compute the first* 3 *terms* (*up to the second degree*) *of the Taylor expansion of* e^{tA} *where* $A = \left[\begin{smallmatrix} 2 & 3 \\ 2 & 2 \end{smallmatrix}\right]$ (*Write as a single matrix*). *Then use it to approximate* $e^{0.1A}$.

Exercise* 3.8.105*: *For any positive integer* n, *find a formula* (*or a recipe*) *for* A^n *for the following matrices*:

a) $\begin{bmatrix} 7 & 4 \\ -5 & -2 \end{bmatrix}$

b) $\begin{bmatrix} -3 & 4 \\ -6 & -7 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3.9 Nonhomogeneous systems

Note: 3 lectures (may have to skip a little), somewhat different from §5.7 in [EP], §7.9 in [BD]

3.9.1 First order constant coefficient

Integrating factor

Let us first focus on the nonhomogeneous first order equation

$$\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t),$$

where A is a constant matrix. The first method we look at is the *integrating factor method*. For simplicity we rewrite the equation as

$$\vec{x}'(t) + P\vec{x}(t) = \vec{f}(t),$$

where $P = -A$. We multiply both sides of the equation by e^{tP} (being mindful that we are dealing with matrices that may not commute) to obtain

$$e^{tP}\vec{x}'(t) + e^{tP}P\vec{x}(t) = e^{tP}\vec{f}(t).$$

We notice that $Pe^{tP} = e^{tP}P$. This fact follows by writing down the series definition of e^{tP}:

$$\begin{aligned} Pe^{tP} &= P\left(I + tP + \frac{1}{2}(tP)^2 + \cdots\right) = P + tP^2 + \frac{1}{2}t^2P^3 + \cdots = \\ &= \left(I + tP + \frac{1}{2}(tP)^2 + \cdots\right)P = e^{tP}P. \end{aligned}$$

So $\frac{d}{dt}\left(e^{tP}\right) = Pe^{tP} = e^{tP}P$. The product rule says

$$\frac{d}{dt}\left(e^{tP}\vec{x}(t)\right) = e^{tP}\vec{x}'(t) + e^{tP}P\vec{x}(t),$$

and so

$$\frac{d}{dt}\left(e^{tP}\vec{x}(t)\right) = e^{tP}\vec{f}(t).$$

We can now integrate. That is, we integrate each component of the vector separately

$$e^{tP}\vec{x}(t) = \int e^{tP}\vec{f}(t)\,dt + \vec{c}.$$

Recall from Exercise 3.8.7 that $(e^{tP})^{-1} = e^{-tP}$. Therefore, we obtain

$$\vec{x}(t) = e^{-tP}\int e^{tP}\vec{f}(t)\,dt + e^{-tP}\vec{c}.$$

Perhaps it is better understood as a definite integral. In this case it will be easy to also solve for the initial conditions. Consider the equation with initial conditions

$$\vec{x}'(t) + P\vec{x}(t) = \vec{f}(t), \qquad \vec{x}(0) = \vec{b}.$$

The solution can then be written as

$$\boxed{\vec{x}(t) = e^{-tP} \int_0^t e^{sP} \vec{f}(s)\, ds + e^{-tP}\vec{b}.} \tag{3.6}$$

Again, the integration means that each component of the vector $e^{sP}\vec{f}(s)$ is integrated separately. It is not hard to see that (3.6) really does satisfy the initial condition $\vec{x}(0) = \vec{b}$.

$$\vec{x}(0) = e^{-0P} \int_0^0 e^{sP} \vec{f}(s)\, ds + e^{-0P}\vec{b} = I\vec{b} = \vec{b}.$$

Example 3.9.1: Suppose that we have the system

$$\begin{aligned} x_1' + 5x_1 - 3x_2 &= e^t, \\ x_2' + 3x_1 - x_2 &= 0, \end{aligned}$$

with initial conditions $x_1(0) = 1, x_2(0) = 0$.

Let us write the system as

$$\vec{x}' + \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} e^t \\ 0 \end{bmatrix}, \qquad \vec{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

The matrix $P = \left[\begin{smallmatrix} 5 & -3 \\ 3 & -1 \end{smallmatrix}\right]$ has a doubled eigenvalue 2 with defect 1, and we leave it as an exercise to double check we computed e^{tP} correctly. Once we have e^{tP}, we find e^{-tP}, simply by negating t.

$$e^{tP} = \begin{bmatrix} (1+3t)\,e^{2t} & -3te^{2t} \\ 3te^{2t} & (1-3t)\,e^{2t} \end{bmatrix}, \qquad e^{-tP} = \begin{bmatrix} (1-3t)\,e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)\,e^{-2t} \end{bmatrix}.$$

Instead of computing the whole formula at once, let us do it in stages. First

$$\begin{aligned} \int_0^t e^{sP}\vec{f}(s)\, ds &= \int_0^t \begin{bmatrix} (1+3s)\,e^{2s} & -3se^{2s} \\ 3se^{2s} & (1-3s)\,e^{2s} \end{bmatrix} \begin{bmatrix} e^s \\ 0 \end{bmatrix} ds \\ &= \int_0^t \begin{bmatrix} (1+3s)\,e^{3s} \\ 3se^{3s} \end{bmatrix} ds \\ &= \begin{bmatrix} \int_0^t (1+3s)\,e^{3s}\, ds \\ \int_0^t 3se^{3s}\, ds \end{bmatrix} \\ &= \begin{bmatrix} te^{3t} \\ \frac{(3t-1)\,e^{3t}+1}{3} \end{bmatrix} \qquad \text{(used integration by parts).} \end{aligned}$$

Then

$$
\begin{aligned}
\vec{x}(t) &= e^{-tP} \int_0^t e^{sP} \vec{f}(s)\, ds + e^{-tP} \vec{b} \\
&= \begin{bmatrix} (1-3t)\, e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)\, e^{-2t} \end{bmatrix} \begin{bmatrix} te^{3t} \\ \frac{(3t-1)\, e^{3t}+1}{3} \end{bmatrix} + \begin{bmatrix} (1-3t)\, e^{-2t} & 3te^{-2t} \\ -3te^{-2t} & (1+3t)\, e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} te^{-2t} \\ -\frac{e^t}{3} + \left(\frac{1}{3} + t\right) e^{-2t} \end{bmatrix} + \begin{bmatrix} (1-3t)\, e^{-2t} \\ -3te^{-2t} \end{bmatrix} \\
&= \begin{bmatrix} (1-2t)\, e^{-2t} \\ -\frac{e^t}{3} + \left(\frac{1}{3} - 2t\right) e^{-2t} \end{bmatrix}.
\end{aligned}
$$

Phew!

Let us check that this really works.

$$x_1' + 5x_1 - 3x_2 = (4te^{-2t} - 4e^{-2t}) + 5(1-2t)\, e^{-2t} + e^t - (1-6t)\, e^{-2t} = e^t.$$

Similarly (exercise) $x_2' + 3x_1 - x_2 = 0$. The initial conditions are also satisfied (exercise).

For systems, the integrating factor method only works if P does not depend on t, that is, P is constant. The problem is that in general

$$\frac{d}{dt}\left[e^{\int P(t)\, dt}\right] \neq P(t)\, e^{\int P(t)\, dt},$$

because matrix multiplication is not commutative.

Eigenvector decomposition

For the next method, note that eigenvectors of a matrix give the directions in which the matrix acts like a scalar. If we solve the system along these directions, the computations are simpler as we treat the matrix as a scalar. We then put those solutions together to get the general solution for the system.

Take the equation

$$\vec{x}'(t) = A\vec{x}(t) + \vec{f}(t). \tag{3.7}$$

Assume A has n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$. Write

$$\vec{x}(t) = \vec{v}_1\, \xi_1(t) + \vec{v}_2\, \xi_2(t) + \cdots + \vec{v}_n\, \xi_n(t). \tag{3.8}$$

That is, we wish to write our solution as a linear combination of eigenvectors of A. If we solve for the scalar functions ξ_1 through ξ_n, we have our solution $\vec{x}$. Let us decompose $\vec{f}$ in terms of the eigenvectors as well. We wish to write

$$\vec{f}(t) = \vec{v}_1\, g_1(t) + \vec{v}_2\, g_2(t) + \cdots + \vec{v}_n\, g_n(t). \tag{3.9}$$

That is, we wish to find g_1 through g_n that satisfy (3.9). Since all the eigenvectors are independent, the matrix $E = [\, \vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_n \,]$ is invertible. Write the equation (3.9) as

$\vec{f} = E\vec{g}$, where the components of $\vec{g}$ are the functions g_1 through g_n. Then $\vec{g} = E^{-1}\vec{f}$. Hence it is always possible to find $\vec{g}$ when there are n linearly independent eigenvectors.

We plug (3.8) into (3.7), and note that $A\vec{v}_k = \lambda_k \vec{v}_k$:

$$\begin{aligned}
\overbrace{\vec{v}_1\xi_1' + \vec{v}_2\xi_2' + \cdots + \vec{v}_n\xi_n'}^{\vec{x}'} &= \overbrace{A\left(\vec{v}_1\xi_1 + \vec{v}_2\xi_2 + \cdots + \vec{v}_n\xi_n\right)}^{A\vec{x}} + \overbrace{\vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n}^{\vec{f}} \\
&= A\vec{v}_1\xi_1 + A\vec{v}_2\xi_2 + \cdots + A\vec{v}_n\xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n \\
&= \vec{v}_1\lambda_1\xi_1 + \vec{v}_2\lambda_2\xi_2 + \cdots + \vec{v}_n\lambda_n\xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n \\
&= \vec{v}_1(\lambda_1\xi_1 + g_1) + \vec{v}_2(\lambda_2\xi_2 + g_2) + \cdots + \vec{v}_n(\lambda_n\xi_n + g_n).
\end{aligned}$$

If we identify the coefficients of the vectors $\vec{v}_1$ through $\vec{v}_n$, we get the equations

$$\begin{aligned}
\xi_1' &= \lambda_1\xi_1 + g_1, \\
\xi_2' &= \lambda_2\xi_2 + g_2, \\
&\vdots \\
\xi_n' &= \lambda_n\xi_n + g_n.
\end{aligned}$$

Each one of these equations is independent of the others. They are all linear first order equations and can easily be solved by the standard integrating factor method for single equations. That is, for the k^{th} equation we write

$$\xi_k'(t) - \lambda_k\xi_k(t) = g_k(t).$$

We use the integrating factor $e^{-\lambda_k t}$ to find that

$$\frac{d}{dt}\left[\xi_k(t)\, e^{-\lambda_k t}\right] = e^{-\lambda_k t} g_k(t).$$

We integrate and solve for ξ_k to get

$$\xi_k(t) = e^{\lambda_k t}\int e^{-\lambda_k t} g_k(t)\, dt + C_k e^{\lambda_k t}.$$

If we are looking for just any particular solution, we can set C_k to be zero. If we leave these constants in, we get the general solution. Write $\vec{x}(t) = \vec{v}_1\xi_1(t) + \vec{v}_2\xi_2(t) + \cdots + \vec{v}_n\xi_n(t)$, and we are done.

As always, it is perhaps better to write these integrals as definite integrals. Suppose that we have an initial condition $\vec{x}(0) = \vec{b}$. Take $\vec{a} = E^{-1}\vec{b}$ to find $\vec{b} = \vec{v}_1 a_1 + \vec{v}_2 a_2 + \cdots + \vec{v}_n a_n$, just like before. Then if we write

$$\boxed{\xi_k(t) = e^{\lambda_k t}\int_0^t e^{-\lambda_k s} g_k(s)\, ds + a_k e^{\lambda_k t},}$$

we get the particular solution $\vec{x}(t) = \vec{v}_1\xi_1(t) + \vec{v}_2\xi_2(t) + \cdots + \vec{v}_n\xi_n(t)$ satisfying $\vec{x}(0) = \vec{b}$, because $\xi_k(0) = a_k$.

Let us remark that the technique we just outlined is the eigenvalue method applied to nonhomogeneous systems. If a system is homogeneous, that is, if $\vec{f} = \vec{0}$, then the equations we get are $\xi_k' = \lambda_k \xi_k$, and so $\xi_k = C_k e^{\lambda_k t}$ are the solutions and that's precisely what we got in § 3.4.

Example 3.9.2: Let $A = \left[\begin{smallmatrix} 1 & 3 \\ 3 & 1 \end{smallmatrix}\right]$. Solve $\vec{x}' = A\vec{x} + \vec{f}$ where $\vec{f}(t) = \left[\begin{smallmatrix} 2e^t \\ 2t \end{smallmatrix}\right]$ for $\vec{x}(0) = \left[\begin{smallmatrix} 3/16 \\ -5/16 \end{smallmatrix}\right]$.

The eigenvalues of A are -2 and 4 and corresponding eigenvectors are $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ respectively. This calculation is left as an exercise. We write down the matrix E of the eigenvectors and compute its inverse (using the inverse formula for 2×2 matrices)

$$E = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad E^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

We are looking for a solution of the form $\vec{x} = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]\xi_1 + \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]\xi_2$. We first need to write $\vec{f}$ in terms of the eigenvectors. That is we wish to write $\vec{f} = \left[\begin{smallmatrix} 2e^t \\ 2t \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] g_1 + \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] g_2$. Thus

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1}\begin{bmatrix} 2e^t \\ 2t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2e^t \\ 2t \end{bmatrix} = \begin{bmatrix} e^t - t \\ e^t + t \end{bmatrix}.$$

So $g_1 = e^t - t$ and $g_2 = e^t + t$.

We further need to write $\vec{x}(0)$ in terms of the eigenvectors. That is, we wish to write $\vec{x}(0) = \left[\begin{smallmatrix} 3/16 \\ -5/16 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] a_1 + \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] a_2$. Hence

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = E^{-1}\begin{bmatrix} 3/16 \\ -5/16 \end{bmatrix} = \begin{bmatrix} 1/4 \\ -1/16 \end{bmatrix}.$$

So $a_1 = 1/4$ and $a_2 = -1/16$. We plug our $\vec{x}$ into the equation and get

$$\begin{aligned} \overbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\xi_1' + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\xi_2'}^{\vec{x}'} &= \overbrace{A\begin{bmatrix} 1 \\ -1 \end{bmatrix}\xi_1 + A\begin{bmatrix} 1 \\ 1 \end{bmatrix}\xi_2}^{A\vec{x}} + \overbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}g_1 + \begin{bmatrix} 1 \\ 1 \end{bmatrix}g_2}^{\vec{f}} \\ &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}(-2\xi_1) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}4\xi_2 + \begin{bmatrix} 1 \\ -1 \end{bmatrix}(e^t - t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}(e^t + t). \end{aligned}$$

We get the two equations

$$\begin{aligned} \xi_1' &= -2\xi_1 + e^t - t, \qquad & \text{where } \xi_1(0) = a_1 = \frac{1}{4}, \\ \xi_2' &= 4\xi_2 + e^t + t, \qquad & \text{where } \xi_2(0) = a_2 = \frac{-1}{16}. \end{aligned}$$

We solve with integrating factor. Computation of the integral is left as an exercise to the student. You will need integration by parts.

$$\xi_1 = e^{-2t} \int e^{2t} \left(e^t - t\right) dt + C_1 e^{-2t} = \frac{e^t}{3} - \frac{t}{2} + \frac{1}{4} + C_1 e^{-2t}.$$

C_1 is the constant of integration. As $\xi_1(0) = 1/4$, then $1/4 = 1/3 + 1/4 + C_1$ and hence $C_1 = -1/3$. Similarly

$$\xi_2 = e^{4t} \int e^{-4t} \left(e^t + t\right) dt + C_2 e^{4t} = -\frac{e^t}{3} - \frac{t}{4} - \frac{1}{16} + C_2 e^{4t}.$$

As $\xi_2(0) = -1/16$ we have $-1/16 = -1/3 - 1/16 + C_2$ and hence $C_2 = 1/3$. The solution is

$$\vec{x}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \underbrace{\left(\frac{e^t - e^{-2t}}{3} + \frac{1-2t}{4}\right)}_{\xi_1} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \underbrace{\left(\frac{e^{4t} - e^t}{3} - \frac{4t+1}{16}\right)}_{\xi_2} = \begin{bmatrix} \frac{e^{4t}-e^{-2t}}{3} + \frac{3-12t}{16} \\ \frac{e^{-2t}+e^{4t}-2e^t}{3} + \frac{4t-5}{16} \end{bmatrix}.$$

That is, $x_1 = \frac{e^{4t}-e^{-2t}}{3} + \frac{3-12t}{16}$ and $x_2 = \frac{e^{-2t}+e^{4t}-2e^t}{3} + \frac{4t-5}{16}$.

***Exercise* 3.9.1:** *Check that x_1 and x_2 solve the problem. Check both that they satisfy the differential equation and that they satisfy the initial conditions.*

Undetermined coefficients

We also have the method of undetermined coefficients for systems. The only difference here is that we have to use unknown vectors rather than just numbers. Same caveats apply to undetermined coefficients for systems as for single equations. This method does not always work. Furthermore, if the right-hand side is complicated, we have to solve for lots of variables. Each element of an unknown vector is an unknown number. So in system of 3 equations if we have say 4 unknown vectors (this would not be uncommon), then we already have 12 unknown numbers that we need to solve for. The method can turn into a lot of tedious work if done by hand. As this method is essentially the same as it is for single equations, let us just do an example.

Example 3.9.3: Let $A = \left[\begin{smallmatrix} -1 & 0 \\ -2 & 1 \end{smallmatrix}\right]$. Find a particular solution of $\vec{x}' = A\vec{x} + \vec{f}$ where $\vec{f}(t) = \left[\begin{smallmatrix} e^t \\ t \end{smallmatrix}\right]$.

Note that we can solve this system in an easier way (can you see how?), but for the purposes of the example, let us use the eigenvalue method plus undetermined coefficients.

The eigenvalues of A are -1 and 1 and corresponding eigenvectors are $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ respectively. Hence our complementary solution is

$$\vec{x}_c = \alpha_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + \alpha_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^t,$$

for some arbitrary constants α_1 and α_2.

We would want to guess a particular solution of

$$\vec{x} = \vec{a} e^t + \vec{b} t + \vec{c}.$$

However, something of the form $\vec{a}e^t$ appears in the complementary solution. Because we do not yet know if the vector $\vec{a}$ is a multiple of $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$, we do not know if a conflict arises. It is possible that there is no conflict, but to be safe we should also try $\vec{b}te^t$. Here we find the crux of the difference between a single equation and systems. We try *both* terms $\vec{a}e^t$ and $\vec{b}te^t$ in the solution, not just the term $\vec{b}te^t$. Therefore, we try

$$\vec{x} = \vec{a}e^t + \vec{b}te^t + \vec{c}t + \vec{d}.$$

Thus we have 8 unknowns. We write $\vec{a} = \left[\begin{smallmatrix}a_1\\a_2\end{smallmatrix}\right]$, $\vec{b} = \left[\begin{smallmatrix}b_1\\b_2\end{smallmatrix}\right]$, $\vec{c} = \left[\begin{smallmatrix}c_1\\c_2\end{smallmatrix}\right]$, and $\vec{d} = \left[\begin{smallmatrix}d_1\\d_2\end{smallmatrix}\right]$. We plug $\vec{x}$ into the equation. First let us compute $\vec{x}'$.

$$\vec{x}' = \left(\vec{a} + \vec{b}\right) e^t + \vec{b}te^t + \vec{c} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} te^t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Now $\vec{x}'$ must equal $A\vec{x} + \vec{f}$, which is

$$\begin{aligned} A\vec{x} + \vec{f} &= A\vec{a}e^t + A\vec{b}te^t + A\vec{c}t + A\vec{d} + \vec{f} \\ &= \begin{bmatrix} -a_1 \\ -2a_1 + a_2 \end{bmatrix} e^t + \begin{bmatrix} -b_1 \\ -2b_1 + b_2 \end{bmatrix} te^t + \begin{bmatrix} -c_1 \\ -2c_1 + c_2 \end{bmatrix} t + \begin{bmatrix} -d_1 \\ -2d_1 + d_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t \\ &= \begin{bmatrix} -a_1 + 1 \\ -2a_1 + a_2 \end{bmatrix} e^t + \begin{bmatrix} -b_1 \\ -2b_1 + b_2 \end{bmatrix} te^t + \begin{bmatrix} -c_1 \\ -2c_1 + c_2 + 1 \end{bmatrix} t + \begin{bmatrix} -d_1 \\ -2d_1 + d_2 \end{bmatrix}. \end{aligned}$$

We identify the coefficients of e^t, te^t, t and any constant vectors in $\vec{x}'$ and in $A\vec{x} + \vec{f}$ to find the equations:

$$\begin{aligned} a_1 + b_1 &= -a_1 + 1, & 0 &= -c_1, \\ a_2 + b_2 &= -2a_1 + a_2, & 0 &= -2c_1 + c_2 + 1, \\ b_1 &= -b_1, & c_1 &= -d_1, \\ b_2 &= -2b_1 + b_2, & c_2 &= -2d_1 + d_2. \end{aligned}$$

We could write the 8×9 augmented matrix and start row reduction, but it is easier to just solve the equations in an ad hoc manner. Immediately we see that $b_1 = 0$, $c_1 = 0$, $d_1 = 0$. Plugging these back in, we get that $c_2 = -1$ and $d_2 = -1$. The remaining equations that tell us something are

$$\begin{aligned} a_1 &= -a_1 + 1, \\ a_2 + b_2 &= -2a_1 + a_2. \end{aligned}$$

So $a_1 = 1/2$ and $b_2 = -1$. Finally, a_2 can be arbitrary and still satisfy the equations. We are looking for just a single solution so presumably the simplest one is when $a_2 = 0$. Therefore,

$$\vec{x} = \vec{a}e^t + \vec{b}te^t + \vec{c}t + \vec{d} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^t \\ -te^t - t - 1 \end{bmatrix}.$$

That is, $x_1 = \frac{1}{2}e^t$, $x_2 = -te^t - t - 1$. We would add this to the complementary solution to get the general solution of the problem. Notice that both $\vec{a}e^t$ and $\vec{b}te^t$ were really needed.

***Exercise* 3.9.2:** *Check that x_1 and x_2 solve the problem. Try setting $a_2 = 1$ and check we get a solution as well. What is the difference between the two solutions we obtained (one with $a_2 = 0$ and one with $a_2 = 1$)?*

As you can see, other than the handling of conflicts, undetermined coefficients works exactly the same as it did for single equations. However, the computations can get out of hand pretty quickly for systems. The equation we considered was pretty simple.

3.9.2 First order variable coefficient

Variation of parameters

Just as for a single equation, there is the method of variation of parameters. For constant coefficient systems, it is essentially the same thing as the integrating factor method we discussed earlier. However, this method works for any linear system, even if it is not constant coefficient, provided we somehow solve the associated homogeneous problem.

Suppose we have the equation

$$\vec{x}\,' = A(t)\,\vec{x} + \vec{f}(t). \tag{3.10}$$

Further, suppose we solved the associated homogeneous equation $\vec{x}\,' = A(t)\,\vec{x}$ and found a fundamental matrix solution $X(t)$. The general solution to the associated homogeneous equation is $X(t)\vec{c}$ for a constant vector $\vec{c}$. Just like for variation of parameters for single equation we try the solution to the nonhomogeneous equation of the form

$$\vec{x}_p = X(t)\,\vec{u}(t),$$

where $\vec{u}(t)$ is a vector-valued function instead of a constant. We substitute $\vec{x}_p$ into (3.10) to obtain

$$\underbrace{X'(t)\,\vec{u}(t) + X(t)\,\vec{u}\,'(t)}_{\vec{x}_p\,'(t)} = \underbrace{A(t)\,X(t)\,\vec{u}(t)}_{A(t)\vec{x}_p(t)} + \vec{f}(t).$$

But $X(t)$ is a fundamental matrix solution to the homogeneous problem. So $X'(t) = A(t)X(t)$, and

$$\cancel{X'(t)\,\vec{u}(t)} + X(t)\,\vec{u}\,'(t) = \cancel{X'(t)\,\vec{u}(t)} + \vec{f}(t).$$

Hence $X(t)\,\vec{u}\,'(t) = \vec{f}(t)$. If we compute $[X(t)]^{-1}$, then $\vec{u}\,'(t) = [X(t)]^{-1}\,\vec{f}(t)$. We integrate to obtain $\vec{u}$ and we have the particular solution $\vec{x}_p = X(t)\,\vec{u}(t)$. Let us write this as a formula

$$\boxed{\vec{x}_p = X(t) \int [X(t)]^{-1}\,\vec{f}(t)\,dt.}$$

If A is constant and $X(t) = e^{tA}$, then $[X(t)]^{-1} = e^{-tA}$. We get a solution $\vec{x}_p = e^{tA} \int e^{-tA}\,\vec{f}(t)\,dt$, which is precisely what we got using the integrating factor method.

Example 3.9.4: Find a particular solution to

$$\vec{x}' = \frac{1}{t^2+1}\begin{bmatrix} t & -1 \\ 1 & t \end{bmatrix}\vec{x} + \begin{bmatrix} t \\ 1 \end{bmatrix}(t^2+1). \tag{3.11}$$

Here $A = \frac{1}{t^2+1}\left[\begin{smallmatrix} t & -1 \\ 1 & t \end{smallmatrix}\right]$ is most definitely not constant. Perhaps by a lucky guess, we find that $X = \left[\begin{smallmatrix} 1 & -t \\ t & 1 \end{smallmatrix}\right]$ solves $X'(t) = A(t)X(t)$. Once we know the complementary solution we can easily find a solution to (3.11). First we find

$$[X(t)]^{-1} = \frac{1}{t^2+1}\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}.$$

Next we know a particular solution to (3.11) is

$$\begin{aligned}
\vec{x}_p &= X(t)\int [X(t)]^{-1}\vec{f}(t)\,dt \\
&= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}\int \frac{1}{t^2+1}\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}\begin{bmatrix} t \\ 1 \end{bmatrix}(t^2+1)\,dt \\
&= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}\int \begin{bmatrix} 2t \\ -t^2+1 \end{bmatrix}dt \\
&= \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}\begin{bmatrix} t^2 \\ -\frac{1}{3}t^3+t \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{3}t^4 \\ \frac{2}{3}t^3+t \end{bmatrix}.
\end{aligned}$$

Adding the complementary solution we find the general solution to (3.11):

$$\vec{x} = \begin{bmatrix} 1 & -t \\ t & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{3}t^4 \\ \frac{2}{3}t^3+t \end{bmatrix} = \begin{bmatrix} c_1 - c_2 t + \frac{1}{3}t^4 \\ c_2 + (c_1+1)\,t + \frac{2}{3}t^3 \end{bmatrix}.$$

***Exercise* 3.9.3:** *Check that $x_1 = \frac{1}{3}t^4$ and $x_2 = \frac{2}{3}t^3 + t$ really solve (3.11).*

In the variation of parameters, just like in the integrating factor method we can obtain the general solution by adding in constants of integration. That is, we will add $X(t)\vec{c}$ for a vector of arbitrary constants. But that is precisely the complementary solution.

3.9.3 Second order constant coefficients

Undetermined coefficients

We have already seen a simple example of the method of undetermined coefficients for second order systems in § 3.6. This method is essentially the same as undetermined coefficients for first order systems. There are some simplifications that we can make, as we did in § 3.6. Let the equation be

$$\vec{x}'' = A\vec{x} + \vec{F}(t),$$

where A is a constant matrix. If $\vec{F}(t)$ is of the form $\vec{F}_0 \cos(\omega t)$, then as two derivatives of cosine is again cosine we can try a solution of the form

$$\vec{x}_p = \vec{c} \cos(\omega t),$$

and we do not need to introduce sines.

If the $\vec{F}$ is a sum of cosines, note that we still have the superposition principle. If $\vec{F}(t) = \vec{F}_0 \cos(\omega_0 t) + \vec{F}_1 \cos(\omega_1 t)$, then we would try $\vec{a} \cos(\omega_0 t)$ for the problem $\vec{x}'' = A\vec{x} + \vec{F}_0 \cos(\omega_0 t)$, and we would try $\vec{b} \cos(\omega_1 t)$ for the problem $\vec{x}'' = A\vec{x} + \vec{F}_1 \cos(\omega_1 t)$. Then we sum the solutions.

However, if there is duplication with the complementary solution, or the equation is of the form $\vec{x}'' = A\vec{x}' + B\vec{x} + \vec{F}(t)$, then we need to do the same thing as we do for first order systems.

You will never go wrong with putting in more terms than needed into your guess. You will find that the extra coefficients will turn out to be zero. But it is useful to save some time and effort.

Eigenvector decomposition

If we have the system

$$\vec{x}'' = A\vec{x} + \vec{f}(t),$$

we can do *eigenvector decomposition*, just like for first order systems.

Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ be eigenvectors. Again form the matrix $E = [\, \vec{v}_1 \quad \vec{v}_2 \quad \cdots \quad \vec{v}_n \,]$. Write

$$\vec{x}(t) = \vec{v}_1 \, \xi_1(t) + \vec{v}_2 \, \xi_2(t) + \cdots + \vec{v}_n \, \xi_n(t).$$

Decompose $\vec{f}$ in terms of the eigenvectors

$$\vec{f}(t) = \vec{v}_1 \, g_1(t) + \vec{v}_2 \, g_2(t) + \cdots + \vec{v}_n \, g_n(t),$$

where, again, $\vec{g} = E^{-1} \vec{f}$.

We plug in, and as before we obtain

$$\begin{aligned}
\overbrace{\vec{v}_1 \xi_1'' + \vec{v}_2 \xi_2'' + \cdots + \vec{v}_n \xi_n''}^{\vec{x}''} &= \overbrace{A \left(\vec{v}_1 \xi_1 + \vec{v}_2 \xi_2 + \cdots + \vec{v}_n \xi_n \right)}^{A\vec{x}} + \overbrace{\vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n}^{\vec{f}} \\
&= A\vec{v}_1 \xi_1 + A\vec{v}_2 \xi_2 + \cdots + A\vec{v}_n \xi_n + \vec{v}_1 g_1 + \vec{v}_2 g_2 + \cdots + \vec{v}_n g_n \\
&= \vec{v}_1 \lambda_1 \xi_1 + \vec{v}_2 \lambda_2 \xi_2 + \cdots + \vec{v}_n \lambda_n \xi_n + \vec{v}_1 \, g_1 + \vec{v}_2 \, g_2 + \cdots + \vec{v}_n \, g_n \\
&= \vec{v}_1 (\lambda_1 \xi_1 + g_1) + \vec{v}_2 (\lambda_2 \xi_2 + g_2) + \cdots + \vec{v}_n (\lambda_n \xi_n + g_n).
\end{aligned}$$

We identify the coefficients of the eigenvectors to get the equations

$$\begin{aligned} \xi_1'' &= \lambda_1 \xi_1 + g_1, \\ \xi_2'' &= \lambda_2 \xi_2 + g_2, \\ &\vdots \\ \xi_n'' &= \lambda_n \xi_n + g_n. \end{aligned}$$

Each one of these equations is independent of the others. We solve each equation using the methods of chapter 2. We write $\vec{x}(t) = \vec{v}_1 \xi_1(t) + \vec{v}_2 \xi_2(t) + \cdots + \vec{v}_n \xi_n(t)$, and we are done; we have a particular solution. We find the general solutions for ξ_1 through ξ_n, and again $\vec{x}(t) = \vec{v}_1 \xi_1(t) + \vec{v}_2\, \xi_2(t) + \cdots + \vec{v}_n \xi_n(t)$ is the general solution (and not just a particular solution).

Example 3.9.5: Let us do the example from § 3.6 using this method. The equation is

$$\vec{x}'' = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \cos(3t).$$

The eigenvalues are -1 and -4, with eigenvectors $\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$. Therefore $E = \left[\begin{smallmatrix} 1 & 1 \\ 2 & -1 \end{smallmatrix}\right]$ and $E^{-1} = \frac{1}{3}\left[\begin{smallmatrix} 1 & 1 \\ 2 & -1 \end{smallmatrix}\right]$. Therefore,

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = E^{-1} \vec{f}(t) = \frac{1}{3} \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2\cos(3t) \end{bmatrix} = \begin{bmatrix} \frac{2}{3}\cos(3t) \\ \frac{-2}{3}\cos(3t) \end{bmatrix}.$$

So after the whole song and dance of plugging in, the equations we get are

$$\xi_1'' = -\xi_1 + \frac{2}{3}\cos(3t), \qquad \xi_2'' = -4\,\xi_2 - \frac{2}{3}\cos(3t).$$

For each equation we use the method of undetermined coefficients. We try $C_1 \cos(3t)$ for the first equation and $C_2 \cos(3t)$ for the second equation. We plug in to get

$$\begin{aligned} -9C_1 \cos(3t) &= -C_1 \cos(3t) + \frac{2}{3}\cos(3t), \\ -9C_2 \cos(3t) &= -4C_2 \cos(3t) - \frac{2}{3}\cos(3t). \end{aligned}$$

We solve each of these equations separately. We get $-9C_1 = -C_1 + 2/3$ and $-9C_2 = -4C_2 - 2/3$. And hence $C_1 = -1/12$ and $C_2 = 2/15$. So our particular solution is

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \left(\frac{-1}{12} \cos(3t) \right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(\frac{2}{15} \cos(3t) \right) = \begin{bmatrix} 1/20 \\ -3/10 \end{bmatrix} \cos(3t).$$

This solution matches what we got previously in § 3.6.

3.9.4 Exercises

Exercise* 3.9.4*: *Find a particular solution to* $x' = x + 2y + 2t$, $y' = 3x + 2y - 4$,

a) *using integrating factor method,* b) *using eigenvector decomposition,*

c) *using undetermined coefficients.*

Exercise* 3.9.5*: *Find the general solution to* $x' = 4x + y - 1$, $y' = x + 4y - e^t$,

a) *using integrating factor method,* b) *using eigenvector decomposition,*

c) *using undetermined coefficients.*

Exercise* 3.9.6*: *Find the general solution to* $x_1'' = -6x_1 + 3x_2 + \cos(t)$, $x_2'' = 2x_1 - 7x_2 + 3\cos(t)$,

a) *using eigenvector decomposition,* b) *using undetermined coefficients.*

Exercise* 3.9.7*: *Find the general solution to* $x_1'' = -6x_1 + 3x_2 + \cos(2t)$, $x_2'' = 2x_1 - 7x_2 + 3\cos(2t)$,

a) *using eigenvector decomposition,* b) *using undetermined coefficients.*

Exercise* 3.9.8*: *Take the equation* $\vec{x}' = \begin{bmatrix} \frac{1}{t} & -1 \\ 1 & \frac{1}{t} \end{bmatrix} \vec{x} + \begin{bmatrix} t^2 \\ -t \end{bmatrix}$.

a) *Check that* $\vec{x}_c = c_1 \begin{bmatrix} t \sin t \\ -t \cos t \end{bmatrix} + c_2 \begin{bmatrix} t \cos t \\ t \sin t \end{bmatrix}$ *is the complementary solution.*

b) *Use variation of parameters to find a particular solution.*

Exercise* 3.9.101*: *Find a particular solution to* $x' = 5x + 4y + t$, $y' = x + 8y - t$,

a) *using integrating factor method,* b) *using eigenvector decomposition,*

c) *using undetermined coefficients.*

Exercise* 3.9.102*: *Find a particular solution to* $x' = y + e^t$, $y' = x + e^t$,

a) *using integrating factor method,* b) *using eigenvector decomposition,*

c) *using undetermined coefficients.*

Exercise* 3.9.103*: *Solve* $x_1' = x_2 + t$, $x_2' = x_1 + t$ *with initial conditions* $x_1(0) = 1$, $x_2(0) = 2$, *using eigenvector decomposition.*

Exercise* 3.9.104*: *Solve* $x_1'' = -3x_1 + x_2 + t$, $x_2'' = 9x_1 + 5x_2 + \cos(t)$ *with initial conditions* $x_1(0) = 0$, $x_2(0) = 0$, $x_1'(0) = 0$, $x_2'(0) = 0$, *using eigenvector decomposition.*

Chapter 6

The Laplace transform

6.1 The Laplace transform

Note: 1.5–2 lectures, §10.1 in [EP], §6.1 and parts of §6.2 in [BD]

6.1.1 The transform

In this chapter we will discuss the Laplace transform[†]. The Laplace transform is a very efficient method to solve certain ODE or PDE problems. The transform takes a differential equation and turns it into an algebraic equation. If the algebraic equation can be solved, applying the inverse transform gives us our desired solution. The Laplace transform also has applications in the analysis of electrical circuits, NMR spectroscopy, signal processing, and elsewhere. Finally, understanding the Laplace transform will also help with understanding the related Fourier transform, which, however, requires more understanding of complex numbers. We will not cover the Fourier transform.

The Laplace transform also gives a lot of insight into the nature of the equations we are dealing with. It can be seen as converting between the time and the frequency domain. For example, take the standard equation

$$mx''(t) + cx'(t) + kx(t) = f(t).$$

We can think of t as time and $f(t)$ as incoming signal. The Laplace transform will convert the equation from a differential equation in time to an algebraic (no derivatives) equation, where the new independent variable s is the frequency.

We can think of the *Laplace transform* as a black box. It eats functions and spits out functions in a new variable. We write $\mathcal{L}\{f(t)\} = F(s)$ for the Laplace transform of $f(t)$. It is common to write lower case letters for functions in the time domain and upper case letters for functions in the frequency domain. We use the same letter to denote that one

[†]Just like the Laplace equation and the Laplacian, the Laplace transform is also named after Pierre-Simon, marquis de Laplace (1749–1827).

function is the Laplace transform of the other. For example $F(s)$ is the Laplace transform of $f(t)$. Let us define the transform.

$$\mathcal{L}\{f(t)\} = F(s) \overset{\text{def}}{=} \int_0^\infty e^{-st} f(t)\, dt.$$

We note that we are only considering $t \geq 0$ in the transform. Of course, if we think of t as time there is no problem, we are generally interested in finding out what will happen in the future (Laplace transform is one place where it is safe to ignore the past). Let us compute some simple transforms.

Example 6.1.1: Suppose $f(t) = 1$, then

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_{t=0}^\infty = \lim_{h\to\infty} \left[\frac{e^{-st}}{-s}\right]_{t=0}^h = \lim_{h\to\infty}\left(\frac{e^{-sh}}{-s} - \frac{1}{-s}\right) = \frac{1}{s}.$$

The limit (the improper integral) only exists if $s > 0$. So $\mathcal{L}\{1\}$ is only defined for $s > 0$.

Example 6.1.2: Suppose $f(t) = e^{-at}$, then

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at}\, dt = \int_0^\infty e^{-(s+a)t}\, dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{t=0}^\infty = \frac{1}{s+a}.$$

The limit only exists if $s + a > 0$. So $\mathcal{L}\{e^{-at}\}$ is only defined for $s + a > 0$.

Example 6.1.3: Suppose $f(t) = t$, then using integration by parts

$$\begin{aligned} \mathcal{L}\{t\} &= \int_0^\infty e^{-st} t\, dt \\ &= \left[\frac{-te^{-st}}{s}\right]_{t=0}^\infty + \frac{1}{s}\int_0^\infty e^{-st}\, dt \\ &= 0 + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_{t=0}^\infty \\ &= \frac{1}{s^2}. \end{aligned}$$

Again, the limit only exists if $s > 0$.

Example 6.1.4: A common function is the *unit step function*, which is sometimes called the *Heaviside function**. This function is generally given as

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

*The function is named after the English mathematician, engineer, and physicist Oliver Heaviside (1850–1925). Only by coincidence is the function "heavy" on "one side."

Let us find the Laplace transform of $u(t-a)$, where $a \geq 0$ is some constant. That is, the function that is 0 for $t < a$ and 1 for $t \geq a$.

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt = \int_a^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_{t=a}^\infty = \frac{e^{-as}}{s},$$

where of course $s > 0$ (and $a \geq 0$ as we said before).

By applying similar procedures we can compute the transforms of many elementary functions. Many basic transforms are listed in Table 6.1.

$f(t)$	$\mathcal{L}\{f(t)\}$	$f(t)$	$\mathcal{L}\{f(t)\}$
C	$\frac{C}{s}$	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
t	$\frac{1}{s^2}$	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
t^2	$\frac{2}{s^3}$	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
t^3	$\frac{6}{s^4}$	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
t^n	$\frac{n!}{s^{n+1}}$	$u(t-a)$	$\frac{e^{-as}}{s}$
e^{-at}	$\frac{1}{s+a}$		

Table 6.1: Some Laplace transforms (C, ω, and a are constants).

***Exercise* 6.1.1:** *Verify Table 6.1.*

Since the transform is defined by an integral. We can use the linearity properties of the integral. For example, suppose C is a constant, then

$$\mathcal{L}\{Cf(t)\} = \int_0^\infty e^{-st} Cf(t)\, dt = C\int_0^\infty e^{-st} f(t)\, dt = C\mathcal{L}\{f(t)\}.$$

So we can "pull out" a constant out of the transform. Similarly we have linearity. Since linearity is very important we state it as a theorem.

Theorem 6.1.1 (Linearity of the Laplace transform)**.** *Suppose that A, B, and C are constants, then*

$$\boxed{\mathcal{L}\{Af(t)+Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\},}$$

and in particular

$$\mathcal{L}\{Cf(t)\} = C\mathcal{L}\{f(t)\}.$$

***Exercise* 6.1.2:** *Verify the theorem. That is, show that $\mathcal{L}\{Af(t)+Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$.*

These rules together with Table 6.1 on the preceding page make it easy to find the Laplace transform of a whole lot of functions already. But be careful. It is a common mistake to think that the Laplace transform of a product is the product of the transforms. In general

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

It must also be noted that not all functions have a Laplace transform. For example, the function $\frac{1}{t}$ does not have a Laplace transform as the integral diverges for all s. Similarly, $\tan t$ or e^{t^2} do not have Laplace transforms.

6.1.2 Existence and uniqueness

When does the Laplace transform exist? A function $f(t)$ is of *exponential order* as t goes to infinity if

$$|f(t)| \leq Me^{ct},$$

for some constants M and c, for sufficiently large t (say for all $t > t_0$ for some t_0). The simplest way to check this condition is to try and compute

$$\lim_{t\to\infty} \frac{f(t)}{e^{ct}}.$$

If the limit exists and is finite (usually zero), then $f(t)$ is of exponential order.

***Exercise* 6.1.3:** *Use L'Hopital's rule from calculus to show that a polynomial is of exponential order. Hint: Note that a sum of two exponential order functions is also of exponential order. Then show that t^n is of exponential order for any n.*

For an exponential order function we have existence and uniqueness of the Laplace transform.

Theorem 6.1.2 (Existence)**.** *Let $f(t)$ be continuous and of exponential order for a certain constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.*

The existence is not difficult to see. Let $f(t)$ be of exponential order, that is $|f(t)| \leq Me^{ct}$ for all $t > 0$ (for simplicity $t_0 = 0$). Let $s > c$, or in other words $(c - s) < 0$. By the comparison theorem from calculus, the improper integral defining $\mathcal{L}\{f(t)\}$ exists if the following integral exists

$$\int_0^\infty e^{-st}(Me^{ct})\,dt = M\int_0^\infty e^{(c-s)t}\,dt = M\left[\frac{e^{(c-s)t}}{c-s}\right]_{t=0}^\infty = \frac{M}{c-s}.$$

The transform also exists for some other functions that are not of exponential order, but that will not be relevant to us. Before dealing with uniqueness, let us note that for exponential order functions we obtain that their Laplace transform decays at infinity:

$$\lim_{s\to\infty} F(s) = 0.$$

Theorem 6.1.3 (Uniqueness)**.** *Let $f(t)$ and $g(t)$ be continuous and of exponential order. Suppose that there exists a constant C, such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.*

Both theorems hold for piecewise continuous functions as well. Recall that piecewise continuous means that the function is continuous except perhaps at a discrete set of points, where it has jump discontinuities like the Heaviside function. Uniqueness, however, does not "see" values at the discontinuities. So we can only conclude that $f(t) = g(t)$ outside of discontinuities. For example, the unit step function is sometimes defined using $u(0) = 1/2$. This new step function, however, has the exact same Laplace transform as the one we defined earlier where $u(0) = 1$.

6.1.3 The inverse transform

As we said, the Laplace transform will allow us to convert a differential equation into an algebraic equation. Once we solve the algebraic equation in the frequency domain we will want to get back to the time domain, as that is what we are interested in. Given a function $F(s)$, we wish to find a function $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$. Theorem 6.1.3 says that the solution $f(t)$ is unique. So we can without fear make the following definition.

Suppose $F(s) = \mathcal{L}\{f(t)\}$ for some function $f(t)$. Define the *inverse Laplace transform* as

$$\mathcal{L}^{-1}\{F(s)\} \overset{\text{def}}{=} f(t).$$

There is an integral formula for the inverse, but it is not as simple as the transform itself—it requires complex numbers and path integrals. For us it will suffice to compute the inverse using Table 6.1 on page 205.

Example 6.1.5: Take $F(s) = \frac{1}{s+1}$. Find the inverse Laplace transform.

We look at the table to find

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}.$$

As the Laplace transform is linear, the inverse Laplace transform is also linear. That is,

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}.$$

Of course, we also have $\mathcal{L}^{-1}\{AF(s)\} = A\mathcal{L}^{-1}\{F(s)\}$. Let us demonstrate how linearity can be used.

Example 6.1.6: Take $F(s) = \frac{s^2+s+1}{s^3+s}$. Find the inverse Laplace transform.

First we use the *method of partial fractions* to write F in a form where we can use Table 6.1 on page 205. We factor the denominator as $s(s^2 + 1)$ and write

$$\frac{s^2 + s + 1}{s^3 + s} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}.$$

Putting the right-hand side over a common denominator and equating the numerators we get $A(s^2 + 1) + s(Bs + C) = s^2 + s + 1$. Expanding and equating coefficients we obtain

$A + B = 1$, $C = 1$, $A = 1$, and thus $B = 0$. In other words,

$$F(s) = \frac{s^2 + s + 1}{s^3 + s} = \frac{1}{s} + \frac{1}{s^2 + 1}.$$

By linearity of the inverse Laplace transform we get

$$\mathcal{L}^{-1}\left\{\frac{s^2 + s + 1}{s^3 + s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = 1 + \sin t.$$

Another useful property is the so-called *shifting property* or the *first shifting property*

$$\boxed{\mathcal{L}\{e^{-at} f(t)\} = F(s + a),}$$

where $F(s)$ is the Laplace transform of $f(t)$.

***Exercise* 6.1.4:** *Derive the first shifting property from the definition of the Laplace transform.*

The shifting property can be used, for example, when the denominator is a more complicated quadratic that may come up in the method of partial fractions. We complete the square and write such quadratics as $(s + a)^2 + b$ and then use the shifting property.

Example 6.1.7: Find $\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+8}\right\}$.

First we complete the square to make the denominator $(s + 2)^2 + 4$. Next we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4}\right\} = \frac{1}{2}\sin(2t).$$

Putting it all together with the shifting property, we find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 4s + 8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s + 2)^2 + 4}\right\} = \frac{1}{2} e^{-2t}\sin(2t).$$

In general, we want to be able to apply the inverse Laplace transform to rational functions, that is functions of the form

$$\frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomials. Since normally, for the functions that we are considering, the Laplace transform goes to zero as $s \to \infty$, it is not hard to see that the degree of $F(s)$ must be smaller than that of $G(s)$. Such rational functions are called *proper rational functions* and we can always apply the method of partial fractions. Of course this means we need to be able to factor the denominator into linear and quadratic terms, which involves finding the roots of the denominator.

6.1.4 The Laplace transform with Python

We show how to compute Laplace and inverse Laplace transforms below. The expression $\theta(t-3)$ in the sympy output represents the Heaviside unit step function $u(t-3)$.

```
from resources306 import *
from sympy.integrals import laplace_transform as L, inverse_laplace_transform as Linv
```

```
s,t = sp.symbols('s t')
w = sp.symbols('w')
L( sp.sin(w*t), t, s )[0]
```

$$\frac{w}{s^2+w^2}$$

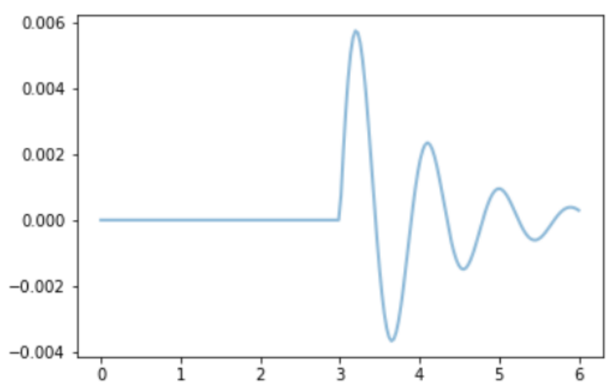
```
F = sp.exp(-3*(s+1))/((s+1)**2+49)
F
```

$$\frac{e^{-3s-3}}{(s+1)^2+49}$$

```
f = Linv(F,s,t)
f
```

$$\frac{e^{-t}\sin(7t-21)\theta(t-3)}{7}$$

```
expressionplot(f, t,0,6 , lw=2)
```



6.1.5 Exercises

Exercise 6.1.5: *Find the Laplace transform of* $3+t^5+\sin(\pi t)$.

Exercise 6.1.6: *Find the Laplace transform of* $a+bt+ct^2$ *for some constants* a, b, *and* c.

Exercise 6.1.7: *Find the Laplace transform of* $A\cos(\omega t)+B\sin(\omega t)$.

Exercise 6.1.8: *Find the Laplace transform of* $\cos^2(\omega t)$.

Exercise 6.1.9: *Find the inverse Laplace transform of* $\frac{4}{s^2-9}$.

Exercise 6.1.10: *Find the inverse Laplace transform of* $\frac{2s}{s^2-1}$.

***Exercise* 6.1.11:** *Find the inverse Laplace transform of* $\frac{1}{(s-1)^2(s+1)}$.

***Exercise* 6.1.12:** *Find the Laplace transform of* $f(t) = \begin{cases} t & \text{if } t \geq 1, \\ 0 & \text{if } t < 1. \end{cases}$

***Exercise* 6.1.13:** *Find the inverse Laplace transform of* $\frac{s}{(s^2+s+2)(s+4)}$.

***Exercise* 6.1.14:** *Find the Laplace transform of* $\sin\big(\omega(t-a)\big)$.

***Exercise* 6.1.15:** *Find the Laplace transform of* $t\sin(\omega t)$. *Hint: Several integrations by parts.*

***Exercise* 6.1.51:** *Apply the definition to find the Laplace transform of each of the following functions:*

a) $f(t) = 2$

b) $f(t) = e^{5t}$

c) $f(t) = \frac{2}{3e^{3t}}$

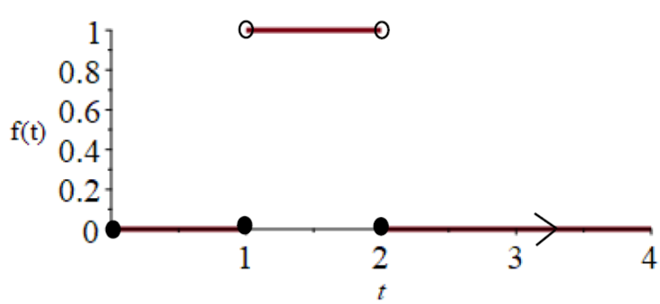
d) $f(t) = t + 1$

e) $f(t) = 5 - 2t$

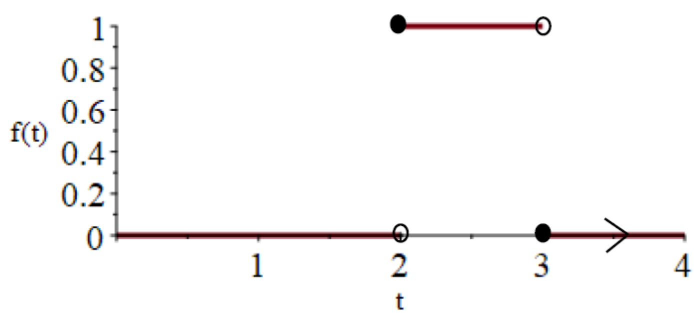
f) $f(t) = e^{\pi t + 2}$

***Exercise* 6.1.52:** *Apply the definition to find the Laplace transform of each of the following functions, represented graphically by:*

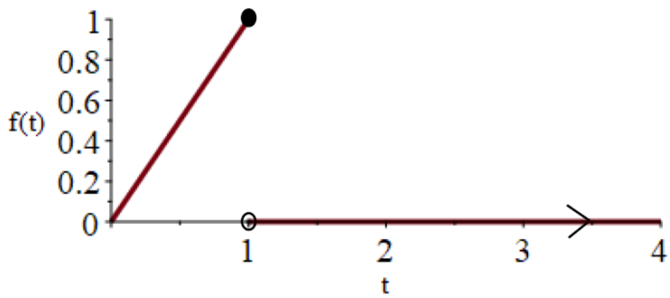
a)



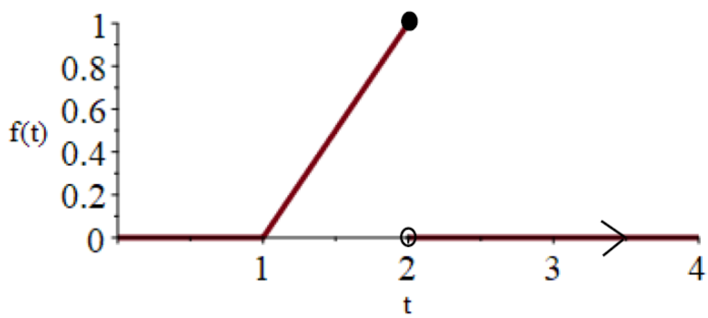
b)



c)



d)



***Exercise* 6.1.101:** *Find the Laplace transform of* $4(t+1)^2$.

***Exercise* 6.1.102:** *Find the inverse Laplace transform of* $\frac{8}{s^3(s+2)}$.

***Exercise* 6.1.103:** *Find the Laplace transform of* te^{-t} (*Hint*: *integrate by parts*).

***Exercise* 6.1.104:** *Find the Laplace transform of* $\sin(t)e^{-t}$ (*Hint*: *integrate by parts*).

***Exercise* 6.1.151:** *Using linearity and the transforms for* t^n, e^{-at}, $\sin \omega t$, *and* $\cos \omega t$, *find the Laplace transform for each of the following functions*:

a) $f(t) = 2 - 5\cos 3t + \frac{t^4}{2} - 6e^{7t}$

b) $f(t) = \frac{1}{3e^{\pi t}} + 2\sin \pi t + (2t+1)^2$

c) $f(t) = 3e^{-5t+2} - \frac{3t^3}{4} + \sin^2 2t$

d) $f(t) = 4\cos \sqrt{2}\,t - \frac{2}{3} + (t^2+t)^2$

e) $f(t) = \frac{2\sin 3t}{5} + \frac{t^5}{6} - \cos^2 3t$

***Exercise* 6.1.152:** *Using linearity and the inverse transforms for* $\frac{n!}{s^{n+1}}$, $\frac{1}{s+a}$, $\frac{\omega}{s^2+\omega^2}$ *and* $\frac{s}{s^2+\omega^2}$, *find the inverse Laplace transform for each of the following functions*:

a) $F(s) = \frac{3}{s} - \frac{2}{s-6} + \frac{5}{s^2+9}$

b) $F(s) = \frac{1}{4s^4} + \frac{3}{7(s+\pi)} + \frac{2s}{s^2+4}$

c) $F(s) = \frac{1}{2s+6} - \frac{3}{4s^3} + \frac{3s+2}{s^2+9}$

d) $F(s) = \frac{1}{s} - \frac{2}{3s^2} + \frac{5}{s-\pi^2}$

e) $F(s) = \frac{1}{6s^5} + \frac{5s-7}{s^2+3}$

Exercise* 6.1.153*: *Use the method of partial fractions to find the inverse Laplace transform of the following functions*:

a) $F(s) = \frac{3}{s(s-2)}$

b) $F(s) = \frac{2s+1}{s(s^2+4)}$

c) $F(s) = \frac{s+4}{s^2-s-6}$

d) $F(s) = \frac{s^2+s+2}{s^2(s+1)}$

e) $F(s) = \frac{3s^3+s}{s^4+3s^2+2}$

Exercise* 6.1.154*: *Apply the first shifting property* (*translation of the transform along the s-axis*) *to find the Laplace transform of the following functions*:

a) $f(t) = t^4 e^{3t}$

b) $f(t) = 3t^3 e^{-2t}$

c) $f(t) = e^{4t} \cos 3t$

d) $f(t) = 2e^{-3t} \sin 5t$

e) $f(t) = 3e^{-\pi t} \cos 2\pi t$

Exercise* 6.1.155*: *Apply the first shifting property to find the inverse Laplace transform of the following functions*:

a) $F(s) = \frac{1}{(s-2)^4}$

b) $F(s) = \frac{1}{2(s+\pi)^2}$

c) $F(s) = \frac{s-3}{s^2-6s+25}$

d) $F(s) = \frac{1}{s^2+8s+16}$

e) $F(s) = \frac{4s+3}{s^2+4s+13}$

f) $F(s) = \frac{3s+4}{s^2-2s+5}$

6.2 Transforms of derivatives and ODEs

Note: 2 lectures, §7.2–7.3 in [EP], §6.2 and §6.3 in [BD]

6.2.1 Transforms of derivatives

Let us see how the Laplace transform is used for differential equations. First let us try to find the Laplace transform of a function that is a derivative. Suppose $g(t)$ is a differentiable function of exponential order, that is, $|g(t)| \leq Me^{ct}$ for some M and c. So $\mathcal{L}\{g(t)\}$ exists, and what is more, $\lim_{t\to\infty} e^{-st} g(t) = 0$ when $s > c$. Then

$$\mathcal{L}\{g'(t)\} = \int_0^\infty e^{-st} g'(t)\, dt = \Big[e^{-st} g(t)\Big]_{t=0}^{\infty} - \int_0^\infty (-s)\, e^{-st} g(t)\, dt = -g(0) + s\mathcal{L}\{g(t)\}.$$

We repeat this procedure for higher derivatives. The results are listed in Table 6.2. The procedure also works for piecewise smooth functions, that is functions that are piecewise continuous with a piecewise continuous derivative.

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2g(0) - sg'(0) - g''(0)$

Table 6.2: Laplace transforms of derivatives ($G(s) = \mathcal{L}\{g(t)\}$ as usual).

***Exercise* 6.2.1:** *Verify Table 6.2.*

6.2.2 Solving ODEs with the Laplace transform

Notice that the Laplace transform turns differentiation into multiplication by s. Let us see how to apply this fact to differential equations.

Example 6.2.1: Take the equation

$$x''(t) + x(t) = \cos(2t), \quad x(0) = 0, \quad x'(0) = 1.$$

We will take the Laplace transform of both sides. By $X(s)$ we will, as usual, denote the Laplace transform of $x(t)$.

$$\begin{aligned} \mathcal{L}\{x''(t) + x(t)\} &= \mathcal{L}\{\cos(2t)\}, \\ s^2X(s) - sx(0) - x'(0) + X(s) &= \frac{s}{s^2+4}. \end{aligned}$$

We plug in the initial conditions now—this makes the computations more streamlined—to obtain

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2+4}.$$

We solve for $X(s)$,

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1}.$$

We use partial fractions (exercise) to write

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}.$$

Now take the inverse Laplace transform to obtain

$$x(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + \sin(t).$$

The procedure for linear constant coefficient equations is as follows. We take an ordinary differential equation in the time variable t. We apply the Laplace transform to transform the equation into an algebraic (non differential) equation in the frequency domain. All the $x(t)$, $x'(t)$, $x''(t)$, and so on, will be converted to $X(s)$, $sX(s) - x(0)$, $s^2X(s) - sx(0) - x'(0)$, and so on. We solve the equation for $X(s)$. Then taking the inverse transform, if possible, we find $x(t)$.

It should be noted that since not every function has a Laplace transform, not every equation can be solved in this manner. Also if the equation is not a linear constant coefficient ODE, then by applying the Laplace transform we may not obtain an algebraic equation.

6.2.3 Using the Heaviside function

Before we move on to more general equations than those we could solve before, we want to consider the Heaviside function. See Figure 6.1 on the next page for the graph.

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

This function is useful for putting together functions, or cutting functions off. Most commonly it is used as $u(t-a)$ for some constant a. This just shifts the graph to the right by a. That is, it is a function that is 0 when $t < a$ and 1 when $t \geq a$. Suppose for example that $f(t)$ is a "signal" and you started receiving the signal $\sin t$ at time $t = \pi$. The function $f(t)$ should then be defined as

$$f(t) = \begin{cases} 0 & \text{if } t < \pi, \\ \sin t & \text{if } t \geq \pi. \end{cases}$$

Using the Heaviside function, $f(t)$ can be written as

$$f(t) = u(t-\pi)\sin t.$$

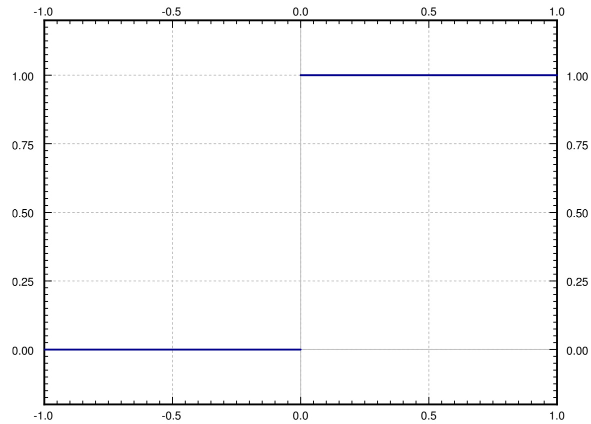


Figure 6.1: Plot of the Heaviside (unit step) function $u(t)$.

Similarly the step function that is 1 on the interval $[1, 2)$ and zero everywhere else can be written as

$$u(t-1) - u(t-2).$$

The Heaviside function is useful to define functions defined piecewise. If you want to define $f(t)$ such that $f(t) = t$ when t is in $[0, 1]$, $f(t) = -t + 2$ when t is in $[1, 2]$, and $f(t) = 0$ otherwise, then you can use the expression

$$f(t) = t\,\big(u(t) - u(t-1)\big) + (-t+2)\,\big(u(t-1) - u(t-2)\big).$$

Hence it is useful to know how the Heaviside function interacts with the Laplace transform. We have already seen that

$$\mathcal{L}\big\{u(t-a)\big\} = \frac{e^{-as}}{s}.$$

This can be generalized into a *shifting property* or *second shifting property*.

$$\boxed{\mathcal{L}\big\{f(t-a)\,u(t-a)\big\} = e^{-as}\mathcal{L}\big\{f(t)\big\}.} \tag{6.1}$$

Example 6.2.2: Suppose that the forcing function is not periodic. For example, suppose that we had a mass-spring system

$$x''(t) + x(t) = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

where $f(t) = 1$ if $1 \le t < 5$ and zero otherwise. We could imagine a mass-spring system, where a rocket is fired for 4 seconds starting at $t = 1$. Or perhaps an RLC circuit, where the voltage is raised at a constant rate for 4 seconds starting at $t = 1$, and then held steady again starting at $t = 5$.

We can write $f(t) = u(t-1) - u(t-5)$. We transform the equation and we plug in the initial conditions as before to obtain

$$s^2 X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

We solve for $X(s)$ to obtain

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}.$$

We leave it as an exercise to the reader to show that

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t.$$

In other words $\mathcal{L}\{1-\cos t\} = \frac{1}{s(s^2+1)}$. So using (6.1) we find

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-s}\mathcal{L}\{1-\cos t\}\right\} = \big(1-\cos(t-1)\big)\,u(t-1).$$

Similarly

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{e^{-5s}\mathcal{L}\{1-\cos t\}\right\} = \big(1-\cos(t-5)\big)\,u(t-5).$$

Hence, the solution is

$$x(t) = \big(1-\cos(t-1)\big)\,u(t-1) - \big(1-\cos(t-5)\big)\,u(t-5).$$

The plot of this solution is given in Figure 6.2 on the facing page.

6.2.4 Transfer functions

Laplace transform leads to the following useful concept for studying the steady state behavior of a linear system. Suppose we have an equation of the form

$$Lx = f(t),$$

where L is a linear constant coefficient differential operator. Then $f(t)$ is usually thought of as input of the system and $x(t)$ is thought of as the output of the system. For example, for a mass-spring system the input is the forcing function and output is the behavior of the mass. We would like to have a convenient way to study the behavior of the system for different inputs.

Let us suppose that all the initial conditions are zero and take the Laplace transform of the equation, we obtain the equation

$$A(s)X(s) = F(s).$$

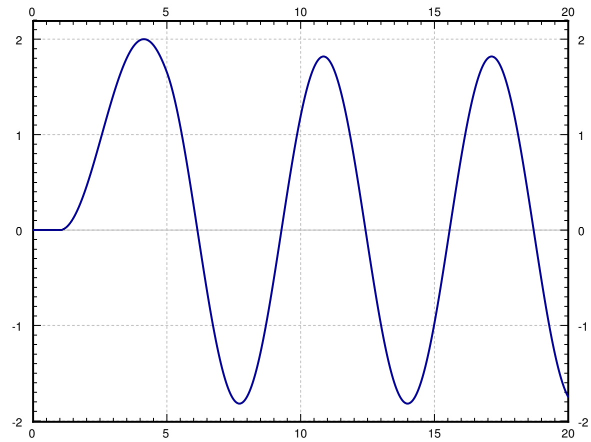


Figure 6.2: Plot of $x(t)$.

Solving for the ratio $X(s)/F(s)$ we obtain the so-called *transfer function* $H(s) = 1/A(s)$.

$$H(s) = \frac{X(s)}{F(s)}.$$

In other words, $X(s) = H(s)F(s)$. We obtain an algebraic dependence of the output of the system based on the input. We can now easily study the steady state behavior of the system given different inputs by simply multiplying by the transfer function.

Example 6.2.3: Given $x'' + \omega_0^2 x = f(t)$, let us find the transfer function (assuming the initial conditions are zero).

First, we take the Laplace transform of the equation.

$$s^2 X(s) + \omega_0^2 X(s) = F(s).$$

Now we solve for the transfer function $X(s)/F(s)$.

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + \omega_0^2}.$$

Let us see how to use the transfer function. Suppose we have the constant input $f(t) = 1$. Hence $F(s) = 1/s$, and

$$X(s) = H(s)F(s) = \frac{1}{s^2 + \omega_0^2}\,\frac{1}{s}.$$

Taking the inverse Laplace transform of $X(s)$ we obtain

$$x(t) = \frac{1 - \cos(\omega_0 t)}{\omega_0^2}.$$

6.2.5 Transforms of integrals

A feature of Laplace transforms is that it is also able to easily deal with integral equations. That is, equations in which integrals rather than derivatives of functions appear. The basic property, which can be proved by applying the definition and doing integration by parts, is

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)\, d\tau\right\} = \frac{1}{s} F(s).}$$

It is sometimes useful (e.g. for computing the inverse transform) to write this as

$$\int_0^t f(\tau)\, d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s} F(s)\right\}.$$

Example 6.2.4: To compute $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$ we could proceed by applying this integration rule.

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{s^2+1}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} d\tau = \int_0^t \sin\tau\, d\tau = 1 - \cos t.$$

Example 6.2.5: An equation containing an integral of the unknown function is called an *integral equation*. For example, take

$$t^2 = \int_0^t e^\tau x(\tau)\, d\tau,$$

where we wish to solve for $x(t)$. We apply the Laplace transform and the shifting property to get

$$\frac{2}{s^3} = \frac{1}{s}\mathcal{L}\{e^t x(t)\} = \frac{1}{s} X(s-1),$$

where $X(s) = \mathcal{L}\{x(t)\}$. Thus

$$X(s-1) = \frac{2}{s^2} \qquad \text{or} \qquad X(s) = \frac{2}{(s+1)^2}.$$

We use the shifting property again

$$x(t) = 2e^{-t}t.$$

6.2.6 Exercises

Exercise 6.2.2: *Using the Heaviside function write down the piecewise function that is* 0 *for* $t < 0$, t^2 *for* t *in* $[0,1]$ *and* t *for* $t > 1$.

Exercise 6.2.3: *Using the Laplace transform solve*

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where $m > 0$, $c > 0$, $k > 0$, *and* $c^2 - 4km > 0$ *(system is overdamped).*

***Exercise* 6.2.4:** *Using the Laplace transform solve*

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where $m > 0$, $c > 0$, $k > 0$, *and* $c^2 - 4km < 0$ (*system is underdamped*).

***Exercise* 6.2.5:** *Using the Laplace transform solve*

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b,$$

where $m > 0$, $c > 0$, $k > 0$, *and* $c^2 = 4km$ (*system is critically damped*).

***Exercise* 6.2.6:** *Solve* $x'' + x = u(t-1)$ *for initial conditions* $x(0) = 0$ *and* $x'(0) = 0$.

***Exercise* 6.2.7:** *Show the differentiation of the transform property. Suppose* $\mathcal{L}\{f(t)\} = F(s)$, *then show*

$$\mathcal{L}\{-tf(t)\} = F'(s).$$

Hint: *Differentiate under the integral sign.*

***Exercise* 6.2.8:** *Solve* $x''' + x = t^3u(t-1)$ *for initial conditions* $x(0) = 1$ *and* $x'(0) = 0$, $x''(0) = 0$.

***Exercise* 6.2.9:** *Show the second shifting property*: $\mathcal{L}\{f(t-a)\,u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$.

***Exercise* 6.2.10:** *Let us think of the mass-spring system with a rocket from Example 6.2.2. We noticed that the solution kept oscillating after the rocket stopped running. The amplitude of the oscillation depends on the time that the rocket was fired* (*for* 4 *seconds in the example*).

a) *Find a formula for the amplitude of the resulting oscillation in terms of the amount of time the rocket is fired.*

b) *Is there a nonzero time* (*if so what is it?*) *for which the rocket fires and the resulting oscillation has amplitude* 0 (*the mass is not moving*)?

***Exercise* 6.2.11:** *Define*

$$f(t) = \begin{cases} (t-1)^2 & \text{if } 1 \le t < 2, \\ 3 - t & \text{if } 2 \le t < 3, \\ 0 & \text{otherwise}. \end{cases}$$

a) *Sketch the graph of* $f(t)$.

b) *Write down* $f(t)$ *using the Heaviside function.*

c) *Solve* $x'' + x = f(t)$, $x(0) = 0$, $x'(0) = 0$ *using Laplace transform.*

***Exercise* 6.2.12:** *Find the transfer function for* $mx'' + cx' + kx = f(t)$ (*assuming the initial conditions are zero*).

Exercise 6.2.101: *Using the Heaviside function $u(t)$, write down the function*

$$f(t) = \begin{cases} 0 & \text{if} \quad t < 1, \\ t-1 & \text{if } 1 \leq t < 2, \\ 1 & \text{if } 2 \leq t. \end{cases}$$

Exercise 6.2.102: *Solve $x'' - x = (t^2 - 1)u(t-1)$ for initial conditions $x(0) = 1$, $x'(0) = 2$ using the Laplace transform.*

Exercise 6.2.103: *Find the transfer function for $x' + x = f(t)$ (assuming the initial conditions are zero).*

Exercise 6.2.151: *Use Laplace transforms to solve the following IVPs:*

a) $x'' + 25x = 0;\ x(0) = 1,\ x'(0) = 2$

b) $x'' + 4x = 2;\ x(0) = 3,\ x'(0) = -1$

c) $x'' + 9x = \cos t;\ x(0) = -1,\ x'(0) = 1$

d) $x'' + 5x' + 6x = 3;\ x(0) = x'(0) = 0$

e) $x'' + 2x' + 5x = 0;\ x(0) = 2,\ x'(0) = 1$

f) $x'' + 6x' + 25x = 0;\ x(0) = 1,\ x'(0) = 3$

Exercise 6.2.152: *Apply the second shifting property (step-translation along the t-axis) and Heaviside step functions to find the Laplace transforms of the following functions:*

a) $f(t) = \begin{cases} \cos t & 0 \leq t < 2\pi, \\ 0 & t \geq 2\pi. \end{cases}$

b) $f(t) = \begin{cases} 0 & 0 \leq t < 2, \\ 1 & 2 \leq t < 5, \\ 2 & t \geq 5. \end{cases}$

c) $f(t) = \begin{cases} \sin 2t & 0 \leq t < 3\pi, \\ 0 & t \geq 3\pi. \end{cases}$

d) $f(t) = \begin{cases} 0 & 0 \leq t < 2, \\ \cos \pi t & 2 \leq t < 5, \\ 0 & t \geq 5. \end{cases}$

e) $f(t) = \begin{cases} 0 & 0 \le t < \pi, \\ \sin 3t & \pi \le t < 4\pi, \\ 0 & t \ge 4\pi. \end{cases}$

***Exercise* 6.2.153:** *Apply the second shifting property (step- translation along the t-axis) to find the inverse Laplace transform of the following functions*:

a) $F(s) = \frac{e^{-3s}}{(s-4)}$

b) $F(s) = \frac{e^{-2s}}{s^3}$

c) $F(s) = \frac{e^{-2\pi s}}{s^2+9}$

d) $F(s) = \frac{e^{-s}-e^{-2s}}{s+1}$

e) $F(s) = \frac{s(1+e^{-\pi s})}{s^2+4}$

f) $F(s) = \frac{e^{-2\pi s}-e^{-3\pi s}}{s^2+16}$

***Exercise* 6.2.154:** *Apply the integration property, as in Example 6.2.4, to find the inverse Laplace transform of the following functions*:

a) $F(s) = \frac{1}{s(s+2)}$

b) $F(s) = \frac{2}{s(s^2+9)}$

c) $F(s) = \frac{s+1}{s(s^2+4)}$

d) $F(s) = \frac{1}{s^2(s+3)}$

e) $F(s) = \frac{3}{s^2(s^2+1)}$

f) $F(s) = \frac{2}{s^2(s-\pi)}$

6.3 Convolution

Note: 1 or 1.5 lectures, §7.2 in [EP], §6.6 in [BD]

6.3.1 The convolution

We said that the Laplace transformation of a product is not the product of the transforms. All hope is not lost however. We simply have to use a different type of a "product." Take two functions $f(t)$ and $g(t)$ defined for $t \geq 0$, and define the *convolution*[*] of $f(t)$ and $g(t)$ as

$$\boxed{(f * g)(t) \overset{\text{def}}{=} \int_0^t f(\tau)g(t-\tau)\,d\tau.} \tag{6.2}$$

As you can see, the convolution of two functions of t is another function of t.

Example 6.3.1: Take $f(t) = e^t$ and $g(t) = t$ for $t \geq 0$. Then

$$(f * g)(t) = \int_0^t e^\tau (t-\tau)\,d\tau = e^t - t - 1.$$

To solve the integral we did one integration by parts.

Example 6.3.2: Take $f(t) = \sin(\omega t)$ and $g(t) = \cos(\omega t)$ for $t \geq 0$. Then

$$(f * g)(t) = \int_0^t \sin(\omega\tau)\,\cos\big(\omega(t-\tau)\big)\,d\tau.$$

We apply the identity

$$\cos(\theta)\sin(\psi) = \frac{1}{2}\big(\sin(\theta+\psi) - \sin(\theta-\psi)\big).$$

Hence,

$$\begin{aligned}(f * g)(t) &= \int_0^t \frac{1}{2}\big(\sin(\omega t) - \sin(\omega t - 2\omega\tau)\big)\,d\tau \\ &= \left[\frac{1}{2}\tau\sin(\omega t) + \frac{1}{4\omega}\,\cos(2\omega\tau - \omega t)\right]_{\tau=0}^{t} \\ &= \frac{1}{2}\,t\sin(\omega t).\end{aligned}$$

The formula holds only for $t \geq 0$. We assumed that f and g are zero (or simply not defined) for negative t.

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[*]For those that have seen convolution defined before, you may have seen it defined as $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)\,d\tau$. This definition agrees with (6.2) if you define $f(t)$ and $g(t)$ to be zero for $t < 0$. When discussing the Laplace transform the definition we gave is sufficient. Convolution does occur in many other applications, however, where you may have to use the more general definition with infinities.

The convolution has many properties that make it behave like a product. Let c be a constant and f, g, and h be functions then

$$\begin{aligned} f * g &= g * f, \\ (cf) * g &= f * (cg) = c(f * g), \\ (f * g) * h &= f * (g * h). \end{aligned}$$

The most interesting property for us, and the main result of this section is the following theorem.

Theorem 6.3.1. *Let $f(t)$ and $g(t)$ be of exponential order, then*

$$\boxed{\mathcal{L}\big\{(f * g)(t)\big\} = \mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)\,d\tau\right\} = \mathcal{L}\big\{f(t)\big\}\mathcal{L}\big\{g(t)\big\}.}$$

In other words, the Laplace transform of a convolution is the product of the Laplace transforms. The simplest way to use this result is in reverse.

Example 6.3.3: Suppose we have the function of s defined by

$$\frac{1}{(s+1)s^2} = \frac{1}{s+1}\,\frac{1}{s^2}.$$

We recognize the two entries of Table 6.2. That is

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \qquad \text{and} \qquad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\,\frac{1}{s^2}\right\} = \int_0^t \tau e^{-(t-\tau)}\,d\tau = e^{-t} + t - 1.$$

The calculation of the integral involved an integration by parts.

6.3.2 Solving ODEs

The next example demonstrates the full power of the convolution and the Laplace transform. We can give the solution to the forced oscillation problem for any forcing function as a definite integral.

Example 6.3.4: Find the solution to

$$x'' + \omega_0^2 x = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$.

We first apply the Laplace transform to the equation. Denote the transform of $x(t)$ by $X(s)$ and the transform of $f(t)$ by $F(s)$ as usual.

$$s^2 X(s) + \omega_0^2 X(s) = F(s),$$

or in other words

$$X(s) = F(s)\frac{1}{s^2+\omega_0^2}.$$

We know

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega_0^2}\right\} = \frac{\sin(\omega_0 t)}{\omega_0}.$$

Therefore,

$$x(t) = \int_0^t f(\tau)\frac{\sin\big(\omega_0(t-\tau)\big)}{\omega_0}\,d\tau,$$

or if we reverse the order

$$x(t) = \int_0^t \frac{\sin(\omega_0\tau)}{\omega_0} f(t-\tau)\,d\tau.$$

Let us notice one more feature of this example. We can now see how Laplace transform handles resonance. Suppose that $f(t) = \cos(\omega_0 t)$. Then

$$x(t) = \int_0^t \frac{\sin(\omega_0\tau)}{\omega_0}\cos\big(\omega_0(t-\tau)\big)\,d\tau = \frac{1}{\omega_0}\int_0^t \sin(\omega_0\tau)\cos\big(\omega_0(t-\tau)\big)\,d\tau.$$

We have computed the convolution of sine and cosine in Example 6.3.2. Hence

$$x(t) = \left(\frac{1}{\omega_0}\right)\left(\frac{1}{2}\,t\,\sin(\omega_0 t)\right) = \frac{1}{2\omega_0}\,t\,\sin(\omega_0 t).$$

Note the t in front of the sine. The solution, therefore, grows without bound as t gets large, meaning we get resonance.

Similarly, we can solve any constant coefficient equation with an arbitrary forcing function $f(t)$ as a definite integral using convolution. A definite integral, rather than a closed form solution, is usually enough for most practical purposes. It is not hard to numerically evaluate a definite integral.

6.3.3 Volterra integral equation

A common integral equation is the *Volterra integral equation**

$$x(t) = f(t) + \int_0^t g(t-\tau)x(\tau)\,d\tau,$$

where $f(t)$ and $g(t)$ are known functions and $x(t)$ is an unknown we wish to solve for. To find $x(t)$, we apply the Laplace transform to the equation to obtain

$$X(s) = F(s) + G(s)X(s),$$

*Named for the Italian mathematician Vito Volterra (1860–1940).

where $X(s)$, $F(s)$, and $G(s)$ are the Laplace transforms of $x(t)$, $f(t)$, and $g(t)$ respectively. We find

$$X(s) = \frac{F(s)}{1 - G(s)}.$$

To find $x(t)$ we now need to find the inverse Laplace transform of $X(s)$.

Example 6.3.5: Solve

$$x(t) = e^{-t} + \int_0^t \sinh(t - \tau)x(\tau)\, d\tau.$$

We apply Laplace transform to obtain

$$X(s) = \frac{1}{s+1} + \frac{1}{s^2 - 1} X(s),$$

or

$$X(s) = \frac{\frac{1}{s+1}}{1 - \frac{1}{s^2-1}} = \frac{s-1}{s^2-2} = \frac{s}{s^2-2} - \frac{1}{s^2-2}.$$

It is not hard to apply Table 6.1 on page 205 to find

$$x(t) = \cosh\left(\sqrt{2}\, t\right) - \frac{1}{\sqrt{2}} \sinh\left(\sqrt{2}\, t\right).$$

6.3.4 Exercises

***Exercise* 6.3.1:** *Let $f(t) = t^2$ for $t \geq 0$, and $g(t) = u(t-1)$. Compute $f * g$.*

***Exercise* 6.3.2:** *Let $f(t) = t$ for $t \geq 0$, and $g(t) = \sin t$ for $t \geq 0$. Compute $f * g$.*

***Exercise* 6.3.3:** *Find the solution to*

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0$, $c > 0$, $k > 0$, and $c^2 - 4km > 0$ (system is overdamped). Write the solution as a definite integral.

***Exercise* 6.3.4:** *Find the solution to*

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0$, $c > 0$, $k > 0$, and $c^2 - 4km < 0$ (system is underdamped). Write the solution as a definite integral.

***Exercise* 6.3.5:** *Find the solution to*

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0$, $c > 0$, $k > 0$, and $c^2 = 4km$ (system is critically damped). Write the solution as a definite integral.

***Exercise* 6.3.6:** *Solve*

$$x(t) = e^{-t} + \int_0^t \cos(t-\tau)x(\tau)\,d\tau.$$

***Exercise* 6.3.7:** *Solve*

$$x(t) = \cos t + \int_0^t \cos(t-\tau)x(\tau)\,d\tau.$$

***Exercise* 6.3.8:** *Compute* $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$ *using convolution.*

***Exercise* 6.3.9:** *Write down the solution to* $x'' - 2x = e^{-t^2}$, $x(0) = 0$, $x'(0) = 0$ *as a definite integral. Hint: Do not try to compute the Laplace transform of* e^{-t^2}.

***Exercise* 6.3.51:** *Apply Theorem* 6.3.1 *to find the inverse Laplace transform of each of the following functions. Check each result by showing that* $\mathcal{L}[(f*g)(t)] = \mathcal{L}[f(t)]\mathcal{L}[g(t)] = F(s)G(s) = H(s)$.

a) $H(s) = \frac{1}{(s+2)(s-3)}$

b) $H(s) = \frac{2}{s(s+4)}$

c) $H(s) = \frac{4}{s(s^2+9)}$

d) $H(s) = \frac{1}{(s-2)^2}$

e) $H(s) = \frac{1}{s^2(s+1)}$

***Exercise* 6.3.101:** *Let* $f(t) = \cos t$ *for* $t \geq 0$, *and* $g(t) = e^{-t}$. *Compute* $f * g$.

***Exercise* 6.3.102:** *Compute* $\mathcal{L}^{-1}\left\{\frac{5}{s^4+s^2}\right\}$ *using convolution.*

***Exercise* 6.3.103:** *Solve* $x'' + x = \sin t$, $x(0) = 0$, $x'(0) = 0$ *using convolution.*

***Exercise* 6.3.104:** *Solve* $x''' + x' = f(t)$, $x(0) = 0$, $x'(0) = 0$, $x''(0) = 0$ *using convolution. Write the result as a definite integral.*

***Exercise* 6.3.151:** *Compute the convolution* $f * g$ *for the following functions*:

a) $f(t) = 3,\ g(t) = \cos 2t$

b) $f(t) = (t+3)^2,\ g(t) = 2$

c) $f(t) = e^{-2t},\ g(t) = e^{3t}$

d) $f(t) = t,\ g(t) = \sin t$

e) $f(t) = t,\ g(t) = e^{2t}$

6.4 Dirac delta and impulse response

Note: 1 or 1.5 lecture, §7.6 in [EP], §6.5 in [BD]

6.4.1 Rectangular pulse

Often in applications we study a physical system by putting in a short pulse and then seeing what the system does. The resulting behavior is often called *impulse response*. Let us see what we mean by a pulse. The simplest kind of a pulse is a simple rectangular pulse defined by

$$\varphi(t) = \begin{cases} 0 & \text{if } \quad t < a, \\ M & \text{if } \ a \leq t < b, \\ 0 & \text{if } \ b \leq t. \end{cases}$$

See Figure 6.3 for a graph.

Notice that

$$\varphi(t) = M\big(u(t-a) - u(t-b)\big),$$

where $u(t)$ is the unit step function.

Let us take the Laplace transform of a square pulse,

$$\begin{aligned} \mathcal{L}\{\varphi(t)\} &= \mathcal{L}\big\{M\big(u(t-a) - u(t-b)\big)\big\} \\ &= M\frac{e^{-as} - e^{-bs}}{s}. \end{aligned}$$

For simplicity we let $a = 0$, and it is convenient to set $M = 1/b$ to have

$$\int_0^\infty \varphi(t)\, dt = 1.$$

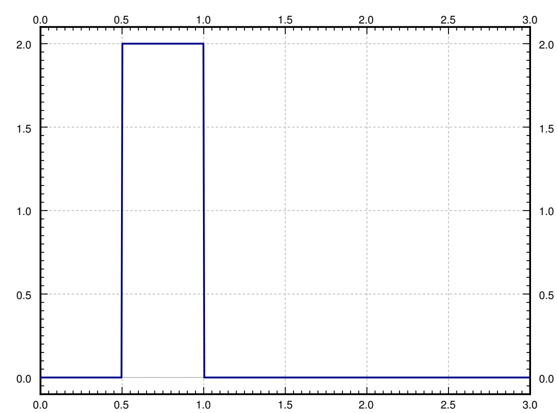


Figure 6.3: Sample square pulse with $a = 0.5$, $b = 1$ and $M = 2$.

That is, to have the pulse have "unit mass." For such a pulse we compute

$$\mathcal{L}\{\varphi(t)\} = \mathcal{L}\left\{\frac{u(t) - u(t-b)}{b}\right\} = \frac{1 - e^{-bs}}{bs}.$$

We generally want b to be very small. That is, we wish to have the pulse be very short and very tall. By letting b go to zero we arrive at the concept of the Dirac delta function.

6.4.2 The delta function

The *Dirac delta function*[*] is not exactly a function; it is sometimes called a *generalized function*. We avoid unnecessary details and simply say that it is an object that does not really make sense unless we integrate it. The motivation is that we would like a "function" $\delta(t)$ such that for any continuous function $f(t)$ we have

$$\boxed{\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0).}$$

The formula should hold if we integrate over any interval that contains 0, not just $(-\infty, \infty)$. So $\delta(t)$ is a "function" with all its "mass" at the single point $t = 0$. In other words, for any interval $[c, d]$

$$\int_{c}^{d} \delta(t)\, dt = \begin{cases} 1 & \text{if the interval } [c,d] \text{ contains 0, i.e. } c \leq 0 \leq d, \\ 0 & \text{otherwise.} \end{cases}$$

Unfortunately there is no such function in the classical sense. You could informally think that $\delta(t)$ is zero for $t \neq 0$ and somehow infinite at $t = 0$.

A good way to think about $\delta(t)$ is as a limit of short pulses whose integral is 1. For example, suppose that we have a square pulse $\varphi(t)$ as above with $a = 0$, $M = 1/b$, that is $\varphi(t) = \frac{u(t)-u(t-b)}{b}$. Compute

$$\int_{-\infty}^{\infty} \varphi(t) f(t)\, dt = \int_{-\infty}^{\infty} \frac{u(t)-u(t-b)}{b} f(t)\, dt = \frac{1}{b} \int_{0}^{b} f(t)\, dt.$$

If $f(t)$ is continuous at $t = 0$, then for very small b, the function $f(t)$ is approximately equal to $f(0)$ on the interval $[0, b]$. We approximate the integral

$$\frac{1}{b} \int_{0}^{b} f(t)\, dt \approx \frac{1}{b} \int_{0}^{b} f(0)\, dt = f(0).$$

Hence,

$$\lim_{b \to 0} \int_{-\infty}^{\infty} \varphi(t) f(t)\, dt = \lim_{b \to 0} \frac{1}{b} \int_{0}^{b} f(t)\, dt = f(0).$$

Let us therefore accept $\delta(t)$ as an object that is possible to integrate. We often want to shift δ to another point, for example $\delta(t-a)$. In that case we have

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt = f(a).$$

Note that $\delta(a-t)$ is the same object as $\delta(t-a)$. In other words, the convolution of $\delta(t)$ with $f(t)$ is again $f(t)$,

$$(f * \delta)(t) = \int_{0}^{t} \delta(t-s) f(s)\, ds = f(t).$$

[*]Named after the English physicist and mathematician Paul Adrien Maurice Dirac (1902–1984).

As we can integrate $\delta(t)$, let us compute its Laplace transform.

$$\boxed{\mathcal{L}\{\delta(t-a)\} = \int_0^\infty e^{-st}\delta(t-a)\,dt = e^{-as}.}$$

In particular,

$$\mathcal{L}\{\delta(t)\} = 1.$$

Remark 6.4.1: Notice that the Laplace transform of $\delta(t-a)$ looks like the Laplace transform of the derivative of the Heaviside function $u(t-a)$, if we could differentiate the Heaviside function. First notice

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}.$$

To obtain what the Laplace transform of the derivative would be we multiply by s, to obtain e^{-as}, which is the Laplace transform of $\delta(t-a)$. We see the same thing using integration,

$$\int_0^t \delta(s-a)\,ds = u(t-a).$$

So in a certain sense

$$\text{“}\quad \frac{d}{dt}\Big[u(t-a)\Big] = \delta(t-a). \quad\text{”}$$

This line of reasoning allows us to talk about derivatives of functions with jump discontinuities. We can think of the derivative of the Heaviside function $u(t-a)$ as being somehow infinite at a, which is precisely our intuitive understanding of the delta function.

Example 6.4.1: Let us compute $\mathcal{L}^{-1}\left\{\frac{s+1}{s}\right\}$. So far we have always looked at proper rational functions in the s variable. That is, the numerator was always of lower degree than the denominator. Not so with $\frac{s+1}{s}$. We write,

$$\mathcal{L}^{-1}\left\{\frac{s+1}{s}\right\} = \mathcal{L}^{-1}\left\{1+\frac{1}{s}\right\} = \mathcal{L}^{-1}\{1\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = \delta(t) + 1.$$

The resulting object is a generalized function and only makes sense when put underneath an integral.

6.4.3 Impulse response

As we said before, in the differential equation $Lx = f(t)$, we think of $f(t)$ as input, and $x(t)$ as the output. Often it is important to find the response to an impulse, and then we use the delta function in place of $f(t)$. The solution to

$$Lx = \delta(t)$$

is called the *impulse response*.

Example 6.4.2: Solve (find the impulse response)

$$x'' + \omega_0^2 x = \delta(t), \quad x(0) = 0, \quad x'(0) = 0. \tag{6.3}$$

We first apply the Laplace transform to the equation. Denote the transform of $x(t)$ by $X(s)$.

$$s^2 X(s) + \omega_0^2 X(s) = 1, \qquad \text{and so} \qquad X(s) = \frac{1}{s^2 + \omega_0^2}.$$

Taking the inverse Laplace transform we obtain

$$x(t) = \frac{\sin(\omega_0 t)}{\omega_0}.$$

Let us notice something about the example above. We showed before that when the input is $f(t)$, then the solution to $Lx = f(t)$ is given by

$$x(t) = \int_0^t f(\tau) \frac{\sin\big(\omega_0(t-\tau)\big)}{\omega_0} \, d\tau.$$

That is, the solution for an arbitrary input is given as convolution with the impulse response. Let us see why. The key is to notice that for functions $x(t)$ and $f(t)$,

$$(x * f)''(t) = \frac{d^2}{dt^2}\left[\int_0^t f(\tau) x(t-\tau) \, d\tau\right] = \int_0^t f(\tau) x''(t-\tau) \, d\tau = (x'' * f)(t).$$

We simply differentiate twice under the integral*, the details are left as an exercise. If we convolve the entire equation (6.3), the left-hand side becomes

$$(x'' + \omega_0^2 x) * f = (x'' * f) + \omega_0^2 (x * f) = (x * f)'' + \omega_0^2 (x * f).$$

The right-hand side becomes

$$(\delta * f)(t) = f(t).$$

Therefore $y(t) = (x * f)(t)$ is the solution to

$$y'' + \omega_0^2 y = f(t).$$

This procedure works in general for other linear equations $Lx = f(t)$. If you determine the impulse response, you also know how to obtain the output $x(t)$ for any input $f(t)$ by simply convolving the impulse response and the input $f(t)$.

*You should really think of the integral going over $(-\infty, \infty)$ rather than over $[0, t]$ and simply assume that $f(t)$ and $x(t)$ are continuous and zero for negative t.

6.4.4 Three-point beam bending

Let us give another quite different example where delta functions turn up. In this case representing point loads on a steel beam. Suppose we have a beam of length L, resting on two simple supports at the ends. Let x denote the position on the beam, and let $y(x)$ denote the deflection of the beam in the vertical direction. The deflection $y(x)$ satisfies the *Euler–Bernoulli equation**,

$$EI \frac{d^4 y}{dx^4} = F(x),$$

where E and I are constants† and $F(x)$ is the force applied per unit length at position x. The situation we are interested in is when the force is applied at a single point as in Figure 6.4.

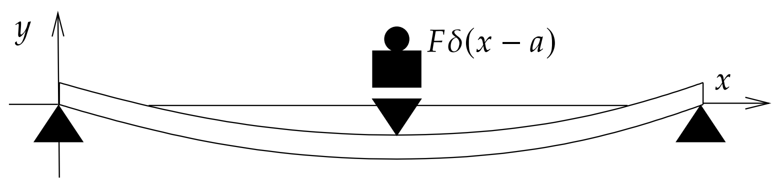


Figure 6.4: Three-point bending.

In this case the equation becomes

$$EI \frac{d^4 y}{dx^4} = -F\delta(x-a),$$

where $x = a$ is the point where the mass is applied. F is the force applied and the minus sign indicates that the force is downward, that is, in the negative y direction. The end points of the beam satisfy the conditions,

$$\begin{aligned} y(0) &= 0, \qquad & y''(0) &= 0, \\ y(L) &= 0, \qquad & y''(L) &= 0. \end{aligned}$$

See § 5.2 for further information about endpoint conditions applied to beams.

Example 6.4.3: Suppose that length of the beam is 2, and suppose that $EI = 1$ for simplicity. Further suppose that the force $F = 1$ is applied at $x = 1$. That is, we have the equation

$$\frac{d^4 y}{dx^4} = -\delta(x-1),$$

and the endpoint conditions are

$$y(0) = 0, \qquad y''(0) = 0, \qquad y(2) = 0, \qquad y''(2) = 0.$$

*Named for the Swiss mathematicians Jacob Bernoulli (1654–1705), Daniel Bernoulli (1700–1782), the nephew of Jacob, and Leonhard Paul Euler (1707–1783).

†E is the elastic modulus and I is the second moment of area. Let us not worry about the details and simply think of these as some given constants.

We could integrate, but using the Laplace transform is even easier. We apply the transform in the x variable rather than the t variable. Let us again denote the transform of $y(x)$ as $Y(s)$.

$$s^4 Y(s) - s^3 y(0) - s^2 y'(0) - s y''(0) - y'''(0) = -e^{-s}.$$

We notice that $y(0) = 0$ and $y''(0) = 0$. Let us call $C_1 = y'(0)$ and $C_2 = y'''(0)$. We solve for $Y(s)$,

$$Y(s) = \frac{-e^{-s}}{s^4} + \frac{C_1}{s^2} + \frac{C_2}{s^4}.$$

We take the inverse Laplace transform utilizing the second shifting property (6.1) to take the inverse of the first term.

$$y(x) = \frac{-(x-1)^3}{6} u(x-1) + C_1 x + \frac{C_2}{6} x^3.$$

We still need to apply two of the endpoint conditions. As the conditions are at $x = 2$ we can simply replace $u(x-1) = 1$ when taking the derivatives. Therefore,

$$0 = y(2) = \frac{-(2-1)^3}{6} + C_1(2) + \frac{C_2}{6} 2^3 = \frac{-1}{6} + 2C_1 + \frac{4}{3} C_2,$$

and

$$0 = y''(2) = \frac{-3 \cdot 2 \cdot (2-1)}{6} + \frac{C_2}{6} 3 \cdot 2 \cdot 2 = -1 + 2C_2.$$

Hence $C_2 = \frac{1}{2}$ and solving for C_1 using the first equation we obtain $C_1 = \frac{-1}{4}$. Our solution for the beam deflection is

$$y(x) = \frac{-(x-1)^3}{6} u(x-1) - \frac{x}{4} + \frac{x^3}{12}.$$

6.4.5 Exercises

***Exercise* 6.4.1:** *Solve (find the impulse response)* $x'' + x' + x = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

***Exercise* 6.4.2:** *Solve (find the impulse response)* $x'' + 2x' + x = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

***Exercise* 6.4.3:** *A pulse can come later and can be bigger. Solve* $x'' + 4x = 4\delta(t-1)$, $x(0) = 0$, $x'(0) = 0$.

***Exercise* 6.4.4:** *Suppose that* $f(t)$ *and* $g(t)$ *are differentiable functions and suppose that* $f(t) = g(t) = 0$ *for all* $t \leq 0$. *Show that*

$$(f * g)'(t) = (f' * g)(t) = (f * g')(t).$$

***Exercise* 6.4.5:** *Suppose that* $Lx = \delta(t)$, $x(0) = 0$, $x'(0) = 0$, *has the solution* $x = e^{-t}$ *for* $t > 0$. *Find the solution to* $Lx = t^2$, $x(0) = 0$, $x'(0) = 0$ *for* $t > 0$.

***Exercise* 6.4.6:** *Compute* $\mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s^2}\right\}$.

Exercise* 6.4.7** (challenging):*** *Solve Example 6.4.3 via integrating* 4 *times in the* x *variable.*

Exercise* 6.4.8*: *Suppose we have a beam of length* 1 *simply supported at the ends and suppose that force* $F = 1$ *is applied at* $x = \frac{3}{4}$ *in the downward direction. Suppose that* $EI = 1$ *for simplicity. Find the beam deflection* $y(x)$*.*

Exercise* 6.4.101*: *Solve (find the impulse response)* $x'' = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise* 6.4.102*: *Solve (find the impulse response)* $x' + ax = \delta(t)$, $x(0) = 0$, $x'(0) = 0$.

Exercise* 6.4.103*: *Suppose that* $Lx = \delta(t)$, $x(0) = 0$, $x'(0) = 0$, *has the solution* $x(t) = \cos(t)$ *for* $t > 0$. *Find (in closed form) the solution to* $Lx = \sin(t)$, $x(0) = 0$, $x'(0) = 0$ *for* $t > 0$.

Exercise* 6.4.104*: *Compute* $\mathcal{L}^{-1}\left\{\frac{s^2}{s^2+1}\right\}$.

Exercise* 6.4.105*: *Compute* $\mathcal{L}^{-1}\left\{\frac{3s^2e^{-s}+2}{s^2}\right\}$.

Exercise* 6.4.151*: *Solve the following IVPs:*

a) $x'' + 16x = \delta(t-3);\ x(0) = 1,\ x'(0) = 0$

b) $x'' + 9x = 1 + \delta(t-4);\ x(0) = x'(0) = 0$

c) $x'' + x' - 6x = 4\delta(t-2);\ x(0) = 0,\ x'(0) = 4$

d) $x'' + 6x' + 9x = \delta(t-4);\ x(0) = 0,\ x'(0) = 2$

e) $x'' + 2x' + 10x = \delta(t-\pi);\ x(0) = x'(0) = 0$

Chapter 7

Power series methods

7.1 Power series

Note: 1 or 1.5 lecture, §8.1 in [EP], §5.1 in [BD]

Many functions can be written in terms of a power series

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k.$$

If we assume that a solution of a differential equation is written as a power series, then perhaps we can use a method reminiscent of undetermined coefficients. That is, we will try to solve for the numbers a_k. Before we can carry out this process, let us review some results and concepts about power series.

7.1.1 Definition

As we said, a *power series* is an expression such as

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots, \tag{7.1}$$

where $a_0, a_1, a_2, \ldots, a_k, \ldots$ and x_0 are constants. Let

$$S_n(x) = \sum_{k=0}^{n} a_k (x - x_0)^k = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \cdots + a_n(x - x_0)^n,$$

denote the so-called *partial sum*. If for some x, the limit

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (x - x_0)^k$$

exists, then we say that the series (7.1) *converges* at x. At $x = x_0$, the series always converges to a_0. When (7.1) converges at any other point $x \neq x_0$, we say that (7.1) is a *convergent power series*, and we write

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (x-x_0)^k .$$

If the series does not converge for any point $x \neq x_0$, we say that the series is *divergent*.

Example 7.1.1: The series

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

is convergent for any x. Recall that $k! = 1 \cdot 2 \cdot 3 \cdots k$ is the factorial. By convention we define $0! = 1$. You may recall that this series converges to e^x.

We say that (7.1) *converges absolutely* at x whenever the limit

$$\lim_{n\to\infty} \sum_{k=0}^{n} |a_k| \, |x-x_0|^k$$

exists. That is, the series $\sum_{k=0}^{\infty} |a_k| \, |x-x_0|^k$ is convergent. If (7.1) converges absolutely at x, then it converges at x. However, the opposite implication is not true.

Example 7.1.2: The series

$$\sum_{k=1}^{\infty} \frac{1}{k} x^k$$

converges absolutely for all x in the interval $(-1,1)$. It converges at $x = -1$, as $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges (conditionally) by the alternating series test. The power series does not converge absolutely at $x = -1$, because $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge. The series diverges at $x = 1$.

7.1.2 Radius of convergence

If a power series converges absolutely at some x_1, then for all x such that $|x - x_0| \leq |x_1 - x_0|$ (that is, x is closer than x_1 to x_0) we have $\left|a_k (x-x_0)^k\right| \leq \left|a_k (x_1-x_0)^k\right|$ for all k. As the numbers $\left|a_k (x_1-x_0)^k\right|$ sum to some finite limit, summing smaller positive numbers $\left|a_k (x-x_0)^k\right|$ must also have a finite limit. Hence, the series must converge absolutely at x.

Theorem 7.1.1. *For a power series (7.1), there exists a number ρ (we allow $\rho = \infty$) called the* radius of convergence *such that the series converges absolutely on the interval $(x_0 - \rho, x_0 + \rho)$ and diverges for $x < x_0 - \rho$ and $x > x_0 + \rho$. We write $\rho = \infty$ if the series converges for all x.*

See Figure 7.1 on the facing page. In Example 7.1.1 the radius of convergence is $\rho = \infty$ as the series converges everywhere. In Example 7.1.2 the radius of convergence is $\rho = 1$. We note that $\rho = 0$ is another way of saying that the series is divergent.

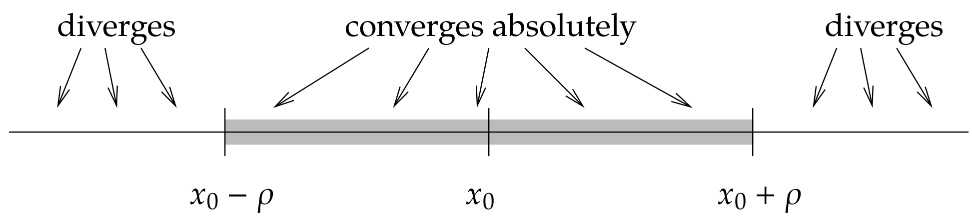


Figure 7.1: Convergence of a power series.

A useful test for convergence of a series is the *ratio test*. Suppose that

$$\sum_{k=0}^{\infty} c_k$$

is a series and the limit

$$L = \lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right|$$

exists. Then the series converges absolutely if $L < 1$ and diverges if $L > 1$.

We apply this test to the series (7.1). Let $c_k = a_k (x - x_0)^k$ in the test. Compute

$$L = \lim_{k\to\infty} \left| \frac{c_{k+1}}{c_k} \right| = \lim_{k\to\infty} \left| \frac{a_{k+1} (x - x_0)^{k+1}}{a_k (x - x_0)^k} \right| = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| |x - x_0|.$$

Define A by

$$A = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right|.$$

Then if $1 > L = A|x - x_0|$ the series (7.1) converges absolutely. If $A = 0$, then the series always converges. If $A > 0$, then the series converges absolutely if $|x - x_0| < 1/A$, and diverges if $|x - x_0| > 1/A$. That is, the radius of convergence is $1/A$.

A similar test is the *root test*. Suppose

$$L = \lim_{k\to\infty} \sqrt[k]{|c_k|}$$

exists. Then $\sum_{k=0}^{\infty} c_k$ converges absolutely if $L < 1$ and diverges if $L > 1$. We can use the same calculation as above to find A. Let us summarize.

Theorem 7.1.2 (Ratio and root tests for power series)**.** *Consider a power series*

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

such that

$$A = \lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| \qquad \text{or} \qquad A = \lim_{k\to\infty} \sqrt[k]{|a_k|}$$

exists. If $A = 0$, then the radius of convergence of the series is ∞. Otherwise, the radius of convergence is $1/A$.

Example 7.1.3: Suppose we have the series

$$\sum_{k=0}^{\infty} 2^{-k}(x-1)^k.$$

First we compute the limit in the ratio test,

$$A = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{2^{-k-1}}{2^{-k}}\right| = \lim_{k\to\infty} 2^{-1} = 1/2.$$

Therefore the radius of convergence is 2, and the series converges absolutely on the interval $(-1,3)$. And we could just as well have used the root test:

$$A = \lim_{k\to\infty} \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{|2^{-k}|} = \lim_{k\to\infty} 2^{-1} = 1/2.$$

Example 7.1.4: Consider

$$\sum_{k=0}^{\infty} \frac{1}{k^k} x^k.$$

Compute the limit for the root test,

$$A = \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left|\frac{1}{k^k}\right|} = \lim_{k\to\infty} \sqrt[k]{\left|\frac{1}{k}\right|^k} = \lim_{k\to\infty} \frac{1}{k} = 0.$$

So the radius of convergence is ∞: the series converges everywhere. The ratio test would also work here.

The root or the ratio test does not always apply. That is the limit of $\left|\frac{a_{k+1}}{a_k}\right|$ or $\sqrt[k]{|a_k|}$ might not exist. There exist more sophisticated ways of finding the radius of convergence, but those would be beyond the scope of this chapter. The two methods above cover many of the series that arise in practice. Often if the root test applies, so does the ratio test, and vice versa, though the limit might be easier to compute in one way than the other.

7.1.3 Analytic functions

Functions represented by power series are called *analytic functions*. Not every function is analytic, although the majority of the functions you have seen in calculus are.

An analytic function $f(x)$ is equal to its *Taylor series** near a point x_0. That is, for x near x_0 we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \tag{7.2}$$

where $f^{(k)}(x_0)$ denotes the k^{th} derivative of $f(x)$ at the point x_0.

*Named after the English mathematician Sir Brook Taylor (1685–1731).

For example, sine is an analytic function and its Taylor series around $x_0 = 0$ is given by

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

In Figure 7.2 we plot $\sin(x)$ and the truncations of the series up to degree 5 and 9. You can see that the approximation is very good for x near 0, but gets worse for x further away from 0. This is what happens in general. To get a good approximation far away from x_0 you need to take more and more terms of the Taylor series.

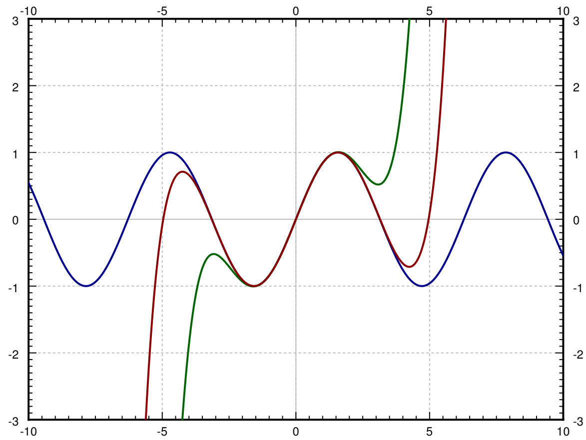


Figure 7.2: The sine function and its Taylor approximations around $x_0 = 0$ of 5^{th} and 9^{th} degree.

7.1.4 Manipulating power series

One of the main properties of power series that we will use is that we can differentiate them term by term. That is, suppose that $\sum a_k (x - x_0)^k$ is a convergent power series. Then for x in the radius of convergence we have

$$\frac{d}{dx}\left[\sum_{k=0}^{\infty} a_k (x - x_0)^k\right] = \sum_{k=1}^{\infty} k a_k (x - x_0)^{k-1}.$$

Notice that the term corresponding to $k = 0$ disappeared as it was constant. The radius of convergence of the differentiated series is the same as that of the original.

Example 7.1.5: Let us show that the exponential $y = e^x$ solves $y' = y$. First write

$$y = e^x = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

Now differentiate

$$y' = \sum_{k=1}^{\infty} k \frac{1}{k!} x^{k-1} = \sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1}.$$

We *reindex* the series by simply replacing k with $k+1$. The series does not change, what changes is simply how we write it. After reindexing the series starts at $k=0$ again.

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \sum_{k+1=1}^{\infty} \frac{1}{\big((k+1)-1\big)!} x^{(k+1)-1} = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

That was precisely the power series for e^x that we started with, so we showed that $\frac{d}{dx}[e^x] = e^x$.

Convergent power series can be added and multiplied together, and multiplied by constants using the following rules. First, we can add series by adding term by term,

$$\left(\sum_{k=0}^{\infty} a_k (x-x_0)^k \right) + \left(\sum_{k=0}^{\infty} b_k (x-x_0)^k \right) = \sum_{k=0}^{\infty} (a_k + b_k)(x-x_0)^k.$$

We can multiply by constants,

$$\alpha \left(\sum_{k=0}^{\infty} a_k (x-x_0)^k \right) = \sum_{k=0}^{\infty} \alpha a_k (x-x_0)^k.$$

We can also multiply series together,

$$\left(\sum_{k=0}^{\infty} a_k (x-x_0)^k \right) \left(\sum_{k=0}^{\infty} b_k (x-x_0)^k \right) = \sum_{k=0}^{\infty} c_k (x-x_0)^k,$$

where $c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$. The radius of convergence of the sum or the product is at least the minimum of the radii of convergence of the two series involved.

7.1.5 Power series for rational functions

Polynomials are simply finite power series. That is, a polynomial is a power series where the a_k are zero for all k large enough. We can always expand a polynomial as a power series about any point x_0 by writing the polynomial as a polynomial in $(x-x_0)$. For example, let us write $2x^2 - 3x + 4$ as a power series around $x_0 = 1$:

$$2x^2 - 3x + 4 = 3 + (x-1) + 2(x-1)^2.$$

In other words $a_0 = 3$, $a_1 = 1$, $a_2 = 2$, and all other $a_k = 0$. To do this, we know that $a_k = 0$ for all $k \geq 3$ as the polynomial is of degree 2. We write $a_0 + a_1(x-1) + a_2(x-1)^2$, we expand, and we solve for a_0, a_1, and a_2. We could have also differentiated at $x = 1$ and used the Taylor series formula (7.2).

Let us look at rational functions, that is, ratios of polynomials. An important fact is that a series for a function only defines the function on an interval even if the function is defined elsewhere. For example, for $-1 < x < 1$ we have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

This series is called the *geometric series*. The ratio test tells us that the radius of convergence is 1. The series diverges for $x \leq -1$ and $x \geq 1$, even though $\frac{1}{1-x}$ is defined for all $x \neq 1$.

We can use the geometric series together with rules for addition and multiplication of power series to expand rational functions around a point, as long as the denominator is not zero at x_0. Note that as for polynomials, we could equivalently use the Taylor series expansion (7.2).

Example 7.1.6: Expand $\frac{x}{1+2x+x^2}$ as a power series around the origin ($x_0 = 0$) and find the radius of convergence.

First, write $1 + 2x + x^2 = (1+x)^2 = \big(1-(-x)\big)^2$. Compute

$$\begin{aligned} \frac{x}{1+2x+x^2} &= x\left(\frac{1}{1-(-x)}\right)^2 \\ &= x\left(\sum_{k=0}^{\infty} (-1)^k x^k\right)^2 \\ &= x\left(\sum_{k=0}^{\infty} c_k x^k\right) \\ &= \sum_{k=0}^{\infty} c_k x^{k+1}, \end{aligned}$$

where to get c_k, we use the formula for the product of series. We obtain, $c_0 = 1$, $c_1 = -1-1 = -2$, $c_2 = 1+1+1 = 3$, etc. Therefore

$$\frac{x}{1+2x+x^2} = \sum_{k=1}^{\infty} (-1)^{k+1} k x^k = x - 2x^2 + 3x^3 - 4x^4 + \cdots$$

The radius of convergence is at least 1. We use the ratio test

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{(-1)^{k+2}(k+1)}{(-1)^{k+1}k}\right| = \lim_{k\to\infty} \frac{k+1}{k} = 1.$$

So the radius of convergence is actually equal to 1.

When the rational function is more complicated, it is also possible to use method of partial fractions. For example, to find the Taylor series for $\frac{x^3+x}{x^2-1}$, we write

$$\frac{x^3+x}{x^2-1} = x + \frac{1}{1+x} - \frac{1}{1-x} = x + \sum_{k=0}^{\infty} (-1)^k x^k - \sum_{k=0}^{\infty} x^k = -x + \sum_{\substack{k=3 \\ k \text{ odd}}}^{\infty} (-2) x^k.$$

7.1.6 Exercises

***Exercise* 7.1.1:** *Is the power series* $\sum_{k=0}^{\infty} e^k x^k$ *convergent? If so, what is the radius of convergence?*

Exercise 7.1.2: *Is the power series* $\sum_{k=0}^{\infty} kx^k$ *convergent? If so, what is the radius of convergence?*

Exercise 7.1.3: *Is the power series* $\sum_{k=0}^{\infty} k!x^k$ *convergent? If so, what is the radius of convergence?*

Exercise 7.1.4: *Is the power series* $\sum_{k=0}^{\infty} \frac{1}{(2k)!}(x-10)^k$ *convergent? If so, what is the radius of convergence?*

Exercise 7.1.5: *Determine the Taylor series for* $\sin x$ *around the point* $x_0 = \pi$.

Exercise 7.1.6: *Determine the Taylor series for* $\ln x$ *around the point* $x_0 = 1$*, and find the radius of convergence.*

Exercise 7.1.7: *Determine the Taylor series and its radius of convergence of* $\frac{1}{1+x}$ *around* $x_0 = 0$.

Exercise 7.1.8: *Determine the Taylor series and its radius of convergence of* $\frac{x}{4-x^2}$ *around* $x_0 = 0$. *Hint: You will not be able to use the ratio test.*

Exercise 7.1.9: *Expand* $x^5 + 5x + 1$ *as a power series around* $x_0 = 5$.

Exercise 7.1.10: *Suppose that the ratio test applies to a series* $\sum_{k=0}^{\infty} a_k x^k$. *Show, using the ratio test, that the radius of convergence of the differentiated series is the same as that of the original series.*

Exercise 7.1.11: *Suppose that* f *is an analytic function such that* $f^{(n)}(0) = n$. *Find* $f(1)$.

Exercise 7.1.51: *For each of the following* 1^{st}*-order DEs:*

i. *Find the solution such that* $y(0) = y_0$ *by separation of variables.*

ii. *Find the solution such that* $y(0) = y_0$ *by the power series method about* $x_0 = 0$.

iii. *Using the power series expansions for the functions* e^x *and* $\frac{1}{1-x}$*, show that the power series of the function* $y(x)$ *found in i. is the same as the power series solution found in ii.*

a) $y' + 3y = 0$

b) $2y' + 5y = 0$

c) $3y' = 2y$

d) $y' + xy = 0$

e) $(x-1)y' + y = 0$

f) $(2 + x)y' + y = 0$

g) $(1 - 3x)y' - 3y = 0$

Exercise* 7.1.101*: *Is the power series* $\displaystyle\sum_{n=1}^{\infty} (0.1)^n x^n$ *convergent? If so, what is the radius of convergence?*

Exercise* 7.1.102** (challenging):*** *Is the power series* $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ *convergent? If so, what is the radius of convergence?*

Exercise* 7.1.103*: *Using the geometric series, expand* $\frac{1}{1-x}$ *around* $x_0 = 2$. *For what* x *does the series converge?*

Exercise* 7.1.104** (challenging):*** *Find the Taylor series for* $x^7 e^x$ *around* $x_0 = 0$.

Exercise* 7.1.105** (challenging):*** *Imagine* f *and* g *are analytic functions such that* $f^{(k)}(0) = g^{(k)}(0)$ *for all large enough* k. *What can you say about* $f(x) - g(x)$*?*

7.2 Series solutions of linear second order ODEs

Note: 1 or 1.5 lecture, §8.2 in [EP], §5.2 and §5.3 in [BD]

Suppose we have a linear second order homogeneous ODE of the form

$$p(x)y'' + q(x)y' + r(x)y = 0.$$

Suppose that $p(x)$, $q(x)$, and $r(x)$ are polynomials. We will try a solution of the form

$$y = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

and solve for the a_k to try to obtain a solution defined in some interval around x_0.

The point x_0 is called an *ordinary point* if $p(x_0) \neq 0$. That is, the functions

$$\frac{q(x)}{p(x)} \qquad \text{and} \qquad \frac{r(x)}{p(x)}$$

are defined for x near x_0. If $p(x_0) = 0$, then we say x_0 is a *singular point*. Handling singular points is harder than ordinary points and so we now focus only on ordinary points.

Example 7.2.1: Let us start with a very simple example

$$y'' - y = 0.$$

Let us try a power series solution near $x_0 = 0$, which is an ordinary point. Every point is an ordinary point in fact, as the equation is constant coefficient. We already know we should obtain exponentials or the hyperbolic sine and cosine, but let us pretend we do not know this.

We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

If we differentiate, the $k = 0$ term is a constant and hence disappears. We therefore get

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

We differentiate yet again to obtain (now the $k = 1$ term disappears)

$$y'' = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}.$$

We reindex the series (replace k with $k + 2$) to obtain

$$y'' = \sum_{k=0}^{\infty} (k+2)\,(k+1)\,a_{k+2} x^k.$$

Now we plug y and y'' into the differential equation

$$\begin{aligned} 0 = y'' - y &= \left(\sum_{k=0}^{\infty} (k+2)(k+1)\, a_{k+2} x^k \right) - \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1)\, a_{k+2} x^k - a_k x^k \right) \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1)\, a_{k+2} - a_k \right) x^k . \end{aligned}$$

As $y'' - y$ is supposed to be equal to 0, we know that the coefficients of the resulting series must be equal to 0. Therefore,

$$(k+2)(k+1)\, a_{k+2} - a_k = 0, \qquad \text{or} \qquad a_{k+2} = \frac{a_k}{(k+2)(k+1)}.$$

The equation above is called a *recurrence relation* for the coefficients of the power series. It did not matter what a_0 or a_1 was. They can be arbitrary. But once we pick a_0 and a_1, then all other coefficients are determined by the recurrence relation.

Let us see what the coefficients must be. First, a_0 and a_1 are arbitrary

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{(3)(2)}, \quad a_4 = \frac{a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)}, \quad a_5 = \frac{a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)}, \quad \ldots$$

So we note that for even k, that is $k = 2n$ we get

$$a_k = a_{2n} = \frac{a_0}{(2n)!},$$

and for odd k, that is $k = 2n+1$ we have

$$a_k = a_{2n+1} = \frac{a_1}{(2n+1)!}.$$

Let us write down the series

$$y = \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty} \left(\frac{a_0}{(2n)!}\, x^{2n} + \frac{a_1}{(2n+1)!}\, x^{2n+1} \right) = a_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!}\, x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\, x^{2n+1}.$$

We recognize the two series as the hyperbolic sine and cosine. Therefore,

$$y = a_0 \cosh x + a_1 \sinh x .$$

Of course, in general we will not be able to recognize the series that appears, since usually there will not be any elementary function that matches it. In that case we will be content with the series.

Example 7.2.2: Let us do a more complex example. Consider *Airy's equation**:

$$y'' - xy = 0,$$

near the point $x_0 = 0$. Note that $x_0 = 0$ is an ordinary point.

*Named after the English mathematician Sir George Biddell Airy (1801–1892).

We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

We differentiate twice (as above) to obtain

$$y'' = \sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}.$$

We plug y into the equation

$$\begin{aligned} 0 = y'' - xy &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - x\left(\sum_{k=0}^{\infty} a_k x^k\right) \\ &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - \left(\sum_{k=0}^{\infty} a_k x^{k+1}\right). \end{aligned}$$

We reindex to make things easier to sum

$$\begin{aligned} 0 = y'' - xy &= \left(2a_2 + \sum_{k=1}^{\infty}(k+2)\,(k+1)\,a_{k+2} x^k\right) - \left(\sum_{k=1}^{\infty} a_{k-1} x^k\right) \\ &= 2a_2 + \sum_{k=1}^{\infty}\Big((k+2)\,(k+1)\,a_{k+2} - a_{k-1}\Big)\,x^k. \end{aligned}$$

Again $y'' - xy$ is supposed to be 0, so $a_2 = 0$, and

$$(k+2)\,(k+1)\,a_{k+2} - a_{k-1} = 0, \qquad \text{or} \qquad a_{k+2} = \frac{a_{k-1}}{(k+2)(k+1)}.$$

We jump in steps of three. First, since $a_2 = 0$ we must have , $a_5 = 0$, $a_8 = 0$, $a_{11} = 0$, etc. In general, $a_{3n+2} = 0$.

The constants a_0 and a_1 are arbitrary and we obtain

$$a_3 = \frac{a_0}{(3)(2)}, \quad a_4 = \frac{a_1}{(4)(3)}, \quad a_6 = \frac{a_3}{(6)(5)} = \frac{a_0}{(6)(5)(3)(2)}, \quad a_7 = \frac{a_4}{(7)(6)} = \frac{a_1}{(7)(6)(4)(3)}, \quad \ldots$$

For a_k where k is a multiple of 3, that is $k = 3n$ we notice that

$$a_{3n} = \frac{a_0}{(2)(3)(5)(6)\cdots(3n-1)(3n)}.$$

For a_k where $k = 3n+1$, we notice

$$a_{3n+1} = \frac{a_1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}.$$

In other words, if we write down the series for y, it has two parts

$$\begin{aligned} y &= \left(a_0 + \frac{a_0}{6}x^3 + \frac{a_0}{180}x^6 + \cdots + \frac{a_0}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n} + \cdots\right) \\ &\quad + \left(a_1 x + \frac{a_1}{12}x^4 + \frac{a_1}{504}x^7 + \cdots + \frac{a_1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}x^{3n+1} + \cdots\right) \\ &= a_0\left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots + \frac{1}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n} + \cdots\right) \\ &\quad + a_1\left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots + \frac{1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}x^{3n+1} + \cdots\right). \end{aligned}$$

We define

$$\begin{aligned} y_1(x) &= 1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots + \frac{1}{(2)(3)(5)(6)\cdots(3n-1)(3n)}x^{3n} + \cdots, \\ y_2(x) &= x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots + \frac{1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}x^{3n+1} + \cdots, \end{aligned}$$

and write the general solution to the equation as $y(x) = a_0 y_1(x) + a_1 y_2(x)$. If we plug in $x = 0$ into the power series for y_1 and y_2, we find $y_1(0) = 1$ and $y_2(0) = 0$. Similarly, $y_1'(0) = 0$ and $y_2'(0) = 1$. Therefore $y = a_0 y_1 + a_1 y_2$ is a solution that satisfies the initial conditions $y(0) = a_0$ and $y'(0) = a_1$.

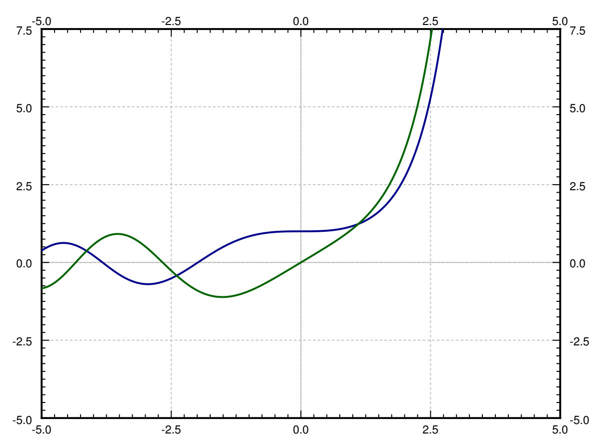


Figure 7.3: The two solutions y_1 and y_2 to Airy's equation.

The functions y_1 and y_2 cannot be written in terms of the elementary functions that you know. See Figure 7.3 for the plot of the solutions y_1 and y_2. These functions have many interesting properties. For example, they are oscillatory for negative x (like solutions to $y'' + y = 0$) and for positive x they grow without bound (like solutions to $y'' - y = 0$).

Sometimes a solution may turn out to be a polynomial.

Example 7.2.3: Let us find a solution to the so-called *Hermite's equation of order n**:

$$y'' - 2xy' + 2ny = 0.$$

Let us find a solution around the point $x_0 = 0$. We try

$$y = \sum_{k=0}^{\infty} a_k x^k.$$

We differentiate (as above) to obtain

$$y' = \sum_{k=1}^{\infty} k a_k x^{k-1},$$
$$y'' = \sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}.$$

Now we plug into the equation

$$\begin{aligned} 0 = y'' - 2xy' + 2ny & \\ &= \left(\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}\right) - 2x\left(\sum_{k=1}^{\infty} k a_k x^{k-1}\right) + 2n\left(\sum_{k=0}^{\infty} a_k x^k\right) \\ &= \left(\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}\right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k\right) + \left(\sum_{k=0}^{\infty} 2n a_k x^k\right) \\ &= \left(2a_2 + \sum_{k=1}^{\infty} (k+2)(k+1)a_{k+2} x^k\right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k\right) + \left(2na_0 + \sum_{k=1}^{\infty} 2n a_k x^k\right) \\ &= 2a_2 + 2na_0 + \sum_{k=1}^{\infty} \big((k+2)(k+1)a_{k+2} - 2ka_k + 2na_k\big)x^k. \end{aligned}$$

As $y'' - 2xy' + 2ny = 0$ we have

$$(k+2)(k+1)a_{k+2} + (-2k+2n)a_k = 0, \qquad \text{or} \qquad a_{k+2} = \frac{(2k-2n)}{(k+2)(k+1)} a_k.$$

This recurrence relation actually includes $a_2 = -na_0$ (which comes about from $2a_2 + 2na_0 = 0$). Again a_0 and a_1 are arbitrary.

$$a_2 = \frac{-2n}{(2)(1)} a_0, \qquad a_3 = \frac{2(1-n)}{(3)(2)} a_1,$$
$$a_4 = \frac{2(2-n)}{(4)(3)} a_2 = \frac{2^2(2-n)(-n)}{(4)(3)(2)(1)} a_0,$$

*Named after the French mathematician Charles Hermite (1822–1901).

$$a_5 = \frac{2(3-n)}{(5)(4)} a_3 = \frac{2^2(3-n)(1-n)}{(5)(4)(3)(2)} a_1, \quad \ldots$$

Let us separate the even and odd coefficients. We find that

$$\begin{aligned} a_{2m} &= \frac{2^m(-n)(2-n)\cdots(2m-2-n)}{(2m)!}, \\ a_{2m+1} &= \frac{2^m(1-n)(3-n)\cdots(2m-1-n)}{(2m+1)!}. \end{aligned}$$

Let us write down the two series, one with the even powers and one with the odd.

$$\begin{aligned} y_1(x) &= 1 + \frac{2(-n)}{2!}x^2 + \frac{2^2(-n)(2-n)}{4!}x^4 + \frac{2^3(-n)(2-n)(4-n)}{6!}x^6 + \cdots, \\ y_2(x) &= x + \frac{2(1-n)}{3!}x^3 + \frac{2^2(1-n)(3-n)}{5!}x^5 + \frac{2^3(1-n)(3-n)(5-n)}{7!}x^7 + \cdots. \end{aligned}$$

We then write

$$y(x) = a_0 y_1(x) + a_1 y_2(x).$$

We remark that if n is a positive even integer, then $y_1(x)$ is a polynomial as all the coefficients in the series beyond a certain degree are zero. If n is a positive odd integer, then $y_2(x)$ is a polynomial. For example, if $n = 4$, then

$$y_1(x) = 1 + \frac{2(-4)}{2!}x^2 + \frac{2^2(-4)(2-4)}{4!}x^4 = 1 - 4x^2 + \frac{4}{3}x^4.$$

7.2.1 Exercises

In the following exercises, when asked to solve an equation using power series methods, you should find the first few terms of the series, and if possible find a general formula for the k^{th} coefficient.

***Exercise* 7.2.1:** *Use power series methods to solve $y'' + y = 0$ at the point $x_0 = 1$.*

***Exercise* 7.2.2:** *Use power series methods to solve $y'' + 4xy = 0$ at the point $x_0 = 0$.*

***Exercise* 7.2.3:** *Use power series methods to solve $y'' - xy = 0$ at the point $x_0 = 1$.*

***Exercise* 7.2.4:** *Use power series methods to solve $y'' + x^2y = 0$ at the point $x_0 = 0$.*

***Exercise* 7.2.5:** *The methods work for other orders than second order. Try the methods of this section to solve the first order system $y' - xy = 0$ at the point $x_0 = 0$.*

***Exercise* 7.2.6** (Chebyshev's equation of order p)**:**

a) *Solve $(1 - x^2)y'' - xy' + p^2y = 0$ using power series methods at $x_0 = 0$.*

b) *For what p is there a polynomial solution?*

Exercise* 7.2.7*: *Find a polynomial solution to* $(x^2 + 1)y'' - 2xy' + 2y = 0$ *using power series methods.*

Exercise* 7.2.8*:

a) *Use power series methods to solve* $(1 - x)y'' + y = 0$ *at the point* $x_0 = 0$.

b) *Use the solution to part* a) *to find a solution for* $xy'' + y = 0$ *around the point* $x_0 = 1$.

Exercise* 7.2.101*: *Use power series methods to solve* $y'' + 2x^3y = 0$ *at the point* $x_0 = 0$.

Exercise* 7.2.102** (challenging):*** *Power series methods also work for nonhomogeneous equations.*

a) *Use power series methods to solve* $y'' - xy = \frac{1}{1-x}$ *at the point* $x_0 = 0$. *Hint: Recall the geometric series.*

b) *Now solve for the initial condition* $y(0) = 0$, $y'(0) = 0$.

Exercise* 7.2.103*: *Attempt to solve* $x^2y'' - y = 0$ *at* $x_0 = 0$ *using the power series method of this section* (x_0 *is a singular point*). *Can you find at least one solution? Can you find more than one solution?*

7.3 Singular points and the method of Frobenius

Note: 1 or 1.5 lectures, §8.4 and §8.5 in [EP], §5.4–§5.7 in [BD]

While behavior of ODEs at singular points is more complicated, certain singular points are not especially difficult to solve. Let us look at some examples before giving a general method. We may be lucky and obtain a power series solution using the method of the previous section, but in general we may have to try other things.

7.3.1 Examples

Example 7.3.1: Let us first look at a simple first order equation

$$2xy' - y = 0.$$

Note that $x = 0$ is a singular point. If we try to plug in

$$y = \sum_{k=0}^{\infty} a_k x^k,$$

we obtain

$$\begin{aligned} 0 = 2xy' - y &= 2x \left(\sum_{k=1}^{\infty} k a_k x^{k-1} \right) - \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= a_0 + \sum_{k=1}^{\infty} (2k a_k - a_k)\, x^k. \end{aligned}$$

First, $a_0 = 0$. Next, the only way to solve $0 = 2ka_k - a_k = (2k-1)\,a_k$ for $k = 1, 2, 3, \ldots$ is for $a_k = 0$ for all k. Therefore we only get the trivial solution $y = 0$. We need a nonzero solution to get the general solution.

Let us try $y = x^r$ for some real number r. Consequently our solution—if we can find one—may only make sense for positive x. Then $y' = rx^{r-1}$. So

$$0 = 2xy' - y = 2xrx^{r-1} - x^r = (2r-1)x^r.$$

Therefore $r = 1/2$, or in other words $y = x^{1/2}$. Multiplying by a constant, the general solution for positive x is

$$y = Cx^{1/2}.$$

If $C \neq 0$, then the derivative of the solution "blows up" at $x = 0$ (the singular point). There is only one solution that is differentiable at $x = 0$ and that's the trivial solution $y = 0$.

Not every problem with a singular point has a solution of the form $y = x^r$, of course. But perhaps we can combine the methods. What we will do is to try a solution of the form

$$y = x^r f(x)$$

where $f(x)$ is an analytic function.

Example 7.3.2: Consider the equation

$$4x^2y'' - 4x^2y' + (1-2x)y = 0,$$

and again note that $x = 0$ is a singular point.

Let us try

$$y = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r},$$

where r is a real number, not necessarily an integer. Again if such a solution exists, it may only exist for positive x. First let us find the derivatives

$$y' = \sum_{k=0}^{\infty} (k+r)\, a_k x^{k+r-1},$$

$$y'' = \sum_{k=0}^{\infty} (k+r)\,(k+r-1)\, a_k x^{k+r-2}.$$

Plugging into our equation we obtain

$$\begin{aligned}
0 &= 4x^2y'' - 4x^2y' + (1-2x)y \\
&= 4x^2 \left(\sum_{k=0}^{\infty} (k+r)\,(k+r-1)\, a_k x^{k+r-2} \right) - 4x^2 \left(\sum_{k=0}^{\infty} (k+r)\, a_k x^{k+r-1} \right) + (1-2x) \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) \\
&= \left(\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\, a_k x^{k+r} \right) \\
&\qquad - \left(\sum_{k=0}^{\infty} 4(k+r)\, a_k x^{k+r+1} \right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) - \left(\sum_{k=0}^{\infty} 2a_k x^{k+r+1} \right) \\
&= \left(\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\, a_k x^{k+r} \right) \\
&\qquad - \left(\sum_{k=1}^{\infty} 4(k+r-1)\, a_{k-1} x^{k+r} \right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) - \left(\sum_{k=1}^{\infty} 2a_{k-1} x^{k+r} \right) \\
&= 4r(r-1)\, a_0 x^r + a_0 x^r + \sum_{k=1}^{\infty} \Big(4(k+r)\,(k+r-1)\, a_k - 4(k+r-1)\, a_{k-1} + a_k - 2a_{k-1} \Big)\, x^{k+r} \\
&= \big(4r(r-1)+1\big)\, a_0 x^r + \sum_{k=1}^{\infty} \Big(\big(4(k+r)\,(k+r-1)+1\big)\, a_k - \big(4(k+r-1)+2\big)\, a_{k-1} \Big)\, x^{k+r}.
\end{aligned}$$

To have a solution we must first have $\big(4r(r-1)+1\big)\, a_0 = 0$. Supposing that $a_0 \neq 0$ we obtain

$$4r(r-1) + 1 = 0.$$

This equation is called the *indicial equation*. This particular indicial equation has a double root at $r = 1/2$.

OK, so we know what r has to be. That knowledge we obtained simply by looking at the coefficient of x^r. All other coefficients of x^{k+r} also have to be zero so

$$\big(4(k+r)(k+r-1)+1\big)\,a_k - \big(4(k+r-1)+2\big)\,a_{k-1} = 0.$$

If we plug in $r = 1/2$ and solve for a_k, we get

$$a_k = \frac{4(k+1/2-1)+2}{4(k+1/2)(k+1/2-1)+1}\,a_{k-1} = \frac{1}{k}\,a_{k-1}.$$

Let us set $a_0 = 1$. Then

$$a_1 = \frac{1}{1}a_0 = 1, \qquad a_2 = \frac{1}{2}a_1 = \frac{1}{2}, \qquad a_3 = \frac{1}{3}a_2 = \frac{1}{3\cdot 2}, \qquad a_4 = \frac{1}{4}a_3 = \frac{1}{4\cdot 3\cdot 2}, \qquad \cdots$$

Extrapolating, we notice that

$$a_k = \frac{1}{k(k-1)(k-2)\cdots 3\cdot 2} = \frac{1}{k!}.$$

In other words,

$$y = \sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+1/2} = x^{1/2} \sum_{k=0}^{\infty} \frac{1}{k!} x^k = x^{1/2} e^x.$$

That was lucky! In general, we will not be able to write the series in terms of elementary functions.

We have one solution, let us call it $y_1 = x^{1/2}e^x$. But what about a second solution? If we want a general solution, we need two linearly independent solutions. Picking a_0 to be a different constant only gets us a constant multiple of y_1, and we do not have any other r to try; we only have one solution to the indicial equation. Well, there are powers of x floating around and we are taking derivatives, perhaps the logarithm (the antiderivative of x^{-1}) is around as well. It turns out we want to try for another solution of the form

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+r} + (\ln x) y_1,$$

which in our case is

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+1/2} + (\ln x) x^{1/2} e^x.$$

We now differentiate this equation, substitute into the differential equation and solve for b_k. A long computation ensues and we obtain some recursion relation for b_k. The reader can (and should) try this to obtain for example the first three terms

$$b_1 = b_0 - 1, \qquad b_2 = \frac{2b_1 - 1}{4}, \qquad b_3 = \frac{6b_2 - 1}{18}, \qquad \ldots$$

We then fix b_0 and obtain a solution y_2. Then we write the general solution as $y = Ay_1 + By_2$.

7.3.2 The method of Frobenius

Before giving the general method, let us clarify when the method applies. Let

$$p(x)y'' + q(x)y' + r(x)y = 0$$

be an ODE. As before, if $p(x_0) = 0$, then x_0 is a singular point. If, furthermore, the limits

$$\lim_{x \to x_0} \, (x - x_0)\frac{q(x)}{p(x)} \qquad \text{and} \qquad \lim_{x \to x_0} \, (x - x_0)^2\frac{r(x)}{p(x)}$$

both exist and are finite, then we say that x_0 is a *regular singular point*.

Example 7.3.3: Often, and for the rest of this section, $x_0 = 0$. Consider

$$x^2y'' + x(1+x)y' + (\pi + x^2)y = 0.$$

Write

$$\begin{aligned}
\lim_{x \to 0} \, x\frac{q(x)}{p(x)} &= \lim_{x \to 0} \, x\frac{x(1+x)}{x^2} = \lim_{x \to 0} \, (1+x) = 1, \\
\lim_{x \to 0} \, x^2\frac{r(x)}{p(x)} &= \lim_{x \to 0} \, x^2\frac{(\pi + x^2)}{x^2} = \lim_{x \to 0} \, (\pi + x^2) = \pi.
\end{aligned}$$

So $x = 0$ is a regular singular point.

On the other hand if we make the slight change

$$x^2y'' + (1+x)y' + (\pi + x^2)y = 0,$$

then

$$\lim_{x \to 0} \, x\frac{q(x)}{p(x)} = \lim_{x \to 0} \, x\frac{(1+x)}{x^2} = \lim_{x \to 0} \, \frac{1+x}{x} = \text{DNE}.$$

Here DNE stands for *does not exist*. The point 0 is a singular point, but not a regular singular point.

Let us now discuss the general *Method of Frobenius**. We only consider the method at the point $x = 0$ for simplicity. The main idea is the following theorem.

Theorem 7.3.1 (Method of Frobenius)**.** *Suppose that*

$$p(x)y'' + q(x)y' + r(x)y = 0 \tag{7.3}$$

has a regular singular point at $x = 0$. Then there exists at least one solution of the form

$$y = x^r \sum_{k=0}^{\infty} a_k x^k,$$

where $a_0 = 1$. A solution of this form is called a Frobenius-type solution.

*Named after the German mathematician Ferdinand Georg Frobenius (1849–1917).

The method usually breaks down like this:

(i) We seek a Frobenius-type solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}.$$

We plug this y into equation (7.3). We collect terms and write everything as a single series.

(ii) The obtained series must be zero. Setting the first coefficient (usually the coefficient of x^r) in the series to zero we obtain the *indicial equation*, which is a quadratic polynomial in r.

(iii) If the indicial equation has two real roots r_1 and r_2 such that $r_1 - r_2$ is not an integer, then we have two linearly independent Frobenius-type solutions. Using the first root, we plug in

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and we solve for all a_k to obtain the first solution. Then using the second root, we plug in

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k,$$

and solve for all b_k to obtain the second solution.

(iv) If the indicial equation has a doubled root r, then there we find one solution

$$y_1 = x^{r} \sum_{k=0}^{\infty} a_k x^k,$$

and then we obtain a new solution by plugging

$$y_2 = x^{r} \sum_{k=0}^{\infty} b_k x^k + (\ln x) y_1,$$

into equation (7.3) and solving for the constants b_k.

(v) If the indicial equation has two real roots such that $r_1 - r_2$ is an integer, then one solution is

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and the second linearly independent solution is of the form

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k + C(\ln x) y_1,$$

where we plug y_2 into (7.3) and solve for the constants b_k and C.

(vi) Finally, if the indicial equation has complex roots, then solving for a_k in the solution

$$y = x^{r_1} \sum_{k=0}^{\infty} a_k x^k$$

results in a complex-valued function—all the a_k are complex numbers. We obtain our two linearly independent solutions* by taking the real and imaginary parts of y.

The main idea is to find at least one Frobenius-type solution. If we are lucky and find two, we are done. If we only get one, we either use the ideas above or even a different method such as reduction of order (see § 2.1) to obtain a second solution.

7.3.3 Bessel functions

An important class of functions that arises commonly in physics are the *Bessel functions*†. For example, these functions appear when solving the wave equation in two and three dimensions. First consider *Bessel's equation* of order p:

$$x^2 y'' + x y' + \left(x^2 - p^2\right) y = 0.$$

We allow p to be any number, not just an integer, although integers and multiples of $1/2$ are most important in applications.

When we plug

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

into Bessel's equation of order p, we obtain the indicial equation

$$r(r-1) + r - p^2 = (r-p)(r+p) = 0.$$

Therefore we obtain two roots $r_1 = p$ and $r_2 = -p$. If p is not an integer, then following the method of Frobenius and setting $a_0 = 1$, we obtain linearly independent solutions of the form

$$y_1 = x^p \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k! (k+p)(k-1+p) \cdots (2+p)(1+p)},$$

$$y_2 = x^{-p} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k! (k-p)(k-1-p) \cdots (2-p)(1-p)}.$$

*See Joseph L. Neuringera, *The Frobenius method for complex roots of the indicial equation*, International Journal of Mathematical Education in Science and Technology, Volume 9, Issue 1, 1978, 71–77.

†Named after the German astronomer and mathematician Friedrich Wilhelm Bessel (1784–1846).

***Exercise* 7.3.1:**

a) *Verify that the indicial equation of Bessel's equation of order p is $(r-p)(r+p)=0$.*

b) *Suppose p is not an integer. Carry out the computation to obtain the solutions y_1 and y_2 above.*

Bessel functions are convenient constant multiples of y_1 and y_2. First we must define the *gamma function*

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt.$$

Notice that $\Gamma(1)=1$. The gamma function also has a wonderful property

$$\Gamma(x+1) = x\Gamma(x).$$

From this property, it follows that $\Gamma(n)=(n-1)!$ when n is an integer. So the gamma function is a continuous version of the factorial. We compute:

$$\begin{aligned}
\Gamma(k+p+1) &= (k+p)(k-1+p)\cdots(2+p)(1+p)\Gamma(1+p),\\
\Gamma(k-p+1) &= (k-p)(k-1-p)\cdots(2-p)(1-p)\Gamma(1-p).
\end{aligned}$$

***Exercise* 7.3.2:** *Verify the identities above using $\Gamma(x+1)=x\Gamma(x)$.*

We define the *Bessel functions of the first kind* of order p and $-p$ as

$$\begin{aligned}
J_p(x) &= \frac{1}{2^p\Gamma(1+p)}y_1 = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k+p},\\
J_{-p}(x) &= \frac{1}{2^{-p}\Gamma(1-p)}y_2 = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(k-p+1)}\left(\frac{x}{2}\right)^{2k-p}.
\end{aligned}$$

As these are constant multiples of the solutions we found above, these are both solutions to Bessel's equation of order p. The constants are picked for convenience.

When p is not an integer, J_p and J_{-p} are linearly independent. When n is an integer we obtain

$$J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,(k+n)!}\left(\frac{x}{2}\right)^{2k+n}.$$

In this case

$$J_n(x) = (-1)^n J_{-n}(x),$$

and so J_{-n} is not a second linearly independent solution. The other solution is the so-called *Bessel function of second kind*. These make sense only for integer orders n and are defined as limits of linear combinations of $J_p(x)$ and $J_{-p}(x)$, as p approaches n in the following way:

$$Y_n(x) = \lim_{p\to n}\frac{\cos(p\pi)J_p(x) - J_{-p}(x)}{\sin(p\pi)}.$$

Each linear combination of $J_p(x)$ and $J_{-p}(x)$ is a solution to Bessel's equation of order p. Then as we take the limit as p goes to n, we see that $Y_n(x)$ is a solution to Bessel's equation of order n. It also turns out that $Y_n(x)$ and $J_n(x)$ are linearly independent. Therefore when n is an integer, we have the general solution to Bessel's equation of order n:

$$y = AJ_n(x) + BY_n(x),$$

for arbitrary constants A and B. Note that $Y_n(x)$ goes to negative infinity at $x = 0$. Many mathematical software packages have these functions $J_n(x)$ and $Y_n(x)$ defined, so they can be used just like say $\sin(x)$ and $\cos(x)$. In fact, Bessel functions have some similar properties. For example, $-J_1(x)$ is a derivative of $J_0(x)$, and in general the derivative of $J_n(x)$ can be written as a linear combination of $J_{n-1}(x)$ and $J_{n+1}(x)$. Furthermore, these functions oscillate, although they are not periodic. See Figure 7.4 for graphs of Bessel functions.

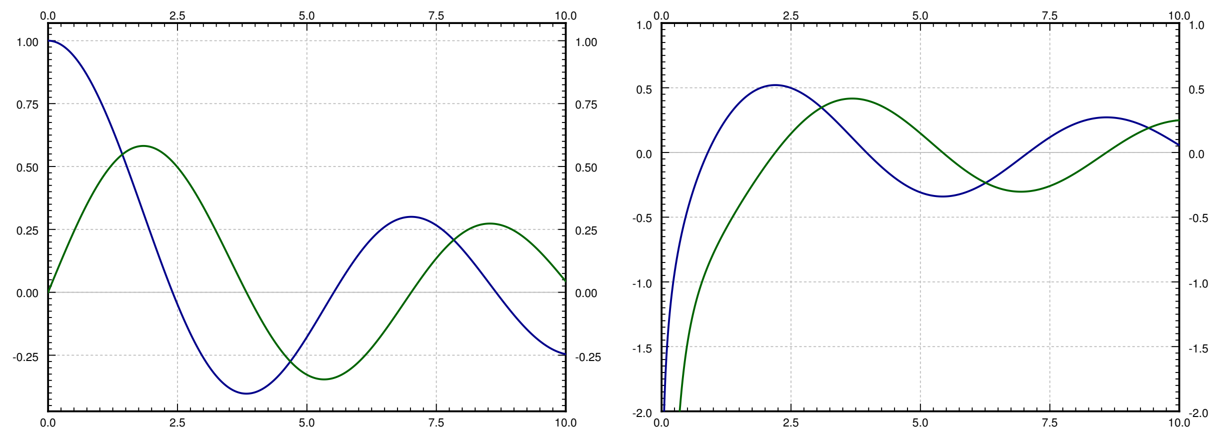


Figure 7.4: Plot of the $J_0(x)$ and $J_1(x)$ in the first graph and $Y_0(x)$ and $Y_1(x)$ in the second graph.

Example 7.3.4: Other equations can sometimes be solved in terms of the Bessel functions. For example, given a positive constant λ,

$$xy'' + y' + \lambda^2 xy = 0,$$

can be changed to $x^2y'' + xy' + \lambda^2x^2y = 0$. Then changing variables $t = \lambda x$, we obtain via chain rule the equation in y and t:

$$t^2y'' + ty' + t^2y = 0,$$

which we recognize as Bessel's equation of order 0. Therefore the general solution is $y(t) = AJ_0(t) + BY_0(t)$, or in terms of x:

$$y = AJ_0(\lambda x) + BY_0(\lambda x).$$

This equation comes up, for example, when finding the fundamental modes of vibration of a circular drum, but we digress.

7.3.4 Exercises

Exercise* 7.3.3*: *Find a particular (Frobenius-type) solution of* $x^2y'' + xy' + (1+x)y = 0$.

Exercise* 7.3.4*: *Find a particular (Frobenius-type) solution of* $xy'' - y = 0$.

Exercise* 7.3.5*: *Find a particular (Frobenius-type) solution of* $y'' + \frac{1}{x}y' - xy = 0$.

Exercise* 7.3.6*: *Find the general solution of* $2xy'' + y' - x^2y = 0$.

Exercise* 7.3.7*: *Find the general solution of* $x^2y'' - xy' - y = 0$.

Exercise* 7.3.8*: *In the following equations classify the point* $x = 0$ *as* ordinary, regular singular, *or* singular but not regular singular.

a) $x^2(1+x^2)y'' + xy = 0$

b) $x^2y'' + y' + y = 0$

c) $xy'' + x^3y' + y = 0$

d) $xy'' + xy' - e^xy = 0$

e) $x^2y'' + x^2y' + x^2y = 0$

Exercise* 7.3.101*: *In the following equations classify the point* $x = 0$ *as* ordinary, regular singular, *or* singular but not regular singular.

a) $y'' + y = 0$

b) $x^3y'' + (1+x)y = 0$

c) $xy'' + x^5y' + y = 0$

d) $\sin(x)y'' - y = 0$

e) $\cos(x)y'' - \sin(x)y = 0$

Exercise* 7.3.102*: *Find the general solution of* $x^2y'' - y = 0$.

Exercise* 7.3.103*: *Find a particular solution of* $x^2y'' + (x - 3/4)y = 0$.

Exercise* 7.3.104** (tricky):*** *Find the general solution of* $x^2y'' - xy' + y = 0$.

Chapter 8

Nonlinear systems

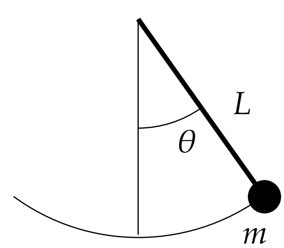
8.1 Linearization, critical points, and equilibria

Note: 1 lecture, §6.1–§6.2 in [EP], §9.2–§9.3 in [BD]

Except for a few brief detours in chapter 1, we considered mostly linear equations. Linear equations suffice in many applications, but in reality most phenomena require nonlinear equations. Nonlinear equations, however, are notoriously more difficult to understand than linear ones, and many strange new phenomena appear when we allow our equations to be nonlinear.

Not to worry, we did not waste all this time studying linear equations. Nonlinear equations can often be approximated by linear ones if we only need a solution "locally," for example, only for a short period of time, or only for certain parameters. Understanding linear equations can also give us qualitative understanding about a more general nonlinear problem. The idea is similar to what you did in calculus in trying to approximate a function by a line with the right slope.

In § 2.4 we looked at the pendulum of length L. The goal was to solve for the angle $\theta(t)$ as a function of the time t. The equation for the setup is the nonlinear equation



$$\theta'' + \frac{g}{L} \sin \theta = 0.$$

Instead of solving this equation, we solved the rather easier linear equation

$$\theta'' + \frac{g}{L} \theta = 0.$$

While the solution to the linear equation is not exactly what we were looking for, it is rather close to the original, as long as the angle θ is small and the time period involved is short.

You might ask: Why don't we just solve the nonlinear problem? Well, it might be very difficult, impractical, or impossible to solve analytically, depending on the equation in question. We may not even be interested in the actual solution, we might only be interested in some qualitative idea of what the solution is doing. For example, what happens as time goes to infinity?

8.1.1 Autonomous systems and phase plane analysis

We restrict our attention to a two-dimensional autonomous system

$$x' = f(x, y), \qquad y' = g(x, y),$$

where $f(x, y)$ and $g(x, y)$ are functions of two variables, and the derivatives are taken with respect to time t. Solutions are functions $x(t)$ and $y(t)$ such that

$$x'(t) = f\big(x(t), y(t)\big), \qquad y'(t) = g\big(x(t), y(t)\big).$$

The way we will analyze the system is very similar to § 1.6, where we studied a single autonomous equation. The ideas in two dimensions are the same, but the behavior can be far more complicated.

It may be best to think of the system of equations as the single vector equation

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} f(x, y) \\ g(x, y) \end{bmatrix}. \tag{8.1}$$

As in § 3.1 we draw the *phase portrait* (or *phase diagram*), where each point (x, y) corresponds to a specific state of the system. We draw the *vector field* given at each point (x, y) by the vector $\left[\begin{smallmatrix} f(x,y) \\ g(x,y) \end{smallmatrix}\right]$. And as before if we find solutions, we draw the trajectories by plotting all points $\big(x(t), y(t)\big)$ for a certain range of t.

Example 8.1.1: Consider the second order equation $x'' = -x + x^2$. Write this equation as a first order nonlinear system

$$x' = y, \qquad y' = -x + x^2.$$

The phase portrait with some trajectories is drawn in Figure 8.1 on the facing page.

From the phase portrait it should be clear that even this simple system has fairly complicated behavior. Some trajectories keep oscillating around the origin, and some go off towards infinity. We will return to this example often, and analyze it completely in this (and the next) section.

If we zoom into the diagram near a point where $\left[\begin{smallmatrix} f(x,y) \\ g(x,y) \end{smallmatrix}\right]$ is not zero, then nearby the arrows point generally in essentially that same direction and have essentially the same magnitude. In other words the behavior is not that interesting near such a point. We are of course assuming that $f(x, y)$ and $g(x, y)$ are continuous.

Let us concentrate on those points in the phase diagram above where the trajectories seem to start, end, or go around. We see two such points: $(0, 0)$ and $(1, 0)$. The trajectories seem to go around the point $(0, 0)$, and they seem to either go in or out of the point $(1, 0)$.

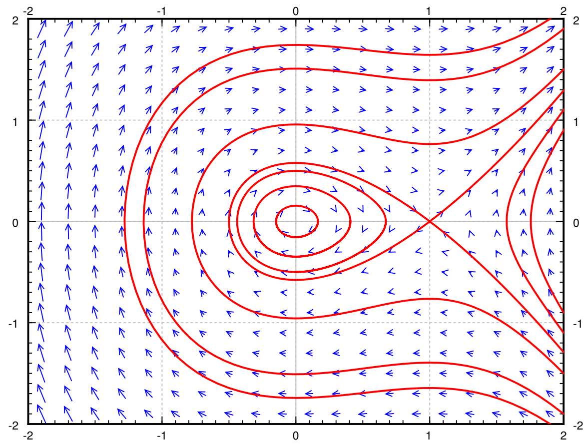


Figure 8.1: Phase portrait with some trajectories of $x' = y$, $y' = -x + x^2$.

These points are precisely those points where the derivatives of both x and y are zero. Let us define the *critical points* as the points (x, y) such that

$$\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = \vec{0}.$$

In other words, these are the points where both $f(x, y) = 0$ and $g(x, y) = 0$.

The critical points are where the behavior of the system is in some sense the most complicated. If $\left[\begin{smallmatrix} f(x,y) \\ g(x,y) \end{smallmatrix}\right]$ is zero, then nearby, the vector can point in any direction whatsoever. Also, the trajectories are either going towards, away from, or around these points, so if we are looking for long-term qualitative behavior of the system, we should look at what is happening near the critical points.

Critical points are also sometimes called *equilibria*, since we have so-called *equilibrium solutions* at critical points. If (x_0, y_0) is a critical point, then we have the solutions

$$x(t) = x_0, \quad y(t) = y_0.$$

In Example 8.1.1 on the preceding page, there are two equilibrium solutions:

$$x(t) = 0, \quad y(t) = 0, \qquad \text{and} \qquad x(t) = 1, \quad y(t) = 0.$$

Compare this discussion on equilibria to the discussion in § 1.6. The underlying concept is exactly the same.

8.1.2 Linearization

In § 3.5 we studied the behavior of a homogeneous linear system of two equations near a critical point. For a linear system of two variables given by an invertible matrix, the only critical point is the origin $(0,0)$. Let us put the understanding we gained in that section to good use understanding what happens near critical points of nonlinear systems.

In calculus we learned to estimate a function by taking its derivative and linearizing. We work similarly with nonlinear systems of ODE. Suppose (x_0, y_0) is a critical point. First change variables to (u,v), so that $(u,v) = (0,0)$ corresponds to (x_0, y_0). That is,

$$u = x - x_0, \qquad v = y - y_0.$$

Next we need to find the derivative. In multivariable calculus you may have seen that the several variables version of the derivative is the *Jacobian matrix**. The Jacobian matrix of the vector-valued function $\begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix}$ at (x_0, y_0) is

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix}.$$

This matrix gives the best linear approximation as u and v (and therefore x and y) vary. We define the *linearization* of the equation (8.1) as the linear system

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Example 8.1.2: Let us keep with the same equations as Example 8.1.1: $x' = y$, $y' = -x + x^2$. There are two critical points, $(0,0)$ and $(1,0)$. The Jacobian matrix at any point is

$$\begin{bmatrix} \frac{\partial f}{\partial x}(x, y) & \frac{\partial f}{\partial y}(x, y) \\ \frac{\partial g}{\partial x}(x, y) & \frac{\partial g}{\partial y}(x, y) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 + 2x & 0 \end{bmatrix}.$$

Therefore at $(0,0)$, we have $u = x$ and $v = y$, and the linearization is

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

At the point $(1,0)$, we have $u = x - 1$ and $v = y$, and the linearization is

$$\begin{bmatrix} u \\ v \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}.$$

The phase diagrams of the two linearizations at the point $(0,0)$ and $(1,0)$ are given in Figure 8.2 on the facing page. Note that the variables are now u and v. Compare Figure 8.2 with Figure 8.1 on the previous page, and look especially at the behavior near the critical points.

*Named for the German mathematician Carl Gustav Jacob Jacobi (1804–1851).

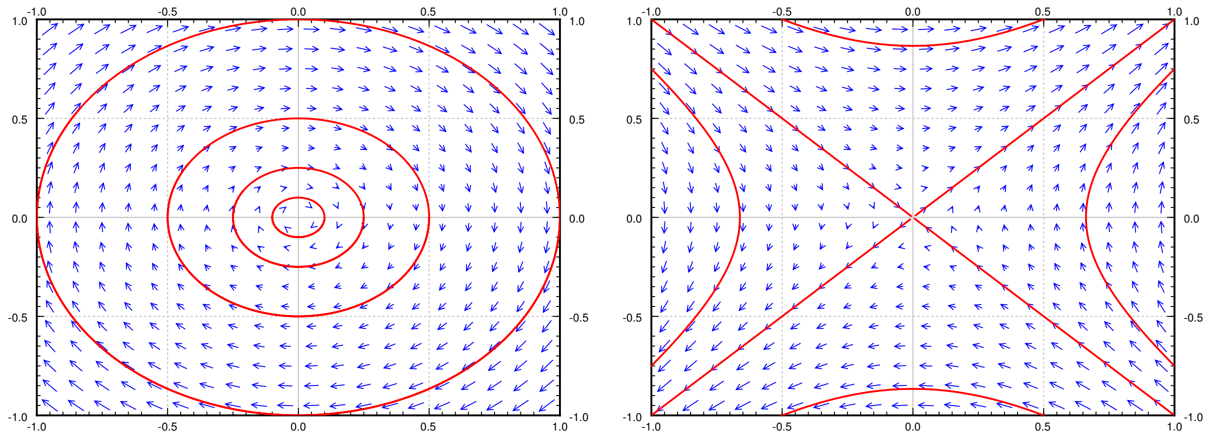


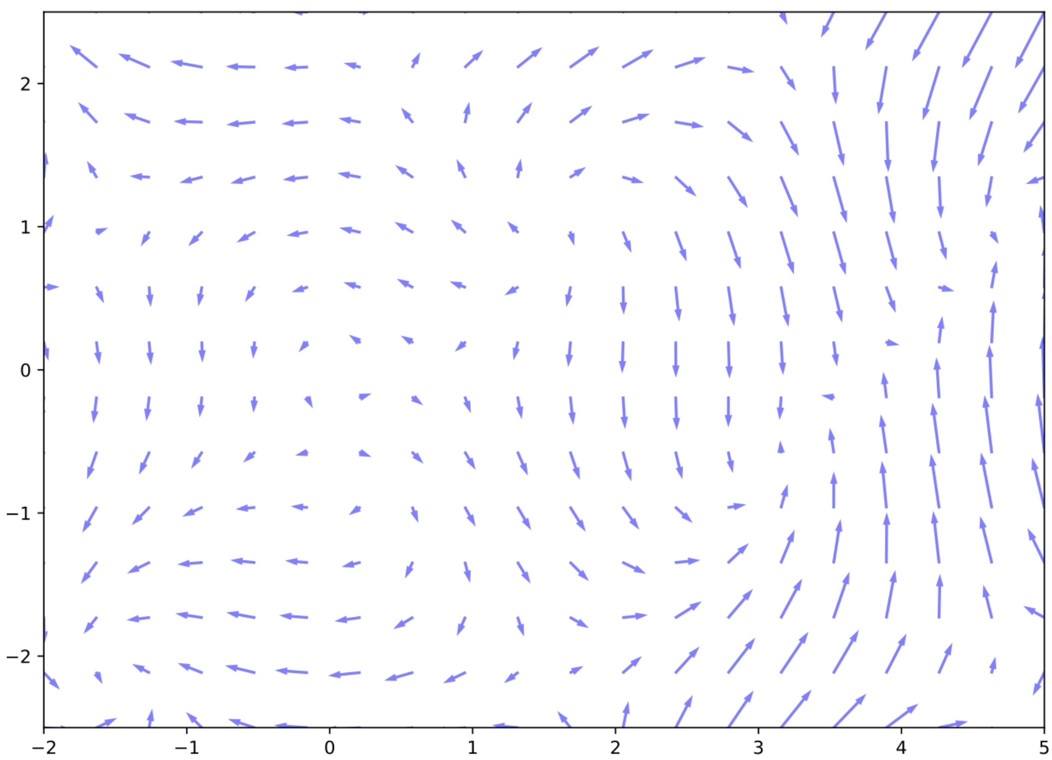
Figure 8.2: Phase diagram with some trajectories of linearizations at the critical points $(0, 0)$ *(left) and* $(1, 0)$ *(right) of* $x' = y$, $y' = -x + x^2$.

8.1.3 Vector fields of 2D systems with Python

The *resources306* module provides a function *fieldplot* to plot the vector field of the two-dimensional autonomous system of ODEs (8.1). It works just like *fieldplotlinear* except that instead of a matrix, it takes a pair of functions of two variables. You supply a Python function that takes the pair (x, y) and returns the pair $(f(x, y), g(x, y))$. You also givethe desired ranges for the horizontal and vertical axes, followed by any desired graphical options. In the example below we use *fieldplot* to make a vector field plot for the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -y\cos(x+y-1) \\ x\cos(x-y+1) \end{bmatrix}. \tag{8.2}$$

```
from resources306 import *
def F(X):
    x,y = X
    return -y*cos(x+y-1), x*cos(x-y+1)
cos = np.cos
plt.figure(figsize=(10,10))
plt.subplot(111,aspect=1)  # optional: make scales same on the two axes
fieldplot(F,-2,5,-2.5,2.5,color='b',alpha=0.5)
```



8.1.4 Exercises

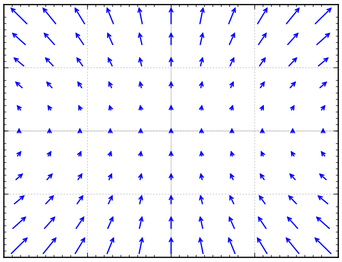
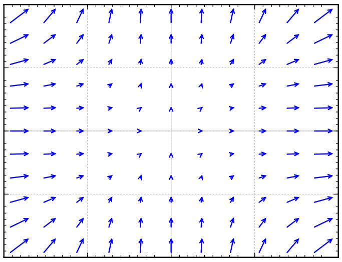
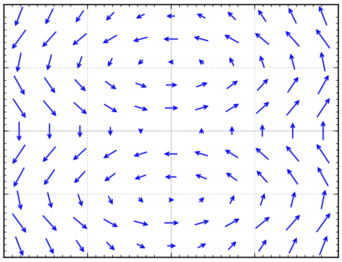
***Exercise* 8.1.1:** *Sketch the phase plane vector field for:*

a) $x' = x^2, \quad y' = y^2,$ *b)* $x' = (x-y)^2, \quad y' = -x,$ *c)* $x' = e^y, \quad y' = e^x.$

***Exercise* 8.1.2:** *Match systems*

1) $x' = x^2, \quad y' = y^2,$ 2) $x' = xy, \quad y' = 1 + y^2,$ 3) $x' = \sin(\pi y), \quad y' = x,$

to the vector fields below. Justify.

a)  *b)*  *c)* 

***Exercise* 8.1.3:** *Find the critical points and linearizations of the following systems.*

a) $x' = x^2 - y^2, \quad y' = x^2 + y^2 - 1,$ *b)* $x' = -y, \quad y' = 3x + yx^2,$

c) $x' = x^2 + y, \quad y' = y^2 + x.$

***Exercise* 8.1.4:** *For the following systems, verify they have critical point at* $(0,0)$, *and find the linearization at* $(0,0)$.

a) $x' = x + 2y + x^2 - y^2, \quad y' = 2y - x^2$ *b)* $x' = -y, \quad y' = x - y^3$

c) $x' = ax + by + f(x,y)$, $y' = cx + dy + g(x,y)$, *where* $f(0,0) = 0$, $g(0,0) = 0$, *and all first partial derivatives of* f *and* g *are also zero at* $(0,0)$, *that is,* $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = \frac{\partial g}{\partial x}(0,0) = \frac{\partial g}{\partial y}(0,0) = 0.$

***Exercise* 8.1.5:** *Take* $x' = (x-y)^2, \quad y' = (x+y)^2.$

a) *Find the set of critical points.*

b) *Sketch a phase diagram and describe the behavior near the critical point(s).*

c) *Find the linearization. Is it helpful in understanding the system?*

***Exercise* 8.1.6:** *Take* $x' = x^2, \quad y' = x^3.$

a) *Find the set of critical points.*

b) *Sketch a phase diagram and describe the behavior near the critical point(s).*

c) *Find the linearization. Is it helpful in understanding the system?*

***Exercise* 8.1.101:** *Find the critical points and linearizations of the following systems.*

a) $x' = \sin(\pi y) + (x-1)^2, \quad y' = y^2 - y,$ *b*) $x' = x + y + y^2, \quad y' = x,$

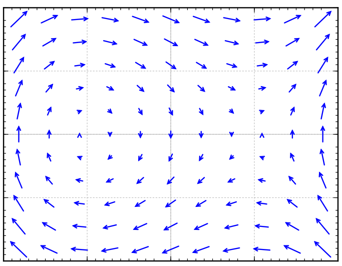
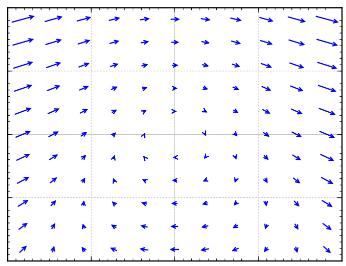
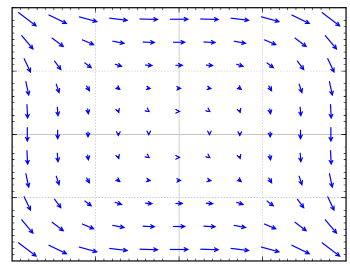
c) $x' = (x-1)^2 + y, \quad y' = x^2 + y.$

***Exercise* 8.1.102:** *Match systems*

1) $x' = y^2, \quad y' = -x^2,$ 2) $x' = y, \quad y' = (x-1)(x+1),$

3) $x' = y + x^2, \quad y' = -x,$

to the vector fields below. Justify.

a)  *b*)  *c*) 

***Exercise* 8.1.103:** *The idea of critical points and linearization works in higher dimensions as well. You simply make the Jacobian matrix bigger by adding more functions and more variables. For the following system of* 3 *equations find the critical points and their linearizations*:

$$x' = x + z^2, \qquad y' = z^2 - y, \qquad z' = z + x^2.$$

***Exercise* 8.1.104:** *Any two-dimensional non-autonomous system* $x' = f(x, y, t)$, $y' = g(x, y, t)$ *can be written as a three-dimensional autonomous system* (*three equations*)*. Write down this autonomous system using the variables* u, v, w.

8.2 Stability and classification of isolated critical points

Note: 1.5–2 lectures, §6.1–§6.2 in [EP], §9.2–§9.3 in [BD]

8.2.1 Isolated critical points and almost linear systems

A critical point is *isolated* if it is the only critical point in some small "neighborhood" of the point. That is, if we zoom in far enough it is the only critical point we see. In the example above, the critical point was isolated. If on the other hand there would be a whole curve of critical points, then it would not be isolated.

A system is called *almost linear* at a critical point (x_0, y_0), if the critical point is isolated and the Jacobian matrix at the point is invertible, or equivalently if the linearized system has an isolated critical point. In such a case, the nonlinear terms are very small and the system behaves like its linearization, at least if we are close to the critical point.

For example, the system in Examples 8.1.1 and 8.1.2 has two isolated critical points $(0, 0)$ and $(0, 1)$, and is almost linear at both critical points as the Jacobian matrices at both points, $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, are invertible.

On the other hand, the system $x' = x^2$, $y' = y^2$ has an isolated critical point at $(0, 0)$, however the Jacobian matrix

$$\begin{bmatrix} 2x & 0 \\ 0 & 2y \end{bmatrix}$$

is zero when $(x, y) = (0, 0)$. So the system is not almost linear. Even a worse example is the system $x' = x$, $y' = x^2$, which does not have isolated critical points; x' and y' are both zero whenever $x = 0$, that is, the entire y-axis.

Fortunately, most often critical points are isolated, and the system is almost linear at the critical points. So if we learn what happens there, we will have figured out the majority of situations that arise in applications.

8.2.2 Stability and classification of isolated critical points

Once we have an isolated critical point, the system is almost linear at that critical point, and we computed the associated linearized system, we can classify what happens to the solutions. We more or less use the classification for linear two-variable systems from § 3.5, with one minor caveat. Let us list the behaviors depending on the eigenvalues of the Jacobian matrix at the critical point in Table 8.1 on the following page. This table is very similar to Table 3.1 on page 163, with the exception of missing "center" points. We will discuss centers later, as they are more complicated.

In the third column, we mark points as *asymptotically stable* or *unstable*. Formally, a *stable critical point* (x_0, y_0) is one where given any small distance ϵ to (x_0, y_0), and any initial condition within a perhaps smaller radius around (x_0, y_0), the trajectory of the system never goes further away from (x_0, y_0) than ϵ. An *unstable critical point* is one that is not

Eigenvalues of the Jacobian matrix	Behavior	Stability
real and both positive	source / unstable node	unstable
real and both negative	sink / stable node	asymptotically stable
real and opposite signs	saddle	unstable
complex with positive real part	spiral source	unstable
complex with negative real part	spiral sink	asymptotically stable

Table 8.1: Behavior of an almost linear system near an isolated critical point.

stable. Informally, a point is stable if we start close to a critical point and follow a trajectory we either go towards, or at least not away from, this critical point.

A stable critical point (x_0, y_0) is called *asymptotically stable* if given any initial condition sufficiently close to (x_0, y_0) and any solution $\big(x(t), y(t)\big)$ satisfying that condition, then

$$\lim_{t\to\infty} \big(x(t), y(t)\big) = (x_0, y_0).$$

That is, the critical point is asymptotically stable if any trajectory for a sufficiently close initial condition goes towards the critical point (x_0, y_0).

Example 8.2.1: Consider $x' = -y - x^2$, $y' = -x + y^2$. See Figure 8.3 on the facing page for the phase diagram. Let us find the critical points. These are the points where $-y - x^2 = 0$ and $-x + y^2 = 0$. The first equation means $y = -x^2$, and so $y^2 = x^4$. Plugging into the second equation we obtain $-x + x^4 = 0$. Factoring we obtain $x(1 - x^3) = 0$. Since we are looking only for real solutions we get either $x = 0$ or $x = 1$. Solving for the corresponding y using $y = -x^2$, we get two critical points, one being $(0, 0)$ and the other being $(1, -1)$. Clearly the critical points are isolated.

Let us compute the Jacobian matrix:

$$\begin{bmatrix} -2x & -1 \\ -1 & 2y \end{bmatrix}.$$

At the point $(0, 0)$ we get the matrix $\left[\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right]$ and so the two eigenvalues are 1 and -1. As the matrix is invertible, the system is almost linear at $(0, 0)$. As the eigenvalues are real and of opposite signs, we get a saddle point, which is an unstable equilibrium point.

At the point $(1, -1)$ we get the matrix $\left[\begin{smallmatrix} -2 & -1 \\ -1 & -2 \end{smallmatrix}\right]$ and computing the eigenvalues we get -1, -3. The matrix is invertible, and so the system is almost linear at $(1, -1)$. As we have real eigenvalues and both negative, the critical point is a sink, and therefore an asymptotically stable equilibrium point. That is, if we start with any point (x_i, y_i) close to $(1, -1)$ as an initial condition and plot a trajectory, it approaches $(1, -1)$. In other words,

$$\lim_{t\to\infty} \big(x(t), y(t)\big) = (1, -1).$$

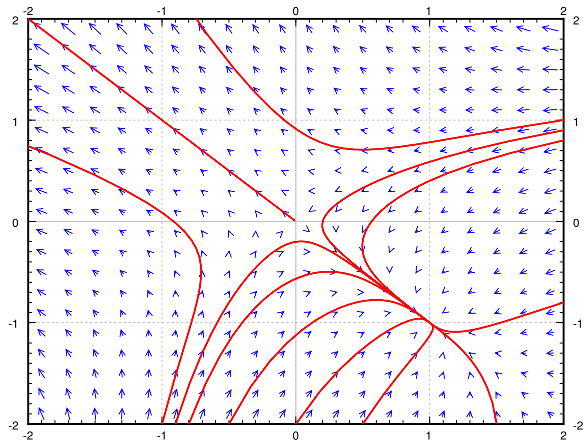


Figure 8.3: The phase portrait with few sample trajectories of $x' = -y - x^2$, $y' = -x + y^2$.

As you can see from the diagram, this behavior is true even for some initial points quite far from $(1, -1)$, but it is definitely not true for all initial points.

Example 8.2.2: Let us look at $x' = y + y^2 e^x$, $y' = x$. First let us find the critical points. These are the points where $y + y^2 e^x = 0$ and $x = 0$. Simplifying we get $0 = y + y^2 = y(y+1)$. So the critical points are $(0, 0)$ and $(0, -1)$, and hence are isolated. Let us compute the Jacobian matrix:

$$\begin{bmatrix} y^2 e^x & 1 + 2ye^x \\ 1 & 0 \end{bmatrix}.$$

At the point $(0, 0)$ we get the matrix $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ and so the two eigenvalues are 1 and -1. As the matrix is invertible, the system is almost linear at $(0, 0)$. And, as the eigenvalues are real and of opposite signs, we get a saddle point, which is an unstable equilibrium point.

At the point $(0, -1)$ we get the matrix $\left[\begin{smallmatrix} 1 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ whose eigenvalues are $\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$. The matrix is invertible, and so the system is almost linear at $(0, -1)$. As we have complex eigenvalues with positive real part, the critical point is a spiral source, and therefore an unstable equilibrium point.

See Figure 8.4 on the next page for the phase diagram. Notice the two critical points, and the behavior of the arrows in the vector field around these points.

8.2.3 The trouble with centers

Recall, a linear system with a center means that trajectories travel in closed elliptical orbits in some direction around the critical point. Such a critical point we call a *center* or a *stable center*. It is not an asymptotically stable critical point, as the trajectories never approach the critical point, but at least if you start sufficiently close to the critical point, you stay close to the critical point. The simplest example of such behavior is the linear system with a center. Another example is the critical point $(0, 0)$ in Example 8.1.1 on page 262.

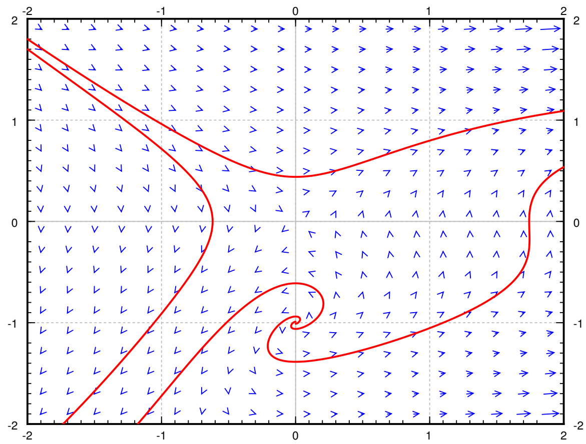


Figure 8.4: The phase portrait with few sample trajectories of $x' = y + y^2 e^x$, $y' = x$.

The trouble with a center in a nonlinear system is that whether the trajectory goes towards or away from the critical point is governed by the sign of the real part of the eigenvalues of the Jacobian matrix, and the Jacobian matrix in a nonlinear system changes from point to point. Since this real part is zero at the critical point itself, it can have either sign nearby, meaning the trajectory could be pulled towards or away from the critical point.

Example 8.2.3: An example of such a problematic behavior is the system $x' = y, y' = -x + y^3$. The only critical point is the origin $(0, 0)$. The Jacobian matrix is

$$\begin{bmatrix} 0 & 1 \\ -1 & 3y^2 \end{bmatrix}.$$

At $(0, 0)$ the Jacobian matrix is $\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$, which has eigenvalues $\pm i$. So the linearization has a center.

Using the quadratic equation, the eigenvalues of the Jacobian matrix at any point (x, y) are

$$\lambda = \frac{3}{2}y^2 \pm i\frac{\sqrt{4 - 9y^4}}{2}.$$

At any point where $y \neq 0$ (so at most points near the origin), the eigenvalues have a positive real part (y^2 can never be negative). This positive real part pulls the trajectory away from the origin. A sample trajectory for an initial condition near the origin is given in Figure 8.5 on the next page.

The moral of the example is that further analysis is needed when the linearization has a center. The analysis will in general be more complicated than in the example above, and is more likely to involve case-by-case consideration. Such a complication should not be surprising to you. By now in your mathematical career, you have seen many places where a simple test is inconclusive, recall for example the second derivative test for maxima or minima, and requires more careful, and perhaps ad hoc analysis of the situation.

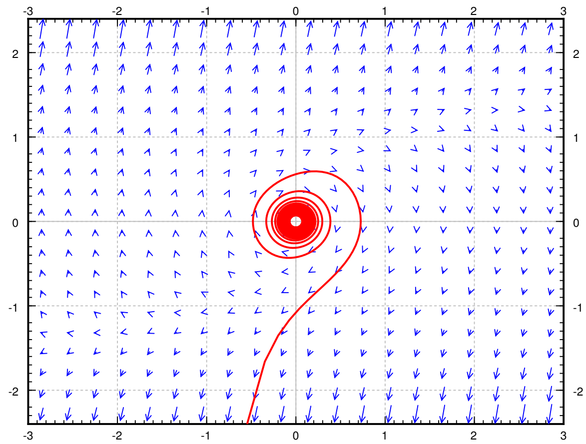


Figure 8.5: An unstable critical point (spiral source) at the origin for $x' = y$, $y' = -x + y^3$, even if the linearization has a center.

8.2.4 Conservative equations

An equation of the form

$$x'' + f(x) = 0$$

for an arbitrary function $f(x)$ is called a *conservative equation*. For example the pendulum equation is a conservative equation. The equations are conservative as there is no friction in the system so the energy in the system is "conserved." Let us write this equation as a system of nonlinear ODE.

$$x' = y, \qquad y' = -f(x).$$

These types of equations have the advantage that we can solve for their trajectories easily.

The trick is to first think of y as a function of x for a moment. Then use the chain rule

$$x'' = y' = \frac{dy}{dx}x' = y\frac{dy}{dx},$$

where the prime indicates a derivative with respect to t. We obtain $y\frac{dy}{dx} + f(x) = 0$. We integrate with respect to x to get $\int y\frac{dy}{dx}\,dx + \int f(x)\,dx = C$. In other words

$$\frac{1}{2}y^2 + \int f(x)\,dx = C.$$

We obtained an implicit equation for the trajectories, with different C giving different trajectories. The value of C is conserved on any trajectory. This expression is sometimes called the *Hamiltonian* or the energy of the system. If you look back to § 1.8, you will notice that $y\frac{dy}{dx} + f(x) = 0$ is an exact equation, and we just found a potential function.

Example 8.2.4: Let us find the trajectories for the equation $x'' + x - x^2 = 0$, which is the equation from Example 8.1.1 on page 262. The corresponding first order system is

$$x' = y, \qquad y' = -x + x^2.$$

Trajectories satisfy

$$\frac{1}{2}y^2 + \frac{1}{2}x^2 - \frac{1}{3}x^3 = C.$$

We solve for y

$$y = \pm\sqrt{-x^2 + \frac{2}{3}x^3 + 2C}.$$

Plotting these graphs we get exactly the trajectories in Figure 8.1 on page 263. In particular we notice that near the origin the trajectories are *closed curves*: they keep going around the origin, never spiraling in or out. Therefore we discovered a way to verify that the critical point at $(0,0)$ is a stable center. The critical point at $(0,1)$ is a saddle as we already noticed. This example is typical for conservative equations.

Consider an arbitrary conservative equation $x'' + f(x) = 0$. All critical points occur when $y = 0$ (the x-axis), that is when $x' = 0$. The critical points are those points on the x-axis where $f(x) = 0$. The trajectories are given by

$$y = \pm\sqrt{-2\int f(x)\,dx + 2C}.$$

So all trajectories are mirrored across the x-axis. In particular, there can be no spiral sources nor sinks. The Jacobian matrix is

$$\begin{bmatrix} 0 & 1 \\ -f'(x) & 0 \end{bmatrix}.$$

The critical point is almost linear if $f'(x) \neq 0$ at the critical point. Let J denote the Jacobian matrix. The eigenvalues of J are solutions to

$$0 = \det(J - \lambda I) = \lambda^2 + f'(x).$$

Therefore $\lambda = \pm\sqrt{-f'(x)}$. In other words, either we get real eigenvalues of opposite signs (if $f'(x) < 0$), or we get purely imaginary eigenvalues (if $f'(x) > 0$). There are only two possibilities for critical points, either an *unstable saddle point*, or a *stable center*. There are never any sinks or sources.

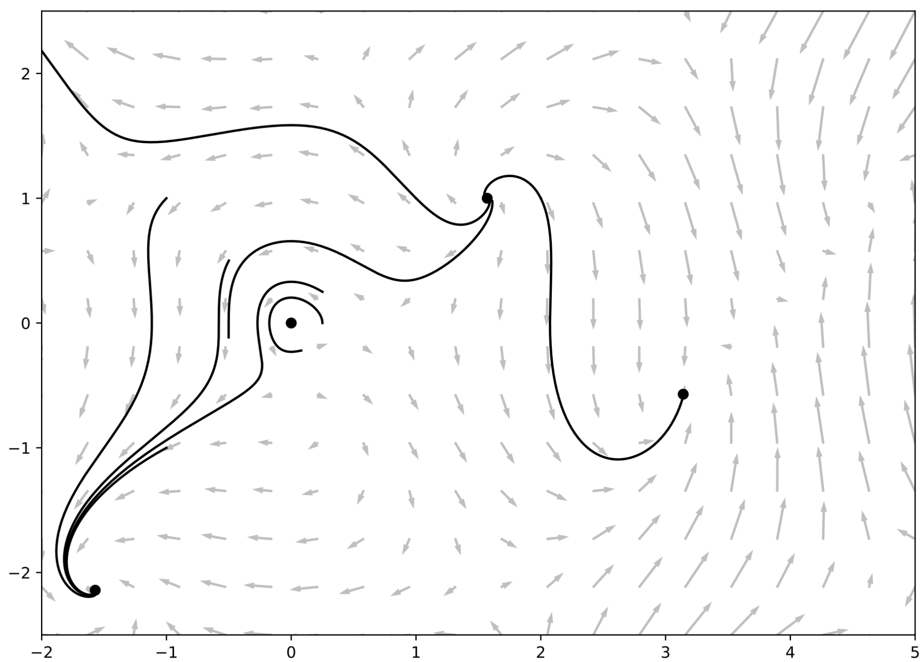
8.2.5 Phase portraits and equilibria with Python

The *resources306* module also provides a function *phaseportrait* to numerically compute and plot solution curves of the two-dimensional autonomous system of ODEs (8.1). This works just like *phaseportraitlinear* described in § 8.1.3, except that instead of a matrix, you supply a Python function that takes the pair (x, y) and returns the pair $(f(x,y), g(x,y))$. This Python function can be the same one used with *fieldplot*. In the example below we use *phaseportrait* to add some solution curves to the example from the previous section. We also add dots to mark some equilibria. These are computed using *fsolve*, which is imported by *resources306* from the module *scipy.optimize*. You give *fsolve* the function whose zero you want, and a rough guess at the location: *fsolve* will try to return an accurate approximation of the zero.

```
from resources306 import *
def F(X):
    x,y = X
    return -y*cos(x+y-1), x*cos(x-y+1)
cos = np.cos
plt.figure(figsize=(10,10))
plt.subplot(111,aspect=1)  # optional: make scales same on the two axes

fieldplot(F,-2,5,-2.5,2.5,color='k',alpha=0.25)

phaseportrait(F, [(.25,0),(.25,.25),(.5,.5,-4,3),(-1,1),(-.5,.5),
                  (-1,-1),(2,1,-3,3),(1,1,-3,3)], color='k' )
x0,y0 = 0,0
x1,y1 = fsolve( F, (1,1) )
x2,y2 = fsolve( F, (3,-1) )
x3,y3 = fsolve( F, (0,4) )
x4,y4 = fsolve( F, (-2,-2) )
plt.plot(x0,y0,'ko', x1,y1,'ko', x2,y2,'ko', x3,y3,'ko', x4,y4,'ko')
```



8.2.6 Exercises

***Exercise* 8.2.1:** *For the systems below, find and classify the critical points, also indicate if the equilibria are stable, asymptotically stable, or unstable.*

a) $x' = -x + 3x^2,\ y' = -y$ *b*) $x' = x^2 + y^2 - 1,\ y' = x$

c) $x' = ye^x,\ y' = y - x + y^2$

Exercise 8.2.2: *Find the implicit equations of the trajectories of the following conservative systems. Next find their critical points (if any) and classify them.*

a) $x'' + x + x^3 = 0$

b) $\theta'' + \sin\theta = 0$

c) $z'' + (z-1)(z+1) = 0$

d) $x'' + x^2 + 1 = 0$

Exercise 8.2.3: *Find and classify the critical point(s) of $x' = -x^2$, $y' = -y^2$.*

Exercise 8.2.4: *Suppose $x' = -xy$, $y' = x^2 - 1 - y$.*

a) *Show there are two spiral sinks at $(-1, 0)$ and $(1, 0)$.*

b) *For any initial point of the form $(0, y_0)$, find what is the trajectory.*

c) *Can a trajectory starting at (x_0, y_0) where $x_0 > 0$ spiral into the critical point at $(-1, 0)$? Why or why not?*

Exercise 8.2.5: *In the example $x' = y$, $y' = y^3 - x$ show that for any trajectory, the distance from the origin is an increasing function. Conclude that the origin behaves like is a spiral source. Hint: Consider $f(t) = \big(x(t)\big)^2 + \big(y(t)\big)^2$ and show it has positive derivative.*

Exercise 8.2.6: *Suppose f is always positive. Find the trajectories of $x'' + f(x') = 0$. Are there any critical points?*

Exercise 8.2.7: *Suppose that $x' = f(x, y)$, $y' = g(x, y)$. Suppose that $g(x, y) > 1$ for all x and y. Are there any critical points? What can we say about the trajectories at t goes to infinity?*

Exercise 8.2.51: *For the system below, find and classify the critical points.*

a) $x' = x - y - x^2 + xy$, $y' = -y - x^2$

b) $x' = -x + \sin y$, $y' = 2x$

c) $x' = x^3 - 4x$, $y' = 3x^3 - 12x + y$

d) $x' = 4y + 5\sin x$, $y' = -3y$

Exercise 8.2.52: *For the systems below:*

i. *Find all critical points (equilibria).*

ii. *Determine the linearized system for each of the points in (i.). Classify each of the critical points (equilibria). If the critical points are* ***not spirals or centers, find all eigenvectors.***

iii. *Sketch the global phase portrait. Include the eigenvectors for the linearized system at each critical point and draw arrows on the solution curves to indicate the direction of flow.*

a) $x' = x^2 - x + y$, $y' = 2x^2 - 2x$

b) $x' = -3x + y,\ y' = -y + x^2$

c) $x' = -x + y^2,\ y' = x + 2y$

d) $x' = 2x + y,\ y' = -y + x^2$

e) $x' = y^2 - 1,\ y' = x^3 - 1$

f) $x' = 3y^2 - 3y + x,\ y' = y^2 - y$

Exercise* 8.2.101*: *For the systems below, find and classify the critical points.*

a) $x' = -x + x^2, y' = y$ b) $x' = y - y^2 - x, y' = -x$ c) $x' = xy, y' = x + y - 1$

Exercise* 8.2.102*: *Find the implicit equations of the trajectories of the following conservative systems. Next find their critical points (if any) and classify them.*

a) $x'' + x^2 = 4$ b) $x'' + e^x = 0$ c) $x'' + (x + 1)e^x = 0$

Exercise* 8.2.103*: *The conservative system* $x'' + x^3 = 0$ *is not almost linear. Classify its critical point(s) nonetheless.*

Exercise* 8.2.104*: *Derive an analogous classification of critical points for equations in one dimension, such as* $x' = f(x)$ *based on the derivative. A point* x_0 *is critical when* $f(x_0) = 0$ *and almost linear if in addition* $f'(x_0) \neq 0$. *Figure out if the critical point is stable or unstable depending on the sign of* $f'(x_0)$. *Explain. Hint: see* § 1.6.

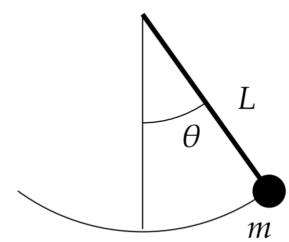
8.3 Applications of nonlinear systems

Note: 2 lectures, §6.3–§6.4 in [EP], §9.3, §9.5 in [BD]

In this section we study two very standard examples of nonlinear systems. First, we look at the nonlinear pendulum equation. We saw the pendulum equation's linearization before, but we noted it was only valid for small angles and short times. Now we find out what happens for large angles. Next, we look at the predator-prey equation, which finds various applications in modeling problems in biology, chemistry, economics, and elsewhere.

8.3.1 Pendulum

The first example we study is the pendulum equation $\theta'' + \frac{g}{L}\sin\theta = 0$. Here, θ is the angular displacement, g is the gravitational acceleration, and L is the length of the pendulum. In this equation we disregard friction, so we are talking about an idealized pendulum.



This equation is a conservative equation, so we can use our analysis of conservative equations from the previous section. Let us change the equation to a two-dimensional system in variables (θ, ω) by introducing the new variable ω:

$$\begin{bmatrix} \theta \\ \omega \end{bmatrix}' = \begin{bmatrix} \omega \\ -\frac{g}{L}\sin\theta \end{bmatrix}.$$

The critical points of this system are when $\omega = 0$ and $-\frac{g}{L}\sin\theta = 0$, or in other words if $\sin\theta = 0$. So the critical points are when $\omega = 0$ and θ is a multiple of π. That is, the points are $\ldots(-2\pi,0),(-\pi,0),(0,0),(\pi,0),(2\pi,0)\ldots$. While there are infinitely many critical points, they are all isolated. Let us compute the Jacobian matrix:

$$\begin{bmatrix} \frac{\partial}{\partial\theta}\Big(\omega\Big) & \frac{\partial}{\partial\omega}\Big(\omega\Big) \\ \frac{\partial}{\partial\theta}\Big(-\frac{g}{L}\sin\theta\Big) & \frac{\partial}{\partial\omega}\Big(-\frac{g}{L}\sin\theta\Big) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{L}\cos\theta & 0 \end{bmatrix}.$$

For conservative equations, there are two types of critical points. Either stable centers, or saddle points. The eigenvalues of the Jacobian matrix are $\lambda = \pm\sqrt{-\frac{g}{L}\cos\theta}$.

The eigenvalues are going to be real when $\cos\theta < 0$. This happens at the odd multiples of π. The eigenvalues are going to be purely imaginary when $\cos\theta > 0$. This happens at the even multiples of π. Therefore the system has a stable center at the points $\ldots(-2\pi,0),(0,0),(2\pi,0)\ldots$, and it has an unstable saddle at the points $\ldots(-3\pi,0),(-\pi,0),(\pi,0),(3\pi,0)\ldots$. Look at the phase diagram in Figure 8.6 on the facing page, where for simplicity we let $\frac{g}{L} = 1$.

In the linearized equation we have only a single critical point, the center at $(0,0)$. Now we see more clearly what we meant when we said the linearization is good for small

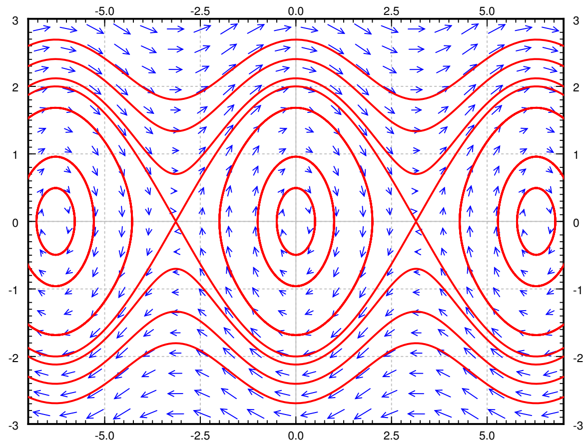


Figure 8.6: Phase plane diagram and some trajectories of the nonlinear pendulum equation.

angles. The horizontal axis is the deflection angle. The vertical axis is the angular velocity of the pendulum. Suppose we start at $\theta = 0$ (no deflection), and we start with a small angular velocity ω. Then the trajectory keeps going around the critical point $(0, 0)$ in an approximate circle. This corresponds to short swings of the pendulum back and forth. When θ stays small, the trajectories really look like circles and hence are very close to our linearization.

When we give the pendulum a big enough push, it goes across the top and keeps spinning about its axis. This behavior corresponds to the wavy curves that do not cross the horizontal axis in the phase diagram. Let us suppose we look at the top curves, when the angular velocity ω is large and positive. Then the pendulum is going around and around its axis. The velocity is going to be large when the pendulum is near the bottom, and the velocity is the smallest when the pendulum is close to the top of its loop.

At each critical point, there is an equilibrium solution. Consider the solution $\theta = 0$; the pendulum is not moving and is hanging straight down. This is a stable place for the pendulum to be, hence this is a *stable* equilibrium.

The other type of equilibrium solution is at the unstable point, for example $\theta = \pi$. Here the pendulum is upside down. Sure you can balance the pendulum this way and it will stay, but this is an *unstable* equilibrium. Even the tiniest push will make the pendulum start swinging wildly.

See Figure 8.7 on the next page for a diagram. The first picture is the stable equilibrium $\theta = 0$. The second picture corresponds to those "almost circles" in the phase diagram around $\theta = 0$ when the angular velocity is small. The next picture is the unstable equilibrium $\theta = \pi$. The last picture corresponds to the wavy lines for large angular velocities.

The quantity

$$\frac{1}{2}\omega^2 - \frac{g}{L}\cos\theta$$

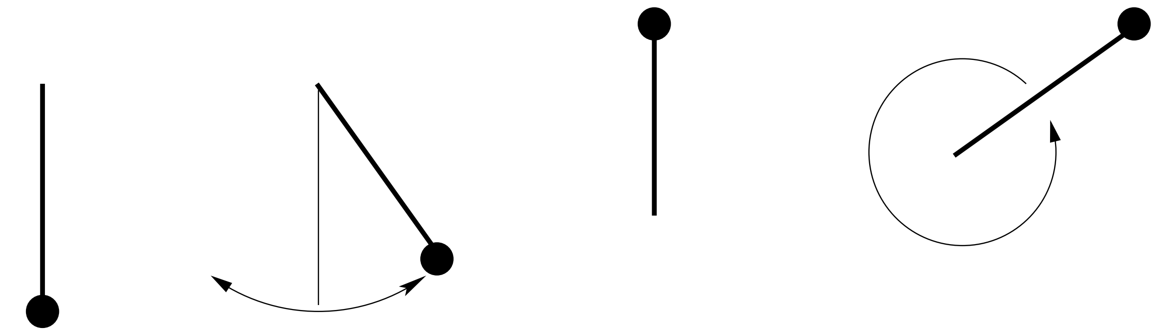


Figure 8.7: Various possibilities for the motion of the pendulum.

is conserved by any solution. This is the energy or the Hamiltonian of the system.

We have a conservative equation and so (exercise) the trajectories are given by

$$\omega = \pm\sqrt{\frac{2g}{L}\cos\theta + C},$$

for various values of C. Let us look at the initial condition of $(\theta_0, 0)$, that is, we take the pendulum to angle θ_0, and just let it go (initial angular velocity 0). We plug the initial conditions into the above and solve for C to obtain

$$C = -\frac{2g}{L}\cos\theta_0.$$

Thus the expression for the trajectory is

$$\omega = \pm\sqrt{\frac{2g}{L}}\sqrt{\cos\theta - \cos\theta_0}.$$

Let us figure out the period. That is, the time it takes for the pendulum to swing back and forth. We notice that the trajectory about the origin in the phase plane is symmetric about both the θ and the ω-axis. That is, in terms of θ, the time it takes from θ_0 to $-\theta_0$ is the same as it takes from $-\theta_0$ back to θ_0. Furthermore, the time it takes from $-\theta_0$ to 0 is the same as to go from 0 to θ_0. Therefore, let us find how long it takes for the pendulum to go from angle 0 to angle θ_0, which is a quarter of the full oscillation and then multiply by 4.

We figure out this time by finding $\frac{dt}{d\theta}$ and integrating from 0 to θ_0. The period is four times this integral. Let us stay in the region where ω is positive. Since $\omega = \frac{d\theta}{dt}$, inverting we get

$$\frac{dt}{d\theta} = \sqrt{\frac{L}{2g}}\frac{1}{\sqrt{\cos\theta - \cos\theta_0}}.$$

Therefore the period T is given by

$$T = 4\sqrt{\frac{L}{2g}} \int_0^{\theta_0} \frac{1}{\sqrt{\cos\theta - \cos\theta_0}} \, d\theta.$$

The integral is an improper integral, and we cannot in general evaluate it symbolically. We must resort to numerical approximation if we want to compute a particular T.

Recall from § 2.4, the linearized equation $\theta'' + \frac{g}{L}\theta = 0$ has period

$$T_{\text{linear}} = 2\pi\sqrt{\frac{L}{g}}.$$

We plot T, T_{linear}, and the relative error $\frac{T - T_{\text{linear}}}{T}$ in Figure 8.8. The relative error says how far is our approximation from the real period percentage-wise. Note that T_{linear} is simply a constant, it does not change with the initial angle θ_0. The actual period T gets larger and larger as θ_0 gets larger. Notice how the relative error is small when θ_0 is small. It is still only 15% when $\theta_0 = \frac{\pi}{2}$, that is, a 90 degree angle. The error is 3.8% when starting at $\frac{\pi}{4}$, a 45 degree angle. At a 5 degree initial angle, the error is only 0.048%.

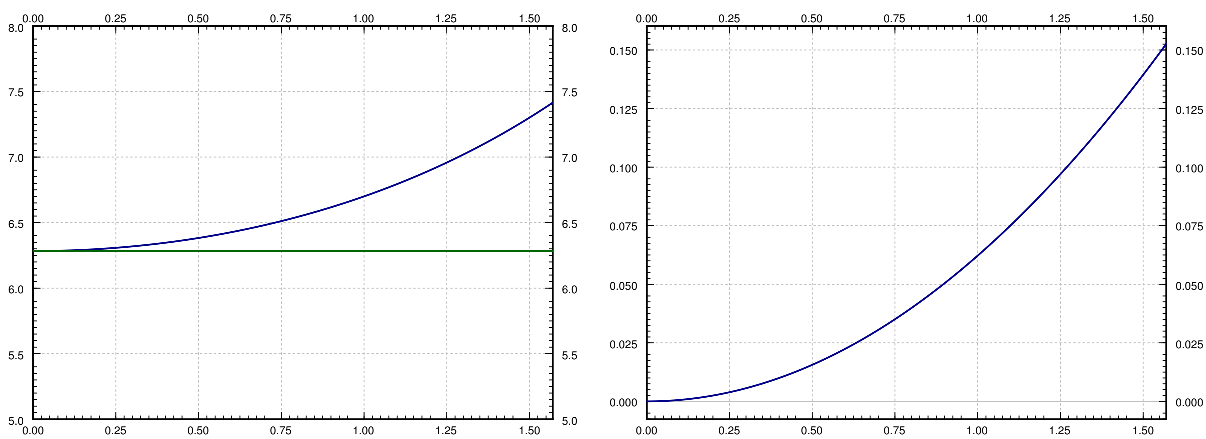


Figure 8.8: The plot of T and T_{linear} with $\frac{g}{L} = 1$ (left), and the plot of the relative error $\frac{T - T_{linear}}{T}$ (right), for θ_0 between 0 and $\pi/2$.

While it is not immediately obvious from the formula, it is true that

$$\lim_{\theta_0 \uparrow \pi} T = \infty.$$

That is, the period goes to infinity as the initial angle approaches the unstable equilibrium point. So if we put the pendulum almost upside down it may take a very long time before it gets down. This is consistent with the limiting behavior, where the exactly upside down pendulum never makes an oscillation, so we could think of that as infinite period.

8.3.2 Predator-prey or Lotka–Volterra systems

One of the most common simple applications of nonlinear systems are the so-called *predator-prey* or *Lotka–Volterra*[*] systems. For example, these systems arise when two species interact, one as the prey and one as the predator. It is then no surprise that the equations also see applications in economics. The system also arises in chemical reactions. In biology, this system of equations explains the natural periodic variations of populations of different species in nature. Before the application of differential equations, these periodic variations in the population baffled biologists.

We keep with the classical example of hares and foxes in a forest, it is the easiest to understand.

$$\begin{aligned} x &= \text{\# of hares (the prey)}, \\ y &= \text{\# of foxes (the predator)}. \end{aligned}$$

When there are a lot of hares, there is plenty of food for the foxes, so the fox population grows. However, when the fox population grows, the foxes eat more hares, so when there are lots of foxes, the hare population should go down, and vice versa. The Lotka–Volterra model proposes that this behavior is described by the system of equations

$$\begin{aligned} x' &= (a - by)x, \\ y' &= (cx - d)y, \end{aligned}$$

where a, b, c, d are some parameters that describe the interaction of the foxes and hares[†]. In this model, these are all positive numbers.

Let us analyze the idea behind this model. The model is a slightly more complicated idea based on the exponential population model. First expand,

$$x' = (a - by)x = ax - byx.$$

The hares are expected to simply grow exponentially in the absence of foxes, that is where the ax term comes in, the growth in population is proportional to the population itself. We are assuming the hares always find enough food and have enough space to reproduce. However, there is another component $-byx$, that is, the population also is decreasing proportionally to the number of foxes. Together we can write the equation as $(a - by)x$, so it is like exponential growth or decay but the constant depends on the number of foxes.

The equation for foxes is very similar, expand again

$$y' = (cx - d)y = cxy - dy.$$

The foxes need food (hares) to reproduce: the more food, the bigger the rate of growth, hence the cxy term. On the other hand, there are natural deaths in the fox population, and hence the $-dy$ term.

[*]Named for the American mathematician, chemist, and statistician Alfred James Lotka (1880–1949) and the Italian mathematician and physicist Vito Volterra (1860–1940).

[†]This interaction does not end well for the hare.

Without further delay, let us start with an explicit example. Suppose the equations are

$$x' = (0.4 - 0.01y)x, \qquad y' = (0.003x - 0.3)y.$$

See Figure 8.9 for the phase portrait. In this example it makes sense to also plot x and y as graphs with respect to time. Therefore the second graph in Figure 8.9 is the graph of x and y on the vertical axis (the prey x is the thinner line with taller peaks), against time on the horizontal axis. The particular solution graphed was with initial conditions of 20 foxes and 50 hares.

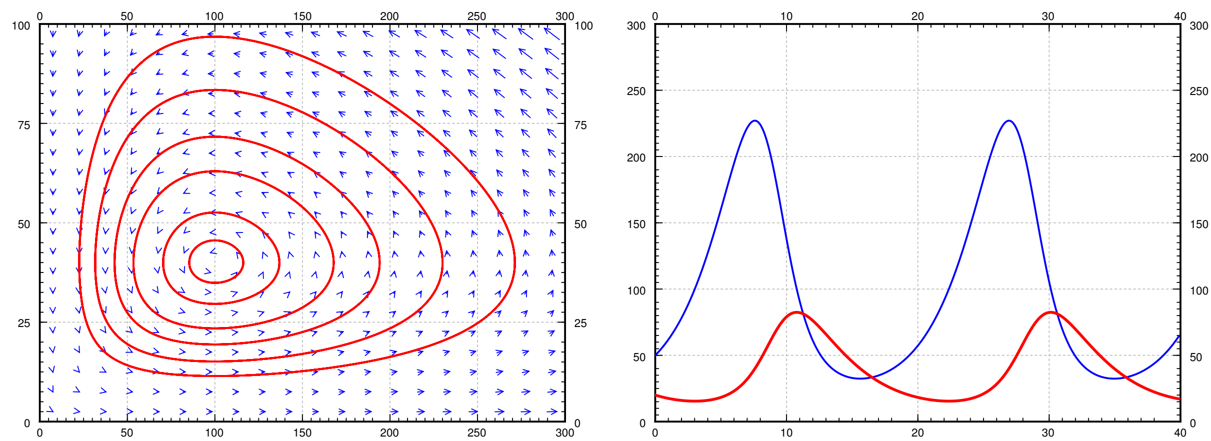


Figure 8.9: The phase portrait (left) and graphs of x and y for a sample solution (right).

Let us analyze what we see on the graphs. We work in the general setting rather than putting in specific numbers. We start with finding the critical points. Set $(a - by)x = 0$, and $(cx - d)y = 0$. The first equation is satisfied if either $x = 0$ or $y = {}^{a}/_{b}$. If $x = 0$, the second equation implies $y = 0$. If $y = {}^{a}/_{b}$, the second equation implies $x = {}^{d}/_{c}$. There are two equilibria: at $(0, 0)$ when there are no animals at all, and at $({}^{d}/_{c}, {}^{a}/_{b})$. In our specific example $x = {}^{d}/_{c} = 100$, and $y = {}^{a}/_{b} = 40$. This is the point where there are 100 hares and 40 foxes.

We compute the Jacobian matrix:

$$\begin{bmatrix} a - by & -bx \\ cy & cx - d \end{bmatrix}.$$

At the origin $(0, 0)$ we get the matrix $\left[\begin{smallmatrix} a & 0 \\ 0 & -d \end{smallmatrix}\right]$, so the eigenvalues are a and $-d$, hence real and of opposite signs. So the critical point at the origin is a saddle. This makes sense. If you started with some foxes but no hares, then the foxes would go extinct, that is, you would approach the origin. If you started with no foxes and a few hares, then the hares would keep multiplying without check, and so you would go away from the origin.

OK, how about the other critical point at $({}^{d}/_{c}, {}^{a}/_{b})$. Here the Jacobian matrix becomes

$$\begin{bmatrix} 0 & -\frac{bd}{c} \\ \frac{ac}{b} & 0 \end{bmatrix}.$$

The eigenvalues satisfy $\lambda^2 + ad = 0$. In other words, $\lambda = \pm i\sqrt{ad}$. The eigenvalues being purely imaginary, we are in the case where we cannot quite decide using only linearization. We could have a stable center, spiral sink, or a spiral source. That is, the equilibrium could be asymptotically stable, stable, or unstable. Of course I gave you a picture above that seems to imply it is a stable center. But never trust a picture only. Perhaps the oscillations are getting larger and larger, but only *very* slowly. Of course this would be bad as it would imply something will go wrong with our population sooner or later. And I only graphed a very specific example with very specific trajectories.

How can we be sure we are in the stable situation? As we said before, in the case of purely imaginary eigenvalues, we have to do a bit more work. Previously we found that for conservative systems, there was a certain quantity that was conserved on the trajectories, and hence the trajectories had to go in closed loops. We can use a similar technique here. We just have to figure out what is the conserved quantity. After some trial and error we find the constant

$$C = \frac{y^a x^d}{e^{cx+by}} = y^a x^d e^{-cx-by}$$

is conserved. Such a quantity is called the *constant of motion*. Let us check C really is a constant of motion. How do we check, you say? Well, a constant is something that does not change with time, so let us compute the derivative with respect to time:

$$C' = a y^{a-1} y' x^d e^{-cx-by} + y^a d x^{d-1} x' e^{-cx-by} + y^a x^d e^{-cx-by}(-cx' - by').$$

Our equations give us what x' and y' are so let us plug those in:

$$\begin{aligned} C' &= a y^{a-1}(cx-d) y x^d e^{-cx-by} + y^a d x^{d-1}(a-by) x e^{-cx-by} \\ &\quad + y^a x^d e^{-cx-by}\big(-c(a-by)x - b(cx-d)y\big) \\ &= y^a x^d e^{-cx-by}\Big(a(cx-d) + d(a-by) + \big(-c(a-by)x - b(cx-d)y\big)\Big) \\ &= 0. \end{aligned}$$

So along the trajectories C is constant. In fact, the expression $C = \frac{y^a x^d}{e^{cx+by}}$ gives us an implicit equation for the trajectories. In any case, once we have found this constant of motion, it must be true that the trajectories are simple curves, that is, the level curves of $\frac{y^a x^d}{e^{cx+by}}$. It turns out, the critical point at $({}^{d}\!/_{c}, {}^{a}\!/_{b})$ is a maximum for C (left as an exercise). So $({}^{d}\!/_{c}, {}^{a}\!/_{b})$ is a stable equilibrium point, and we do not have to worry about the foxes and hares going extinct or their populations exploding.

One blemish on this wonderful model is that the number of foxes and hares are discrete quantities and we are modeling with continuous variables. Our model has no problem with there being 0.1 fox in the forest for example, while in reality that makes no sense. The approximation is a reasonable one as long as the number of foxes and hares are large, but it does not make much sense for small numbers. One must be careful in interpreting any results from such a model.

An interesting consequence (perhaps counterintuitive) of this model is that adding animals to the forest might lead to extinction, because the variations will get too big, and one of the populations will get close to zero. For example, suppose there are 20 foxes and 50 hares as before, but now we bring in more foxes, bringing their number to 200. If we run the computation, we find the number of hares will plummet to just slightly more than 1 hare in the whole forest. In reality that most likely means the hares die out, and then the foxes will die out as well as they will have nothing to eat.

Showing that a system of equations has a stable solution can be a very difficult problem. When Isaac Newton put forth his laws of planetary motions, he proved that a single planet orbiting a single sun is a stable system. But any solar system with more than 1 planet proved very difficult indeed. In fact, such a system behaves chaotically (see § 8.5), meaning small changes in initial conditions lead to very different long-term outcomes. From numerical experimentation and measurements, we know the earth will not fly out into the empty space or crash into the sun, for at least some millions of years or so. But we do not know what happens beyond that.

8.3.3 Exercises

***Exercise* 8.3.1:** *Take the* damped nonlinear pendulum equation $\theta'' + \mu\theta' + (g/L)\sin\theta = 0$ *for some* $\mu > 0$ (*that is, there is some friction*).

a) *Suppose* $\mu = 1$ *and* $g/L = 1$ *for simplicity, find and classify the critical points.*

b) *Do the same for any* $\mu > 0$ *and any* g *and* L, *but such that the damping is small, in particular,* $\mu^2 < 4(g/L)$.

c) *Explain what your findings mean, and if it agrees with what you expect in reality.*

***Exercise* 8.3.2:** *Suppose the hares do not grow exponentially, but logistically. In particular consider*

$$x' = (0.4 - 0.01y)x - \gamma x^2, \qquad y' = (0.003x - 0.3)y.$$

For the following two values of γ, *find and classify all the critical points in the positive quadrant, that is, for* $x \geq 0$ *and* $y \geq 0$. *Then sketch the phase diagram. Discuss the implication for the long term behavior of the population.*

a) $\gamma = 0.001$, *b*) $\gamma = 0.01$.

***Exercise* 8.3.3:**

a) *Suppose* x *and* y *are positive variables. Show* $\frac{yx}{e^{x+y}}$ *attains a maximum at* $(1, 1)$.

b) *Suppose* a, b, c, d *are positive constants, and also suppose* x *and* y *are positive variables. Show* $\frac{y^a x^d}{e^{cx+by}}$ *attains a maximum at* $(d/c, a/b)$.

***Exercise* 8.3.4:** *Suppose that for the pendulum equation we take a trajectory giving the spinning-around motion, for example* $\omega = \sqrt{\frac{2g}{L}\cos\theta + \frac{2g}{L} + \omega_0^2}$*. This is the trajectory where the lowest angular velocity is* ω_0^2*. Find an integral expression for how long it takes the pendulum to go all the way around.*

***Exercise* 8.3.5 (challenging):** *Take the pendulum, suppose the initial position is* $\theta = 0$.

a) *Find the expression for* ω *giving the trajectory with initial condition* $(0, \omega_0)$*. Hint: Figure out what C should be in terms of* ω_0.

b) *Find the crucial angular velocity* ω_1*, such that for any higher initial angular velocity, the pendulum will keep going around its axis, and for any lower initial angular velocity, the pendulum will simply swing back and forth. Hint: When the pendulum doesn't go over the top the expression for* ω *will be undefined for some* θ*s.*

c) *What do you think happens if the initial condition is* $(0, \omega_1)$*, that is, the initial angle is 0, and the initial angular velocity is exactly* ω_1.

***Exercise* 8.3.51:** *For each of the following nonlinear, damped mass-spring equations,*

i. *Write the corresponding* 1^{st}*- order system.*

ii. *Find all critical points (equilibria).*

iii. *Determine the linearized system for each of the points in (i.). Classify each of the critical points. If the critical points are* ***not spirals or centers, find all eigenvectors.***

iv. *Sketch the global phase portrait for the almost-linear system. Include the eigenvectors for the linearized system at each critical point and draw arrows on the solution curves to indicate the direction of flow.*

a) $x'' + 9x - x^3 = 0$

b) $x'' + x' + 2x - x^2 = 0$

c) $x'' + 2x' + 4x - x^3 = 0$

d) $x'' + 9x - 10x^3 + x^5 = 0$

***Exercise* 8.3.52:** *For each of the following nonlinear systems,*

i. *Find all critical points (equilibria).*

ii. *Determine the linearized system for each of the points in (i.). Classify each of the critical points. If the critical points are* ***not spirals or centers, find all eigenvectors.***

iii. *Sketch the phase portrait, showing at least 4 critical points. Include the eigenvectors for the linearized system at each critical point and draw arrows on the solution curves to indicate the direction of flow.*

a) $x' = 2y,\ y' = \sin x - y$

b) $x' = -2x + 4\sin y,\ y' = 2x$

c) $x' = x - y,\ y' = 2\sin x$

d) $x' = 3y,\ y' = \sin(\pi x)$

e) $x' = \sin(\pi y),\ y' = x + y$

Exercise* 8.3.101: Take the damped nonlinear pendulum equation $\theta'' + \mu\theta' + (g/L)\sin\theta = 0$ for some $\mu > 0$ (that is, there is friction). Suppose the friction is large, in particular $\mu^2 > 4(g/L)$.*

a) *Find and classify the critical points.*

b) *Explain what your findings mean, and if it agrees with what you expect in reality.*

Exercise* 8.3.102: Suppose we have the system predator-prey system where the foxes are also killed at a constant rate h (h foxes killed per unit time): $x' = (a - by)x,\ y' = (cx - d)y - h$.*

a) *Find the critical points and the Jacobian matrices of the system.*

b) *Put in the constants $a = 0.4$, $b = 0.01$, $c = 0.003$, $d = 0.3$, $h = 10$. Analyze the critical points. What do you think it says about the forest?*

***Exercise* 8.3.103** (challenging)*: Suppose the foxes never die. That is, we have the system $x' = (a - by)x,\ y' = cxy$. Find the critical points and notice they are not isolated. What will happen to the population in the forest if it starts at some positive numbers. Hint: Think of the constant of motion.*

8.4 Limit cycles

Note: less than 1 lecture, discussed in §6.1 and §6.4 in [EP] , §9.7 in [BD]

For nonlinear systems, trajectories do not simply need to approach or leave a single point. They may in fact approach a larger set, such as a circle or another closed curve.

Example 8.4.1: The *Van der Pol oscillator** is the following equation

$$x'' - \mu(1 - x^2)x' + x = 0,$$

where μ is some positive constant. The Van der Pol oscillator originated with electrical circuits, but finds applications in diverse fields such as biology, seismology, and other physical sciences.

For simplicity, let us use $\mu = 1$. A phase diagram is given in the left-hand plot in Figure 8.10. Notice how the trajectories seem to very quickly settle on a closed curve. On the right-hand side is the plot of a single solution for $t = 0$ to $t = 30$ with initial conditions $x(0) = 0.1$ and $x'(0) = 0.1$. The solution quickly tends to a periodic solution.

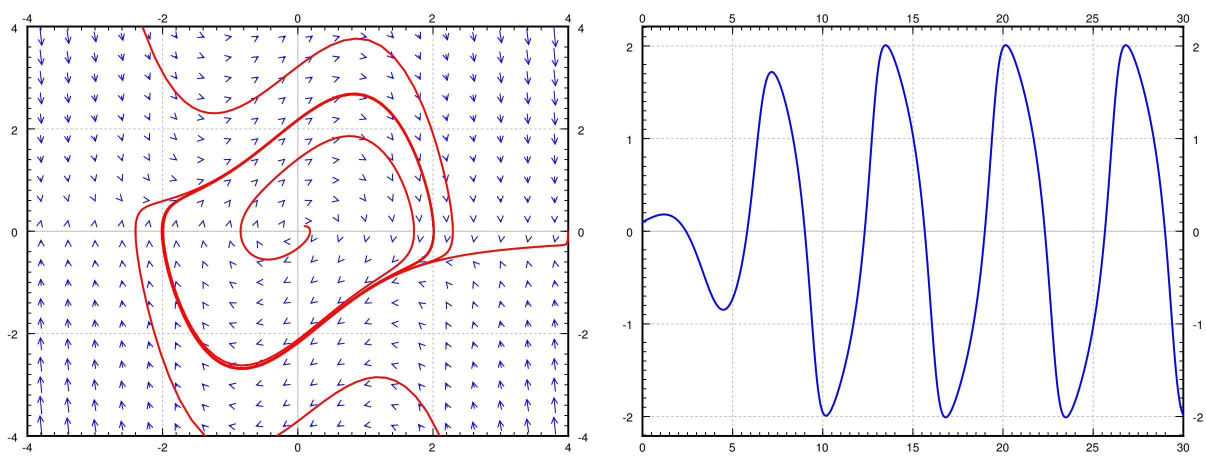


Figure 8.10: The phase portrait (left) and a graph of a sample solution of the Van der Pol oscillator.

The Van der Pol oscillator is an example of so-called *relaxation oscillation*. The word relaxation comes from the sudden jump (the very steep part of the solution). For larger μ the steep part becomes even more pronounced, for small μ the limit cycle looks more like a circle. In fact, setting $\mu = 0$, we get $x'' + x = 0$, which is a linear system with a center and all trajectories become circles.

A trajectory in the phase portrait that is a closed curve (a curve that is a loop) is called a *closed trajectory*. A *limit cycle* is a closed trajectory such that at least one other trajectory spirals into it (or spirals out of it). For example, the closed curve in the phase portrait for

*Named for the Dutch physicist Balthasar van der Pol (1889–1959).

the Van der Pol equation is a limit cycle. If all trajectories that start near the limit cycle spiral into it, the limit cycle is called *asymptotically stable*. The limit cycle in the Van der Pol oscillator is asymptotically stable.

Given a closed trajectory on an autonomous system, any solution that starts on it is periodic. Such a curve is called a *periodic orbit*. More precisely, if $\big(x(t), y(t)\big)$ is a solution such that for some t_0 the point $\big(x(t_0), y(t_0)\big)$ lies on a periodic orbit, then both $x(t)$ and $y(t)$ are periodic functions (with the same period). That is, there is some number P such that $x(t) = x(t+P)$ and $y(t) = y(t+P)$.

Consider the system

$$x' = f(x,y), \qquad y' = g(x,y), \tag{8.3}$$

where the functions f and g have continuous derivatives in some region R in the plane.

Theorem 8.4.1 (Poincaré–Bendixson*). *Suppose R is a closed bounded region (a region in the plane that includes its boundary and does not have points arbitrarily far from the origin). Suppose $\big(x(t), y(t)\big)$ is a solution of (8.3) in R that exists for all $t \geq t_0$. Then either the solution is a periodic function, or the solution tends towards a periodic solution in R.*

The main point of the theorem is that if you find one solution that exists for all t large enough (that is, as t goes to infinity) and stays within a bounded region, then you have found either a periodic orbit, or a solution that spirals towards a limit cycle or tends to a critical point. That is, in the long term, the behavior is very close to a periodic function. Note that a constant solution at a critical point is periodic (with any period). The theorem is more a qualitative statement rather than something to help us in computations. In practice it is hard to find analytic solutions and so hard to show rigorously that they exist for all time. But if we think the solution exists we numerically solve for a large time to approximate the limit cycle. Another caveat is that the theorem only works in two dimensions. In three dimensions and higher, there is simply too much room.

The theorem applies to all solutions in the Van der Pol oscillator. Solutions that start at any point except the origin $(0,0)$ will tend to the periodic solution around the limit cycle, and if the initial condition of $(0,0)$ will lead to the constant solution $x = 0$, $y = 0$.

Example 8.4.2: Consider

$$x' = y + {(x^2+y^2-1)}^2 x, \qquad y' = -x + {(x^2+y^2-1)}^2 y.$$

A vector field along with solutions with initial conditions $(1.02, 0)$, $(0.9, 0)$, and $(0.1, 0)$ are drawn in Figure 8.11 on the next page.

Notice that points on the unit circle (distance one from the origin) satisfy $x^2 + y^2 - 1 = 0$. And $x(t) = \sin(t)$, $y = \cos(t)$ is a solution of the system. Therefore we have a closed trajectory. For points off the unit circle, the second term in x' pushes the solution further away from the y-axis than the system $x' = y$, $y' = -x$, and y' pushes the solution further away from the x-axis than the linear system $x' = y$, $y' = -x$. In other words for all other initial conditions the trajectory will spiral out.

*Ivar Otto Bendixson (1861–1935) was a Swedish mathematician.

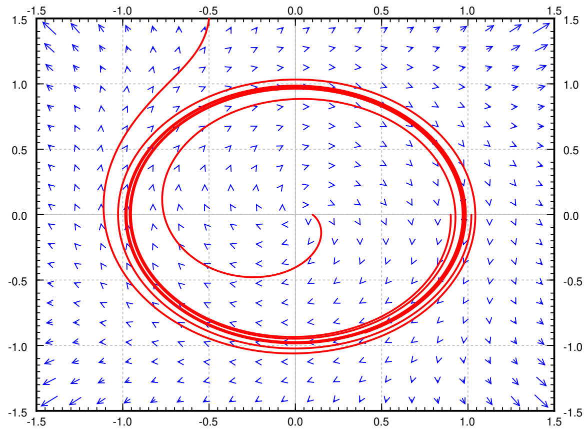


Figure 8.11: Unstable limit cycle example.

This means that for initial conditions inside the unit circle, the solution spirals out towards the periodic solution on the unit circle, and for initial conditions outside the unit circle the solutions spiral off towards infinity. Therefore the unit circle is a limit cycle, but not an asymptotically stable one. The Poincaré–Bendixson Theorem applies to the initial points inside the unit circle, as those solutions stay bounded, but not to those outside, as those solutions go off to infinity.

A very similar analysis applies to the system

$$x' = y + (x^2 + y^2 - 1)x, \qquad y' = -x + (x^2 + y^2 - 1)y.$$

We still obtain a closed trajectory on the unit circle, and points outside the unit circle spiral out to infinity, but now points inside the unit circle spiral towards the critical point at the origin. So this system does not have a limit cycle, even though it has a closed trajectory.

Due to the Picard theorem (Theorem 3.1.1 on page 135) we find that no matter where we are in the plane we can always find a solution a little bit further in time, as long as f and g have continuous derivatives. So if we find a closed trajectory in an autonomous system, then for every initial point inside the closed trajectory, the solution will exist for all time and it will stay bounded (it will stay inside the closed trajectory). So the moment we found the solution above going around the unit circle, we knew that for every initial point inside the circle, the solution exists for all time and the Poincaré–Bendixson theorem applies.

Let us next look for conditions when limit cycles (or periodic orbits) do not exist. We assume the equation (8.3) is defined on a *simply connected region*, that is, a region with no holes we can go around. For example the entire plane is a simply connected region, and so is the inside of the unit disc. However, the entire plane minus a point is not a simply connected domain as it has a “hole” at the origin.

Theorem 8.4.2 (Bendixson–Dulac*). *Suppose R is a simply connected region, and the expression*†

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}$$

is either always positive or always negative on R (except perhaps a small set such as on isolated points or curves) then the system (8.3) has no closed trajectory inside R.

The theorem gives us a way of ruling out the existence of a closed trajectory, and hence a way of ruling out limit cycles. The exception about points or curves means that we can allow the expression to be zero at a few points, or perhaps on a curve, but not on any larger set.

Example 8.4.3: Let us look at $x' = y + y^2 e^x$, $y' = x$ in the entire plane (see Example 8.2.2 on page 271). The entire plane is simply connected and so we can apply the theorem. We compute $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = y^2 e^x + 0$. The function $y^2 e^x$ is always positive except on the line $y = 0$. Therefore, via the theorem, the system has no closed trajectories.

In some books (or the internet) the theorem is not stated carefully and it concludes there are no periodic solutions. That is not quite right. The example above has two critical points and hence it has constant solutions, and constant functions are periodic. The conclusion of the theorem should be that there exist no trajectories that form closed curves. Another way to state the conclusion of the theorem would be to say that there exist no nonconstant periodic solutions that stay in R.

Example 8.4.4: Let us look at a somewhat more complicated example. Take the system $x' = -y - x^2$, $y' = -x + y^2$ (see Example 8.2.1 on page 270). We compute $\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} = -2x + 2y = 2(-x + y)$. This expression takes on both signs, so if we are talking about the whole plane we cannot simply apply the theorem. However, we could apply it on the set where $-x + y \geq 0$. Via the theorem, there is no closed trajectory in that set. Similarly, there is no closed trajectory in the set $-x + y \leq 0$. We cannot conclude (yet) that there is no closed trajectory in the entire plane. Perhaps half of it is in the set where $-x + y \geq 0$ and the other half is in the set where $-x + y \leq 0$.

The key is to look at the line where $-x+y = 0$, or $x = y$. On this line $x' = -y-x^2 = -x-x^2$ and $y' = -x + y^2 = -x + x^2$. In particular, when $x = y$ then $x' \leq y'$. That means that the arrows, the vectors (x', y'), always point into the set where $-x + y \geq 0$. There is no way we can start in the set where $-x + y \geq 0$ and go into the set where $-x + y \leq 0$. Once we are in the set where $-x + y \geq 0$, we stay there. So no closed trajectory can have points in both sets.

Example 8.4.5: Consider $x' = y + (x^2 + y^2 - 1)x$, $y' = -x + (x^2 + y^2 - 1)y$, and consider the region R given by $x^2 + y^2 > \frac{1}{2}$. That is, R is the region outside a circle of radius $\frac{1}{\sqrt{2}}$

*Henri Dulac (1870–1955) was a French mathematician.

†Usually the expression in the Bendixson–Dulac Theorem is $\frac{\partial(\varphi f)}{\partial x} + \frac{\partial(\varphi g)}{\partial y}$ for some continuously differentiable function φ. For simplicity, let us just consider the case $\varphi = 1$.

centered at the origin. Then there is a closed trajectory in R, namely $x = \cos(t)$, $y = \sin(t)$. Furthermore,

$$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial x} = 4x^2 + 4y^2 - 2,$$

which is always positive on R. So what is going on? The Bendixson–Dulac theorem does not apply since the region R is not simply connected—it has a hole, the circle we cut out!

8.4.1 Exercises

Exercise* 8.4.1*: *Show that the following systems have no closed trajectories.*

a) $x' = x^3 + y, \quad y' = y^3 + x^2,$

b) $x' = e^{x-y}, \quad y' = e^{x+y},$

c) $x' = x + 3y^2 - y^3, \quad y' = y^3 + x^2.$

Exercise* 8.4.2*: *Formulate a condition for a* 2*-by-*2 *linear system* $\vec{x}' = A\vec{x}$ *to not be a center using the Bendixson–Dulac theorem. That is, the theorem says something about certain elements of* A*.*

Exercise* 8.4.3*: *Explain why the Bendixson–Dulac Theorem does not apply for any conservative system* $x'' + h(x) = 0$*.*

Exercise* 8.4.4*: *A system such as* $x' = x$, $y' = y$ *has solutions that exist for all time* t*, yet there are no closed trajectories. Explain why the Poincaré–Bendixson Theorem does not apply.*

Exercise* 8.4.5*: *Differential equations can also be given in different coordinate systems. Suppose we have the system* $r' = 1 - r^2$, $\theta' = 1$ *given in polar coordinates. Find all the closed trajectories and check if they are limit cycles and if so, if they are asymptotically stable or not.*

Exercise* 8.4.101*: *Show that the following systems have no closed trajectories.*

a) $x' = x + y^2, \quad y' = y + x^2,$

b) $x' = -x\sin^2(y), \quad y' = e^x,$

c) $x' = xy, \quad y' = x + x^2.$

Exercise* 8.4.102*: *Suppose an autonomous system in the plane has a solution* $x = \cos(t) + e^{-t}$, $y = \sin(t) + e^{-t}$*. What can you say about the system (in particular about limit cycles and periodic solutions)?*

Exercise* 8.4.103*: *Show that the limit cycle of the Van der Pol oscillator (for* $\mu > 0$*) must not lie completely in the set where* $-1 < x < 1$*. Compare with* *Figure* 8.10 *on page* 288*.*

Exercise* 8.4.104*: *Suppose we have the system* $r' = \sin(r)$, $\theta' = 1$ *given in polar coordinates. Find all the closed trajectories.*

8.5 Chaos

Note: 1 lecture, §6.5 in [EP], §9.8 in [BD]

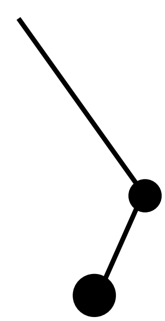
You have surely heard the story about the flap of a butterfly wing in the Amazon causing hurricanes in the North Atlantic. In a prior section, we mentioned that a small change in initial conditions of the planets can lead to very different configuration of the planets in the long term. These are examples of *chaotic systems*. Mathematical chaos is not really chaos, there is precise order behind the scenes. Everything is still deterministic. However a chaotic system is extremely sensitive to initial conditions. This also means even small errors induced via numerical approximation create large errors very quickly, so it is almost impossible to numerically approximate for long times. This is a large part of the trouble, as chaotic systems cannot be in general solved analytically.

Take the weather, the most well-known chaotic system. A small change in the initial conditions (the temperature at every point of the atmosphere for example) produces drastically different predictions in relatively short time, and so we cannot accurately predict weather. And we do not actually know the exact initial conditions. We measure temperatures at a few points with some error, and then we somehow estimate what is in between. There is no way we can accurately measure the effects of every butterfly wing. Then we solve the equations numerically introducing new errors. You should not trust weather prediction more than a few days out.

Chaotic behavior was first noticed by Edward Lorenz* in the 1960s when trying to model thermally induced air convection (movement). Lorentz was looking at the relatively simple system:

$$x' = -10x + 10y, \qquad y' = 28x - y - xz, \qquad z' = -\frac{8}{3}z + xy.$$

A small change in the initial conditions yields a very different solution after a reasonably short time.



A simple example the reader can experiment with, and which displays chaotic behavior, is a double pendulum. The equations for this setup are somewhat complicated, and their derivation is quite tedious, so we will not bother to write them down. The idea is to put a pendulum on the end of another pendulum. The movement of the bottom mass will appear chaotic. This type of chaotic system is a basis for a whole number of office novelty desk toys. It is simple to build a version. Take a piece of a string. Tie two heavy nuts at different points of the string; one at the end, and one a bit above. Now give the bottom nut a little push. As long as the swings are not too big and the string stays tight, you have a double pendulum system.

*Edward Norton Lorenz (1917–2008) was an American mathematician and meteorologist.

8.5.1 Duffing equation and strange attractors

Let us study the so-called *Duffing equation*:

$$x'' + ax' + bx + cx^3 = C\cos(\omega t).$$

Here a, b, c, C, and ω are constants. Except for the cx^3 term, this equation looks like a forced mass-spring system. The cx^3 means the spring does not exactly obey Hooke's law (which no real-world spring actually does obey exactly). When c is not zero, the equation does not have a closed form solution, so we must resort to numerical solutions, as is usual for nonlinear systems. Not all choices of constants and initial conditions exhibit chaotic behavior. Let us study

$$x'' + 0.05x' + x^3 = 8\cos(t).$$

The equation is not autonomous, so we cannot draw the vector field in the phase plane. We can still draw the trajectories. In Figure 8.12 we plot trajectories for t going from 0 to 15, for two very close initial conditions $(x, x') = (2, 3)$ and $(x, x') = (2, 2.9)$, and also the solutions in the (x, t) space. The two trajectories are close at first, but after a while diverge significantly. This sensitivity to initial conditions is precisely what we mean by the system behaving chaotically.

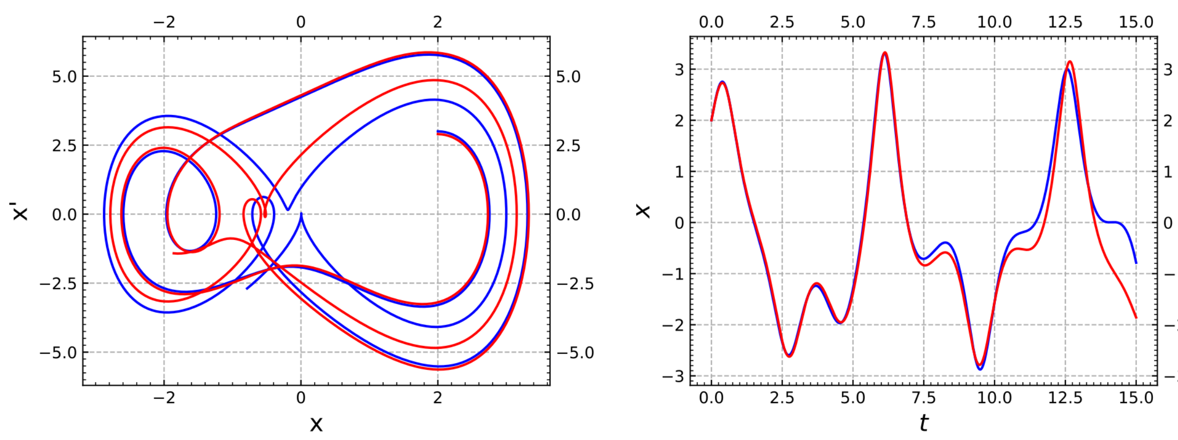


Figure 8.12: On left, two trajectories in phase space for $0 \le t \le 15$*, for the Duffing equation one with initial conditions* $(2, 3)$ *and the other with* $(2, 2.9)$*. On right the two solutions in* (x, t)*-space.*

Let us see the long term behavior. In Figure 8.13 on the next page, we plot the behavior of the system for initial conditions $(2, 3)$ for a longer period of time. It is hard to see any particular pattern in the shape of the solution except that it seems to oscillate, but each oscillation appears quite unique. The oscillation is expected due to the forcing term. We mention that to produce the picture accurately, a ridiculously large number of steps* had to be used in the numerical algorithm, as even small errors quickly propagate in a chaotic system.

*In fact for reference, 30,000 steps were used with the Runge–Kutta algorithm, see exercises in § 1.7.

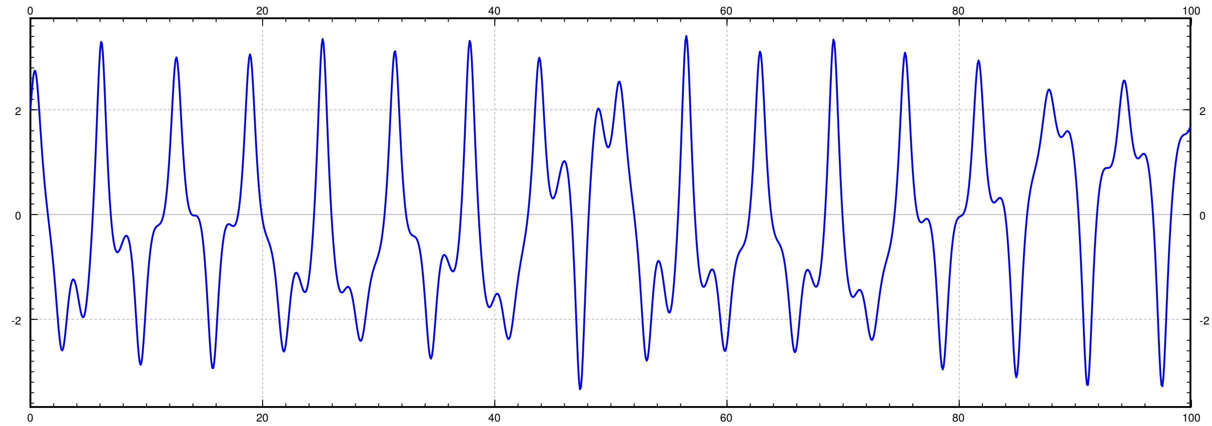


Figure 8.13: The solution to the given Duffing equation for t from 0 to 100.

It is very difficult to analyze chaotic systems, or to find the order behind the madness, but let us try to do something that we did for the standard mass-spring system. One way we analyzed the system is that we figured out what was the long term behavior (not dependent on initial conditions). From the figure above, it is clear that we will not get a nice exact description of the long term behavior for this chaotic system, but perhaps we can find some order to what happens on each "oscillation" and what do these oscillations have in common.

The concept we explore is that of a *Poincaré section**. Instead of looking at t in a certain interval, we look at where the system is at a certain sequence of points in time. Imagine flashing a strobe at a fixed frequency and drawing the points where the solution is during the flashes. The right strobing frequency depends on the system in question. The correct frequency for the forced Duffing equation (and other similar systems) is the frequency of the forcing term. For the Duffing equation above, find a solution $\big(x(t), y(t)\big)$, and look at the points

$$\big(x(0), y(0)\big), \quad \big(x(2\pi), y(2\pi)\big), \quad \big(x(4\pi), y(4\pi)\big), \quad \big(x(6\pi), y(6\pi)\big), \quad \ldots$$

As we are really not interested in the transient part of the solution, that is, the part of the solution that depends on the initial condition, we skip some number of steps in the beginning. For example, we might skip the first 100 such steps and start plotting points at $t = 100(2\pi)$, that is

$$\big(x(200\pi), y(200\pi)\big), \quad \big(x(202\pi), y(202\pi)\big), \quad \big(x(204\pi), y(204\pi)\big), \quad \ldots$$

The plot of these points is the Poincaré section. After plotting enough points, a curious pattern emerges in Figure 8.14 on the following page (the left-hand picture), a so-called *strange attractor*.

*Named for the French polymath Jules Henri Poincaré (1854–1912).

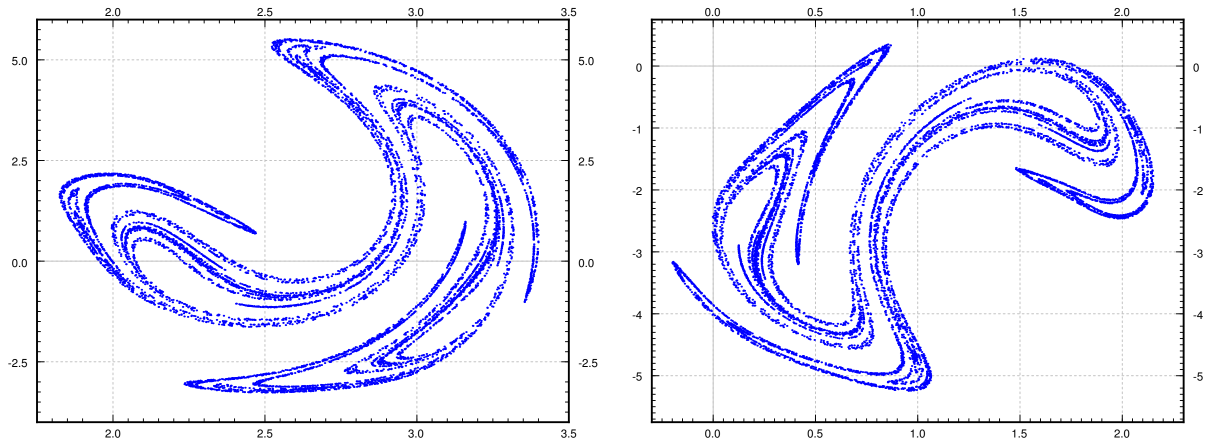


Figure 8.14: Strange attractor. The left plot is with no phase shift, the right plot has phase shift $\pi/4$.

Given a sequence of points, an *attractor* is a set towards which the points in the sequence eventually get closer and closer to, that is, they are attracted. The Poincaré section is not really the attractor itself, but as the points are very close to it, we see its shape. The strange attractor is a very complicated set. It has fractal structure, that is, if you zoom in as far as you want, you keep seeing the same complicated structure.

The initial condition makes no difference. If we start with a different initial condition, the points eventually gravitate towards the attractor, and so as long as we throw away the first few points, we get the same picture. Similarly small errors in the numerical approximations do not matter here.

An amazing thing is that a chaotic system such as the Duffing equation is not random at all. There is a very complicated order to it, and the strange attractor says something about this order. We cannot quite say what state the system will be in eventually, but given the fixed strobing frequency we narrow it down to the points on the attractor.

If we use a phase shift, for example $\pi/4$, and look at the times

$$\pi/4, \quad 2\pi + \pi/4, \quad 4\pi + \pi/4, \quad 6\pi + \pi/4, \quad \ldots$$

we obtain a slightly different attractor. The picture is the right-hand side of Figure 8.14. It is as if we had rotated, moved, and slightly distorted the original. For each phase shift you can find the set of points towards which the system periodically keeps coming back to.

Study the pictures and notice especially the scales—where are these attractors located in the phase plane. Notice the regions where the strange attractor lives and compare it to the plot of the trajectories in Figure 8.12 on page 294.

Let us compare this section to the discussion in § 2.6 about forced oscillations. Take the equation

$$x'' + 2px' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t).$$

This is like the Duffing equation, but with no x^3 term. The steady periodic solution is of the form

$$x = C\cos(\omega t + \gamma).$$

Strobing using the frequency ω, we obtain a single point in the phase space. The attractor in this setting is a single point—an expected result as the system is not chaotic. It was the opposite of chaotic: Any difference induced by the initial conditions dies away very quickly, and we settle into always the same steady periodic motion.

8.5.2 The Lorenz system

In two dimensions to find chaotic behavior, we must study forced, or non-autonomous, systems such as the Duffing equation. The Poincaré–Bendixson Theorem says that a solution to an autonomous two-dimensional system that exists for all time in the future and does not go towards infinity is periodic or tends towards a periodic solution. Hardly the chaotic behavior we are looking for.

In three dimensions even autonomous systems can be chaotic. Let us very briefly return to the Lorenz system

$$x' = -10x + 10y, \qquad y' = 28x - y - xz, \qquad z' = -\frac{8}{3}z + xy.$$

The Lorenz system is an autonomous system in three dimensions exhibiting chaotic behavior. See the Figure 8.15 for a sample trajectory, which is now a curve in three-dimensional space.

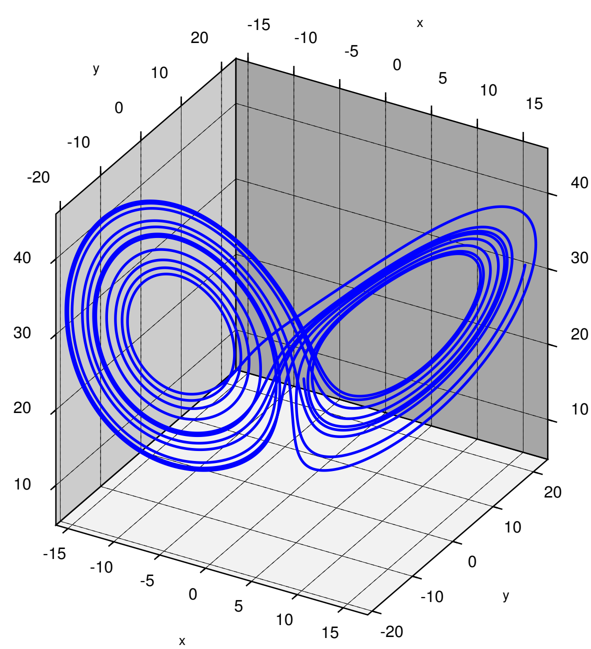


Figure 8.15: A trajectory in the Lorenz system.

The solutions tend to an *attractor* in space, the so-called *Lorenz attractor*. In this case no strobing is necessary. Again we cannot quite see the attractor itself, but if we try to follow a solution for long enough, as in the figure, we get a pretty good picture of what the attractor looks like. The Lorenz attractor is also a strange attractor and has a complicated fractal structure. And, just as for the Duffing equation, what we want to draw is not the whole trajectory, but start drawing the trajectory after a while, once it is close to the attractor.

The path of the trajectory is not simply a repeating figure-eight. The trajectory spins some seemingly random number of times on the left, then spins a number of times on the right, and so on. As this system arose in weather prediction, one can perhaps imagine a few days of warm weather and then a few days of cold weather, where it is not easy to predict when the weather will change, just as it is not really easy to predict far in advance when the solution will jump onto the other side. See Figure 8.16 for a plot of the x component of the solution drawn above. A negative x corresponds to the left "loop" and a positive x corresponds to the right "loop".

Most of the mathematics we studied in this book is quite classical and well understood. On the other hand, chaos, including the Lorenz system, continues to be the subject of current research. Furthermore, chaos has found applications not just in the sciences, but also in art.

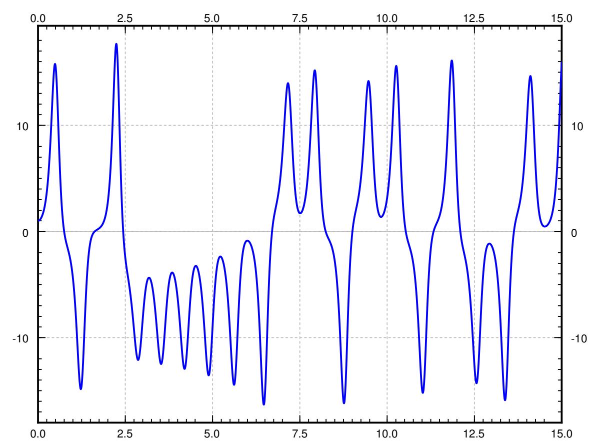


Figure 8.16: Graph of the $x(t)$ component of the solution.

8.5.3 Exercises

Exercise* 8.5.1*: *For the non-chaotic equation* $x'' + 2px' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega t)$*, suppose we strobe with frequency* ω *as we mentioned above. Use the known steady periodic solution to find precisely the point which is the attractor for the Poincaré section.*

Exercise* 8.5.2** (project):*** *A simple fractal attractor can be drawn via the following chaos game. Draw the three vertices of a triangle and label them, say p_1, p_2 and p_3. Draw some random point p (it does not have to be one of the three points above). Roll a die to pick of the p_1, p_2, or p_3 randomly (for example 1 and 4 mean p_1, 2 and 5 mean p_2, and 3 and 6 mean p_3). Suppose we picked p_2, then let p_{new} be the point exactly halfway between p and p_2. Draw this point and let p now refer to this new point p_{new}. Rinse, repeat. Try to be precise and draw as many iterations as possible. Your points will be attracted to the so-called* Sierpinski triangle*. A computer was used to run the game for 10,000 iterations to obtain the picture in Figure 8.17.*

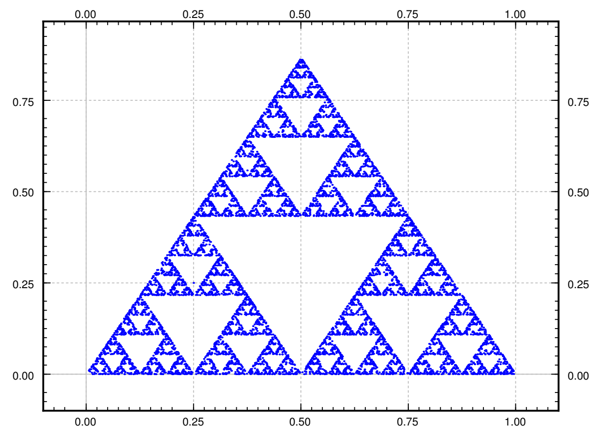


Figure 8.17: 10,000 iterations of the chaos game producing the Sierpinski triangle.

Exercise* 8.5.3** (project):*** *Construct the double pendulum described in the text with a string and two nuts (or heavy beads). Play around with the position of the middle nut, and perhaps use different weight nuts. Describe what you find.*

Exercise* 8.5.4** (computer project):*** *Use a computer software (such as Matlab, Octave, or perhaps even a spreadsheet), plot the solution of the given forced Duffing equation with Euler's method. Plotting the solution for t from 0 to 100 with several different (small) step sizes. Discuss.*

Exercise* 8.5.101*: *Find critical points of the Lorenz system and the associated linearizations.*

Appendix A

Table of Laplace Transforms

The function u is the Heaviside function, δ is the Dirac delta function, and

$$\Gamma(t) = \int_0^\infty e^{-\tau}\tau^{t-1}\,d\tau, \qquad \operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-\tau^2}\,d\tau, \qquad \operatorname{erfc}(t) = 1 - \operatorname{erf}(t).$$

$f(t)$	$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
C	$\frac{C}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^n	$\frac{n!}{s^{n+1}}$
$t^p \quad (p > 0)$	$\frac{\Gamma(p+1)}{s^{p+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$u(t-a)$	$\frac{e^{-as}}{s}$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
$\operatorname{erf}\left(\frac{t}{2a}\right)$	$\frac{1}{s}e^{(as)^2}\operatorname{erfc}(as)$
$\frac{1}{\sqrt{\pi t}}\exp\left(\frac{-a^2}{4t}\right) \quad (a \geq 0)$	$\frac{e^{-as}}{\sqrt{s}}$
$\frac{1}{\sqrt{\pi t}} - ae^{a^2 t}\operatorname{erfc}(a\sqrt{t}) \quad (a > 0)$	$\frac{1}{\sqrt{s}+a}$

$f(t)$	$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$
$af(t) + bg(t)$	$aF(s) + bG(s)$
$f(at) \quad (a > 0)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
$f(t-a)u(t-a)$	$e^{-as}F(s)$
$e^{-at}f(t)$	$F(s+a)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2g(0) - sg'(0) - g''(0)$
$g^{(n)}(t)$	$s^nG(s) - s^{n-1}g(0) - \cdots - g^{(n-1)}(0)$
$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau$	$F(s)G(s)$
$tf(t)$	$-F'(s)$
$t^nf(t)$	$(-1)^nF^{(n)}(s)$
$\int_0^t f(\tau)d\tau$	$\frac{1}{s}F(s)$
$\frac{f(t)}{t}$	$\int_s^\infty F(\sigma)d\sigma$

Further Reading

[BM] Paul W. Berg and James L. McGregor, *Elementary Partial Differential Equations*, Holden-Day, San Francisco, CA, 1966.

[BD] William E. Boyce and Richard C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, 11th edition, John Wiley & Sons Inc., New York, NY, 2017.

[EP] C.H. Edwards and D.E. Penney, *Differential Equations and Boundary Value Problems: Computing and Modeling*, 5th edition, Pearson, 2014.

[F] Stanley J. Farlow, *An Introduction to Differential Equations and Their Applications*, McGraw-Hill, Inc., Princeton, NJ, 1994. (Published also by Dover Publications, 2006.)

[I] E.L. Ince, *Ordinary Differential Equations*, Dover Publications, Inc., New York, NY, 1956.

[T] William F. Trench, *Elementary Differential Equations with Boundary Value Problems*. Books and Monographs. Book 9. 2013. `https://digitalcommons.trinity.edu/mono/9`

Answers to Selected Exercises

0.2.101: Compute $x' = -2e^{-2t}$ and $x'' = 4e^{-2t}$. Then $(4e^{-2t}) + 4(-2e^{-2t}) + 4(e^{-2t}) = 0$.

0.2.102: Yes.

0.2.103: $y = x^r$ is a solution for $r = 0$ and $r = 2$.

0.2.104: $C_1 = 100$, $C_2 = -90$

0.2.105: $\varphi = -9e^{8s}$

0.2.106: a) $x = 9e^{-4t}$ b) $x = \cos(2t) + \sin(2t)$ c) $p = 4e^{3q}$ d) $T = 3\sinh(2x)$

0.2.151:

a) $y = ce^{-\frac{5}{7}x}$

b) $y = c_1 e^{\frac{1}{2}x} + c_2 e^{-4x}$

c) $y = c_1 e^{5x} + c_2 e^{-2x}$

d) $y = c_1 e^{-\frac{2}{3}x} + c_2 e^{3x}$

e) $y = c_1 e^{-x} + c_2 e^{\frac{1}{4}x}$

0.2.152:

a) $C = 5$

b) $C = e^{-2}$

c) $C = \frac{\pi}{4}$

d) $C = -8$

0.3.101: a) PDE, equation, second order, linear, nonhomogeneous, constant coefficient.
b) ODE, equation, first order, linear, nonhomogeneous, not constant coefficient, not autonomous.
c) ODE, equation, seventh order, linear, homogeneous, constant coefficient, autonomous.
d) ODE, equation, second order, linear, nonhomogeneous, constant coefficient, autonomous.
e) ODE, system, second order, nonlinear.
f) PDE, equation, second order, nonlinear.

0.3.102: equation: $a(x)y = b(x)$, solution: $y = \frac{b(x)}{a(x)}$.

0.3.103: $k = 0$ or $k = 1$

1.1.101: $y = e^x + \frac{x^2}{2} + 9$

1.1.102: $x = (3t - 2)^{1/3}$

1.1.103: $x = \sin^{-1}\big(t + 1\big)$

1.1.104: 170

1.1.105: If $n \neq 1$, then $y = \big((1 - n)x + 1\big)^{1/(1-n)}$. If $n = 1$, then $y = e^x$.

1.1.106: The equation is $r' = -C$ for some constant C. The snowball will be completely melted in 25 minutes from time $t = 0$.

1.1.107: $y = Ax^3 + Bx^2 + Cx + D$, so 4 constants.

1.1.151:

a) $y = \frac{x^5}{5} + \frac{e^{2x}}{4} + Cx + D$

b) $y = \frac{x^5}{10} + \frac{x^3}{6} + Cx^2 + Dx + E$

c) $y = -\frac{\sin(2x)}{8} + Cx^2 + Dx + E$

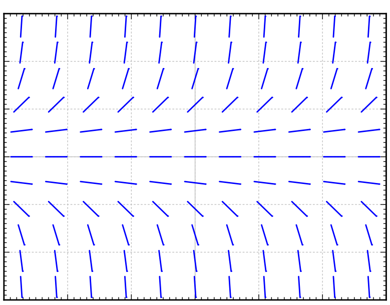
1.1.152:

a) $y = x^3 + x^2 + 2x - 5$

b) $y = 9e^{\frac{1}{3}x} + x - 11$

c) $y = \frac{4}{3}(x + 9)^{3/2} - 5x - 26$

1.2.101:



$y = 0$ is a solution such that $y(0) = 0$.

1.2.102: Yes a solution exists. $y' = f(x, y)$ where $f(x, y) = xy$. The function $f(x, y)$ is continuous and $\frac{\partial f}{\partial y} = x$, which is also continuous near $(0, 0)$. So a solution exists and is unique. (In fact $y = 0$ is the solution).

1.2.103: No, the equation is not defined at $(x, y) = (1, 0)$.

1.2.104: a) $y' = \cos y$, b) $y' = y \cos(x)$, c) $y' = \sin x$. Justification left to reader.

1.2.105: Picard does not apply as f is not continuous at $y = 0$. The equation does not have a continuously differentiable solution. Suppose it did. Notice that $y'(0) = 1$. By the first derivative test, $y(x) > 0$ for small positive x. But then for those x we would have $y'(x) = 0$, so clearly the derivative cannot be continuous.

1.2.106: The solution is $y(x) = \int_{x_0}^{x} f(s)\,ds + y_0$, and this does indeed exist for every x.

1.3.101: $y = Ce^{x^2}$

1.3.102: $x = e^{t^3} + 1$

1.3.103: $x^3 + x = t + 2$

1.3.104: $y = \frac{1}{1-\ln x}$

1.3.105: $\sin(y) = -\cos(x) + C$

1.3.106: The range is approximately 7.45 to 12.15 minutes.

1.3.107: a) $x = \frac{1000e^t}{e^t+24}$. b) 102 rabbits after one month, 861 after 5 months, 999 after 10 months, 1000 after 15 months.

1.3.151: $100e^{-0.00012\times 3315} \approx 67\%$

1.3.152:

a) $\frac{100}{3}\ln 2 \approx 23.1$ yr

b) $250000e^{-3/4} \approx \$118,092$

1.3.153: $7\ln(2)/\ln(5/4) \approx 21.7$ yr

1.3.154:

a) $k = \frac{1}{5}\ln\frac{4}{3} \approx 0.05754$

b) $P' = kP$ with $P(0) = 0.9$ million,
$P(t) = 0.9e^{kt}$ where $k = \frac{1}{5}\ln\frac{4}{3}$

c) $P(20) = 0.9e^{4\ln\frac{4}{3}} = 0.9\left(\frac{4}{3}\right)^4 \approx 2.84$ million

d) $P(30) = 0.9e^{6\ln\frac{4}{3}} = 0.9\left(\frac{4}{3}\right)^6 \approx 5.06$ million

1.3.155:

a) $r = \ln(4/3)/10 \approx 2.88\%$

b) $A_0 = \$40000/e^{8r} = \$40000/(4/3)^{4/5} \approx \$31777$

1.3.156: $\lambda = \ln\left(\frac{m_1}{m_2}\right)/(t_2 - t_1)\ (\mathrm{yr})^{-1}$

1.3.157: $28.8\ln(40)/\ln(2) \approx 153.3$ years.

1.3.158: $y = e^{x^3+x^2-2}$

1.3.159: $y = \frac{9(x-1)}{x+1}$

1.3.160: $y = -\sqrt{\ln(x^2+1)+4}$

1.3.161: $y = 2\sec(x)$

1.3.162: $y = -\sqrt{2xe^x - 2e^x + 27}$

1.4.101: $y = Ce^{-x^3} + 1/3$

1.4.102: $y = 2e^{\cos(2x)+1} + 1$

1.4.103: 250 grams

1.4.104: $P(5) = 1000e^{2\times 5 - 0.05\times 5^2} = 1000e^{8.75} \approx 6.31 \times 10^6$

1.4.105: $Ah' = I - kh$, where k is a constant with units $\mathrm{m}^2 s$.

1.4.151:

a) $v(t) = 70 - 60e^{-kt}$ ft/sec where $k = \ln(2)/3 \approx 0.2310$ (sec)$^{-1}$

b) At $t = 3\ln(6)/\ln(2) \approx 7.755$ sec

1.4.152:

a) $x(t) = 100 - 90e^{-kt}$ where $k = \frac{1}{4}\ln\frac{9}{7}$ (hr)$^{-1}$

b) $x(7) = 100 - 90e^{-k(7)} \approx 42.0\%$

c) $x'(7) = 90ke^{-k(7)} \approx 3.7\%/\mathrm{hr}$

1.4.153:

a) $v(t) = 50(1 - e^{-t/2})$ ft/sec

b) $2\ln 5 \approx 3.2$ sec

c) $100\ln 5 - 80 \approx 81$ ft

1.4.154:

a) $A(t) = \$750{,}000 - \$450{,}000e^{0.04t}$

b) $25\ln\frac{5}{3} \approx 12.8$ years

c) $A_0 = \$750{,}000$

1.4.155:

a) $v(t) = 40 - 39e^{-\frac{1}{10}t}$ m/s

b) $40 - 20e^{1/2} \approx 7.03$ m/s

1.4.156:

a) $C(t) = 100 - 180e^{-kt}$ μg/mL where $k = \frac{1}{3}\ln\frac{16}{13} \approx 0.0692\,(\text{sec})^{-1}$

b) $3\,\frac{\ln(16/11)}{\ln(16/13)} - 3 \approx 2.41$ sec

1.4.157:

a) $2\frac{\ln 15/13}{\ln 13/12} \approx 3.58$ minutes before 2 PM

b) $2\frac{\ln 13/9}{\ln 13/12} \approx 9.19$ minutes after 2 PM

1.4.158: $y(x) = -2 + 6e^{2x}$

1.4.159: $y(x) = 1 + ce^{-x^2}$

1.4.160: $y(x) = e^{2x^3-1}$

1.4.161: $y(x) = 2x^5 + cx^3$

1.4.162: $y(x) = 2\sqrt{x} + cx^{-2}$

1.4.163: $y(x) = 2 + 3e^{-\sin x}$

1.4.164: $y(x) = -\frac{3}{2}(x+2)^{-1} + c(x+2)$

1.4.165: $y(x) = \frac{\ln x}{x} + \frac{3}{x}$

1.4.166: $y(x) = (x+e)e^{\cos 3x}$

1.4.167: $y(x) = \frac{\tan^{-1} x + c}{x^2+1}$

1.4.168: $y(x) = \frac{1}{2}x - \frac{1}{4} + ce^{-2x}$

1.5.101: $y = \frac{2}{3x-2}$

1.5.102: $y = \frac{3-x^2}{2x}$

1.5.103: $y = \left(7e^{3x} + 3x + 1\right)^{1/3}$

1.5.104: $y = \sqrt{x^2 - \ln(C - x)}$

1.5.151: $\frac{y(x)}{y(x)+2x} = Ax^2$, $y(x) = \frac{2Ax^3}{1-Ax^2} = \frac{2x^3}{c-x^2}$, $c = 1/A$

1.5.152: $\sqrt{v} + 1 - \ln(\sqrt{v} + 1) = \frac{1}{2}x + c$, where $v(x) = x + y(x) - 5$

1.5.153: $y(x) = (\frac{3}{5}x^{-2} + cx^3)^{-\frac{1}{3}}$

1.5.154: $y(x) = \tan(x + \frac{\pi}{4}) - x - 7$

1.5.155: $y(x) = -2x^3 - x$

1.5.156: $y(x) = \sin^{-1}(\frac{1}{2}e^{2x} + x - \frac{1}{2})$

1.5.157: $\frac{x^2}{y^2} + \ln\left(\frac{y^2}{x^2}\right) = \ln\left(\frac{1}{x^2}\right) + c$

1.5.158: $y(x) = (\frac{1}{3} + ce^{-5x^3})^{\frac{1}{5}}$

1.5.159: $\ln|1 - e^{2y}| = -2x + c$, $y(x) = \frac{1}{2}\ln\left(1 \pm e^{-2x+c}\right)$

1.5.160: $y(x) = (Ax^6 - x^3)^{\frac{1}{3}}$

1.5.161: $y(x) = (\frac{x^2}{6} + c)^3 e^{-3/x}$

1.6.101: a) 0, 1, 2 are critical points. b) $x = 0$ is unstable (semistable), $x = 1$ is stable, and $x = 2$ is unstable. c) 1

1.6.102: a) There are no critical points. b) ∞

1.6.103: a) $\frac{dx}{dt} = kx(M - x) + A$ b) $\frac{kM+\sqrt{(kM)^2+4Ak}}{2k}$

1.6.104: a) α is a stable critical point, β is an unstable one. b) α, c) α, d) ∞ or DNE.

1.7.101: Approximately: 1.0000, 1.2397, 1.3829

1.7.102: a) 0, 8, 12 b) $x(4) = 16$, so errors are: 16, 8, 4. c) Factors are 0.5, 0.5, 0.5.

1.7.103: a) 0, 0, 0 b) $x = 0$ is a solution so errors are: 0, 0, 0.

1.7.104: a) Improved Euler: $y(1) \approx 3.3897$ for $h = 1/4$, $y(1) \approx 3.4237$ for $h = 1/8$, b) Standard Euler: $y(1) \approx 2.8828$ for $h = 1/4$, $y(1) \approx 3.1316$ for $h = 1/8$, c) $y = 2e^x - x - 1$, so $y(2)$ is approximately 3.4366. d) Approximate errors for improved Euler: 0.046852 for $h = 1/4$, and 0.012881 for $h = 1/8$. For standard Euler: 0.55375 for $h = 1/4$, and 0.30499 for $h = 1/8$. Factor is approximately 0.27 for improved Euler, and 0.55 for standard Euler.

1.8.101: a) $e^{xy} + \sin(x) = C$ b) $x^2 + xy - 2y^2 = C$ c) $e^x + e^y = C$ d) $x^3 + 3xy + y^3 = C$

1.8.102: a) Integrating factor is y, equation becomes $dx + 3y^2\,dy = 0$. b) Integrating factor is e^x, equation becomes $e^x\,dx - e^{-y}\,dy = 0$. c) Integrating factor is y^2, equation becomes $(\cos(x) + y)\,dx + x\,dy = 0$. d) Integrating factor is x, equation becomes $(2xy + y^2)\,dx + (x^2 + 2xy)\,dy = 0$.

1.8.103: a) The equation is $-f(x)\,dx + \frac{1}{g(y)}\,dy$, and this is exact because $M = -f(x)$, $N = \frac{1}{g(y)}$, so $M_y = 0 = N_x$. b) $-x\,dx + \frac{1}{y}\,dy = 0$, leads to potential function $F(x, y) = -\frac{x^2}{2} + \ln|y|$, solving $F(x, y) = C$ leads to the same solution as the example.

2.1.101: Yes. To justify try to find a constant A such that $\sin(x) = Ae^x$ for all x.

2.1.102: No. $e^{x+2} = e^2 e^x$.

2.1.103: $y = 5$

2.1.104: $y = C_1 \ln(x) + C_2$

2.1.105: $y'' - 3y' + 2y = 0$

2.1.151: $y(x) = -2\cos 5x + \frac{3}{5}\sin 5x$

2.1.152: $y(x) = 2e^{2x} + 2e^{-3x}$

2.1.153: $y(x) = -e^{-4x} - 8xe^{-4x}$

2.1.154: $y(x) = \frac{7}{2}e^{-2x} - \frac{1}{2}e^{-4x}$

2.1.155: $y(x) = 2e^{-3x} + 11xe^{-3x}$

2.2.101: $y = C_1 e^{(-2+\sqrt{2})x} + C_2 e^{(-2-\sqrt{2})x}$

2.2.102: $y = C_1 e^{3x} + C_2 x e^{3x}$

2.2.103: $y = e^{-x/4}\cos\big((\sqrt{7}/4)x\big) - \sqrt{7}e^{-x/4}\sin\big((\sqrt{7}/4)x\big)$

2.2.104: $y = \frac{2(a-b)}{5}\, e^{-3x/2} + \frac{3a+2b}{5}\, e^{x}$

2.2.105: $z(t) = 2e^{-t}\cos(t)$

2.2.106: $y = \frac{a\beta-b}{\beta-\alpha} e^{\alpha x} + \frac{b-a\alpha}{\beta-\alpha} e^{\beta x}$

2.2.107: $y'' - y' - 6y = 0$

2.2.151:

a) $y'' + 8y' + 15y = 0$

b) $y'' - 4y' = 0$

c) $y'' + 4y' + 4y = 0$

d) $y'' + 2y' + 2y = 0$

e) $y'' = 0$

2.2.152:

a) $y(x) = C_1\frac{\cos(\ln x)}{x} + C_2\frac{\sin(\ln x)}{x}$

b) $y(x) = C_1x^2 + C_2x^2\ln x$

c) $y(x) = C_1\frac{\cos(2\ln x)}{x} + C_2\frac{\sin(2\ln x)}{x}$

d) $y(x) = C_1x^{\frac{1}{3}} + C_2x^{-4}$

e) $y(x) = C_1x^{-\frac{1}{2}} + C_2x^{-\frac{1}{2}}\ln x$

f) $y(x) = C_1\frac{\cos(\ln x)}{\sqrt{x}} + C_2\frac{\sin(\ln x)}{\sqrt{x}}$

2.3.101: $y = C_1e^x + C_2x^3 + C_3x^2 + C_4x + C_5$

2.3.102: a) $r^3 - 3r^2 + 4r - 12 = 0$ b) $y''' - 3y'' + 4y' - 12y = 0$ c) $y = C_1e^{3x} + C_2\sin(2x) + C_3\cos(2x)$

2.3.103: $y = 0$

2.3.104: No. $e^1e^x - e^{x+1} = 0$.

2.3.105: Yes. (Hint: First note that $\sin(x)$ is bounded. Then note that x and $x\sin(x)$ cannot be multiples of each other.)

2.3.106: $y''' - y'' + y' - y = 0$

2.3.151:

a) $y(x) = C_1 + C_2x + C_3e^{\frac{1}{4}x} + C_4e^{-2x}$

b) $y(x) = C_1e^{-2x} + C_2e^{2x} + C_3\cos 2x + C_4\sin 2x$

c) $y(x) = C_1 e^{-x} + C_2 e^{x} + C_3 e^{-2x} + C_4 e^{2x}$

d) $y(x) = C_1 + C_2 \cos \sqrt{3}x + C_3 \sin \sqrt{3}x + C_4 x \cos \sqrt{3}x + C_5 x \sin \sqrt{3}x$

e) $y(x) = C_1 e^{-x} + C_2 e^{x} + C_3 \cos 3x + C_4 \sin 3x$

f) $y(x) = C_1 + C_2 x + C_3 x^2 + C_4 e^{-4x} + C_5 e^{3x}$

g) $y(x) = C_1 + C_2 e^{-x} + C_3 x e^{-x} + C_4 x^2 e^{-x}$

h) $y(x) = C_1 e^{-2x} + C_2 e^{2x} + C_3 x e^{-2x} + C_4 x e^{2x}$

2.4.101: $k = 8/9$ (and larger)

2.4.102: a) $0.05I'' + 0.1I' + (1/5)I = 0$ b) $I = Ce^{-t}\cos(\sqrt{3}\,t - \gamma)$ c) $I = 10e^{-t}\cos(\sqrt{3}\,t) + \frac{10}{\sqrt{3}}e^{-t}\sin(\sqrt{3}\,t)$

2.4.103: a) $k = 500000$ b) $\frac{1}{5\sqrt{2}} \approx 0.141$ c) 45000 kg d) 11250 kg

2.4.104: $m_0 = \frac{1}{3}$. If $m < m_0$, then the system is overdamped and will not oscillate.

2.4.151:

a) $x(t) = \frac{10}{3}e^{-\frac{1}{2}t} - \frac{4}{3}e^{-2t}$;
Undamped $x_u(t) = 2\cos t + \sin t = \sqrt{5}\cos(t - \gamma)$ where $\gamma = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) \approx 0.464$

b) $x(t) = 2e^{-4t}(\cos(2t) + \sin(2t)) = 2\sqrt{2}e^{-4t}\cos(2t - \frac{\pi}{4})$

c) $x(t) = 2e^{-3t} + 10te^{-3t}$;
Undamped $x_u(t) = 2\cos(3t) + \frac{4}{3}\sin(3t) = \frac{2\sqrt{13}}{3}\cos(3t - \gamma)$ where $\gamma = \cos^{-1}\left(\frac{3}{\sqrt{13}}\right) \approx 0.588$

d) $x(t) = e^{-t/2}(4\cos(2t) + 3\sin(2t)) = 5e^{-t/2}\cos(2t - \gamma)$ where $\gamma = \cos^{-1}\left(\frac{4}{5}\right) \approx 0.644$

e) $x(t) = e^{-3t}(2\cos(4t) + \sin(4t)) = \sqrt{5}e^{-3t}\cos(4t - \gamma)$ where $\gamma = \cos^{-1}\left(\frac{2}{\sqrt{5}}\right) \approx 0.464$

2.5.101: $y = \frac{-16\sin(3x)+6\cos(3x)}{73}$

2.5.102: a) $y = \frac{2e^x+3x^3-9x}{6}$ b) $y = C_1\cos(\sqrt{2}x) + C_2\sin(\sqrt{2}x) + \frac{2e^x+3x^3-9x}{6}$

2.5.103: $y(x) = x^2 - 4x + 6 + e^{-x}(x-5)$

2.5.104: $y = \frac{2xe^x-(e^x+e^{-x})\log(e^{2x}+1)}{4}$

2.5.105: $y = \frac{-\sin(x+c)}{3} + C_1 e^{\sqrt{2}\,x} + C_2 e^{-\sqrt{2}\,x}$

2.5.151:

a) $y(x) = C_1 + C_2 e^{2x}\cos x + C_3 e^{2x}\sin x + Ax^2 + Bx + Cxe^{2x}\sin x + Dxe^{2x}\cos x$

b) $y(x) = C_1 + C_2e^{-2x} + C_3e^{2x} + Ax^3 + Bx^2 + Cx + Dxe^{2x} + Ee^{3x}$

c) $y(x) = C_1+C_2e^{-2x}\cos 3x+C_3e^{-2x}\sin 3x+Ax^3+Bx^2+Cx+Dxe^{-2x}\cos 3x+Exe^{-2x}\sin 3x$

d) $y(x) = C_1 + C_2x + C_3e^{-3x} + C_4e^{x} + Ax^3 + Bx^2 + Cxe^{-3x} + De^{5x}$

e) $y(x) = C_1 + C_2x + C_3e^{-3x} + C_4e^{3x} + Axe^{-3x} + Bx^4 + Cx^3 + Dx^2$

f) $y(x) = C_1 + C_2x + C_3x^2 + C_4e^{4x} + C_5e^{-2x} + Ax^4 + Bx^3 + Cxe^{-2x} + Dxe^{4x}$

g) $y(x) = C_1 + C_2x + C_3x^2 + C_4e^{-x}\cos x + C_5e^{-x}\sin x + Ax^6 + Bx^5 + Cx^4 + Dx^3 + Exe^{-x}\sin x + Fxe^{-x}\cos x$

2.5.152:

a) $y(x) = C_1 + C_2e^{2x} - 2x^2 - 2x + \frac{5}{3}e^{3x}$

b) $y(x) = C_1 + C_2e^{-2x} + C_3e^{x} - \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{8}e^{2x}$

c) $y(x) = C_1e^{-3x}\cos 2x + C_2e^{-3x}\sin 2x + \frac{1}{15}\cos x + \frac{1}{30}\sin x$

d) $y(x) = C_1 + C_2e^{-3x} + C_3e^{2x} - \frac{1}{18}x^3 - \frac{7}{36}x^2 - \frac{13}{108}x + \frac{2}{9}e^{3x}$

2.5.153:

a) $y(x) = \frac{1}{3} - 2e^{2x} - 2x^2 - 2x + \frac{5}{3}e^{3x}$

b) $y(x) = \frac{9}{16} + \frac{11}{48}e^{-2x} + \frac{1}{12}e^{x} - \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{8}e^{2x}$

2.6.101: $\omega = \frac{\sqrt{31}}{4\sqrt{2}} \approx 0.984$ $C(\omega) = \frac{16}{3\sqrt{7}} \approx 2.016$

2.6.102: $x_{sp} = \frac{(\omega_0^2-\omega^2)F_0}{m(2\omega p)^2+m(\omega_0^2-\omega^2)^2}\cos(\omega t) + \frac{2\omega p F_0}{m(2\omega p)^2+m(\omega_0^2-\omega^2)^2}\sin(\omega t) + \frac{A}{k}$, where $p = \frac{c}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$.

2.6.103: a) $\omega = 2$ b) 25

2.6.151:

a) $x(t) = 5\cos 2t + \frac{1}{2}\sin 2t - 3\cos 3t$

b) $x(t) = 3\cos 3t - \frac{6}{5}\sin 3t + \frac{4}{5}\sin 2t$

c) $x(t) = -\frac{9}{7}\cos 4t - \frac{2}{7}\sin 4t + \frac{2}{7}\cos 3t + \frac{5}{7}\sin 3t$

3.1.101: $y_1 = C_1e^{3x}$, $y_2 = y(x) = C_2e^{x} + \frac{C_1}{2}e^{3x}$, $y_3 = y(x) = C_3e^{x} + \frac{C_1}{2}e^{3x}$

3.1.102: $x = \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t}$, $y = \frac{5}{3}e^{2t} + \frac{4}{3}e^{-t}$

3.1.103: $x_1' = x_2$, $x_2' = x_3$, $x_3' = x_1 + t$

3.1.104: $y_3' + y_1 + y_2 = t$, $y_4' + y_1 - y_2 = t^2$, $y_1' = y_3$, $y_2' = y_4$

3.1.105: $x_1 = x_2 = at$. Explanation of the intuition is left to reader.

3.1.106: a) Left to reader. b) $x_1' = \frac{r}{V}(x_2 - x_1)$, $x_2' = \frac{r}{V}x_1 - \frac{r-s}{V}x_2$. c) As t goes to infinity, both x_1 and x_2 go to zero, explanation is left to reader.

3.2.101: -15

3.2.102: -2

3.2.103: $\vec{x} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$

3.2.104: a) $\begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}$ b) $\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$

3.3.101: Yes.

3.3.102: No. $2\begin{bmatrix} \cosh(t) \\ 1 \end{bmatrix} - \begin{bmatrix} e^t \\ 1 \end{bmatrix} - \begin{bmatrix} e^{-t} \\ 1 \end{bmatrix} = \vec{0}$

3.3.103: $\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ t & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$

3.3.104: a) $\vec{x}\,' = \begin{bmatrix} 0 & 2t \\ 0 & 2t \end{bmatrix}\vec{x}$ b) $\vec{x} = \begin{bmatrix} C_2 e^{t^2} + C_1 \\ C_2 e^{t^2} \end{bmatrix}$

3.4.101: a) Eigenvalues: $4, 0, -1$ Eigenvectors: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix}$
b) $\vec{x} = C_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{4t} + C_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + C_3 \begin{bmatrix} 3 \\ 5 \\ -2 \end{bmatrix} e^{-t}$

3.4.102: a) Eigenvalues: $\frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$, Eigenvectors: $\begin{bmatrix} -2 \\ 1-\sqrt{3}i \end{bmatrix}, \begin{bmatrix} -2 \\ 1+\sqrt{3}i \end{bmatrix}$
b) $\vec{x} = C_1 e^{t/2} \begin{bmatrix} -2\cos\left(\frac{\sqrt{3}t}{2}\right) \\ \cos\left(\frac{\sqrt{3}t}{2}\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}t}{2}\right) \end{bmatrix} + C_2 e^{t/2} \begin{bmatrix} -2\sin\left(\frac{\sqrt{3}t}{2}\right) \\ \sin\left(\frac{\sqrt{3}t}{2}\right) - \sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right) \end{bmatrix}$

3.4.103: $\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + C_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$

3.4.104: $\vec{x} = C_1 \begin{bmatrix} \cos(t) \\ -\sin(t) \end{bmatrix} + C_2 \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$

3.4.151:

a) $\vec{x}(t) = C_1 \begin{bmatrix} 1 \\ 7 \end{bmatrix} e^{-4t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}$

b) $\vec{x}(t) = C_1 \begin{bmatrix} 1 \\ -6 \end{bmatrix} e^{-2t} + C_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$

c) $\vec{x}(t) = C_1 \begin{bmatrix} \cos t \\ 2\cos t + \sin t \end{bmatrix} e^{-t} + C_2 \begin{bmatrix} \sin t \\ 2\sin t - \cos t \end{bmatrix} e^{-t}$

d) $\vec{x}(t) = -9 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + 2 \begin{bmatrix} 5 \\ 6 \end{bmatrix} e^{4t}$

e) $\vec{x}(t) = C_1 \begin{bmatrix} 2\cos 4t \\ \cos 4t - 2\sin 4t \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} 2\sin 4t \\ \sin 4t + 2\cos 4t \end{bmatrix} e^{5t}$

f) $\vec{x}(t) = 2 \begin{bmatrix} -\cos 2t \\ \cos 2t - 2\sin 2t \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -\sin 2t \\ \sin 2t + 2\cos 2t \end{bmatrix}$

g) $\vec{x}(t) = -\frac{1}{3} \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-2t} + \frac{7}{3} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t}$

h) $\vec{x}(t) = C_1 \begin{bmatrix} -2\cos 3t \\ 3\cos 3t - 3\sin 3t \end{bmatrix} e^{2t} + C_2 \begin{bmatrix} -2\sin 3t \\ 3\sin 3t + 3\cos 3t \end{bmatrix} e^{2t}$

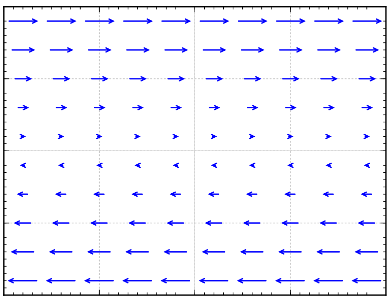
i) $\vec{x}(t) = C_1 \begin{bmatrix} 5\cos 3t \\ -\cos 3t - 3\sin 3t \end{bmatrix} + C_2 \begin{bmatrix} 5\sin 3t \\ -\sin 3t + 3\cos 3t \end{bmatrix}$

j) $\vec{x}(t) = -\begin{bmatrix} 7 \\ -2 \end{bmatrix} e^{3t} + 5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$

3.5.101: a) Two eigenvalues: $\pm\sqrt{2}$ so the behavior is a saddle. b) Two eigenvalues: 1 and 2, so the behavior is a source. c) Two eigenvalues: $\pm 2i$, so the behavior is a center (ellipses). d) Two eigenvalues: -1 and -2, so the behavior is a sink. e) Two eigenvalues: 5 and -3, so the behavior is a saddle.

3.5.102: Spiral source.

3.5.103:



The solution does not move anywhere if $y = 0$. When y is positive, the solution moves (with constant speed) in the positive x direction. When y is negative, the solution moves (with constant speed) in the negative x direction. It is not one of the behaviors we have seen.

Note that the matrix has a double eigenvalue 0 and the general solution is $x = C_1 t + C_2$ and $y = C_1$, which agrees with the description above.

3.6.101: $\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \big(a_1 \cos(\sqrt{3}\, t) + b_1 \sin(\sqrt{3}\, t)\big) + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} \big(a_2 \cos(\sqrt{2}\, t) + b_2 \sin(\sqrt{2}\, t)\big) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \big(a_3 \cos(t) + b_3 \sin(t)\big) + \begin{bmatrix} -1 \\ 1/2 \\ 2/3 \end{bmatrix} \cos(2t)$

3.6.102: $\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \vec{x}\,'' = \begin{bmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{bmatrix} \vec{x}$. Solution: $\vec{x} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \big(a_1 \cos(\sqrt{3k/m}\, t) + b_1 \sin(\sqrt{3k/m}\, t)\big) + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \big(a_2 \cos(\sqrt{k/m}\, t) + b_2 \sin(\sqrt{k/m}\, t)\big) + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \big(a_3 t + b_3\big)$.

3.6.103: $x_2 = (2/5) \cos(\sqrt{1/6}\, t) - (2/5) \cos(t)$

3.7.101: a) $3, 0, 0$ b) No defects. c) $\vec{x} = C_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + C_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

3.7.102: a) $1, 1, 2$

b) Eigenvalue 1 has a defect of 1

c) $\vec{x} = C_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{t} + C_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right) e^{t} + C_3 \begin{bmatrix} 3 \\ 3 \\ -2 \end{bmatrix} e^{2t}$

3.7.103: a) $2, 2, 2$

b) Eigenvalue 2 has a defect of 2

c) $\vec{x} = C_1 \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} e^{2t} + C_2 \left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right) e^{2t} + C_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \frac{t^2}{2} \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \right) e^{2t}$

3.7.104: $A = \begin{bmatrix} 5 & 5 \\ 0 & 5 \end{bmatrix}$

3.7.151:

a) $\vec{x}(t) = C_1 \begin{bmatrix} 4 \\ -4 \end{bmatrix} e^{5t} + C_2\left(\begin{bmatrix} 4 \\ -4 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)e^{5t}$

b) $\vec{x}(t) = C_1 \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{-4t} + C_2\left(\begin{bmatrix} -1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)e^{-4t}$

c) $\vec{x}(t) = \frac{3}{2} \begin{bmatrix} 2 \\ 2 \end{bmatrix} e^{-3t} - 4 \begin{bmatrix} 2t+1 \\ 2t \end{bmatrix} e^{-3t}$

d) $\vec{x}(t) = -5 \begin{bmatrix} -1 \\ -1 \end{bmatrix} e^{7t} - 7 \begin{bmatrix} -t+1 \\ -t \end{bmatrix} e^{7t}$

e) $\vec{x}(t) = C_1 \begin{bmatrix} 2 \\ -4 \end{bmatrix} e^{5t} + C_2\left(\begin{bmatrix} 2 \\ -4 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)e^{5t}$

f) $\vec{x}(t) = C_1 \begin{bmatrix} -2 \\ 4 \end{bmatrix} e^{-6t} + C_2\left(\begin{bmatrix} -2 \\ 4 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)e^{-6t}$

3.8.101: $e^{tA} = \begin{bmatrix} \frac{e^{3t}+e^{-t}}{2} & \frac{e^{-t}-e^{3t}}{2} \\ \frac{e^{-t}-e^{3t}}{2} & \frac{e^{3t}+e^{-t}}{2} \end{bmatrix}$

3.8.102: $e^{tA} = \begin{bmatrix} 2e^{3t}-4e^{2t}+3e^{t} & \frac{3e^{t}}{2}-\frac{3e^{3t}}{2} & -e^{3t}+4e^{2t}-3e^{t} \\ 2e^{t}-2e^{2t} & e^{t} & 2e^{2t}-2e^{t} \\ 2e^{3t}-5e^{2t}+3e^{t} & \frac{3e^{t}}{2}-\frac{3e^{3t}}{2} & -e^{3t}+5e^{2t}-3e^{t} \end{bmatrix}$

3.8.103: a) $e^{tA} = \begin{bmatrix} (t+1)\,e^{2t} & -te^{2t} \\ te^{2t} & (1-t)\,e^{2t} \end{bmatrix}$ b) $\vec{x} = \begin{bmatrix} (1-t)\,e^{2t} \\ (2-t)\,e^{2t} \end{bmatrix}$

3.8.104: $\begin{bmatrix} 1+2t+5t^2 & 3t+6t^2 \\ 2t+4t^2 & 1+2t+5t^2 \end{bmatrix}$ $e^{0.1A} \approx \begin{bmatrix} 1.25 & 0.36 \\ 0.24 & 1.25 \end{bmatrix}$

3.8.105: a) $\begin{bmatrix} 5(3^n) - 2^{n+2} & 4(3^n) - 2^{n+2} \\ 5(2^n) - 5(3^n) & 5(2^n) - 4(3^n) \end{bmatrix}$ b) $\begin{bmatrix} 3 - 2(3^n) & 2(3^n) - 2 \\ 3 - 3^{n+1} & 3^{n+1} - 2 \end{bmatrix}$

c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ if n is even, and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ if n is odd.

3.9.101: The general solution is (particular solutions should agree with one of these): $x(t) = C_1e^{9t} + 4C_2e^{4t} - t/3 - 5/54$, $y(t) = C_1e^{9t} - C_2e^{4t} + t/6 + 7/216$

3.9.102: The general solution is (particular solutions should agree with one of these): $x(t) = C_1e^{t} + C_2e^{-t} + te^{t}$, $y(t) = C_1e^{t} - C_2e^{-t} + te^{t}$

3.9.103: $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left(\frac{5}{2}e^{t} - t - 1\right) + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \frac{-1}{2}e^{-t}$

3.9.104: $\vec{x} = \begin{bmatrix} 1 \\ 9 \end{bmatrix} \left(\left(\frac{1}{140} + \frac{1}{120\sqrt{6}}\right) e^{\sqrt{6}t} + \left(\frac{1}{140} + \frac{1}{120\sqrt{6}}\right) e^{-\sqrt{6}t} - \frac{t}{60} - \frac{\cos(t)}{70} \right)$
$+ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(\frac{-9}{80} \sin(2t) + \frac{1}{30} \cos(2t) + \frac{9t}{40} - \frac{\cos(t)}{30} \right)$

6.1.101: $\frac{8}{s^3} + \frac{8}{s^2} + \frac{4}{s}$

6.1.102: $2t^2 - 2t + 1 - e^{-2t}$

6.1.103: $\frac{1}{(s+1)^2}$

6.1.104: $\frac{1}{s^2+2s+2}$

6.1.151:

a) $F(s) = \frac{2}{s} - \frac{5s}{s^2+9} + \frac{12}{s^5} - \frac{6}{s-7}$

b) $F(s) = \frac{1}{3(s+\pi)} + \frac{2\pi}{s^2+\pi^2} + \frac{8}{s^3} + \frac{4}{s^2} + \frac{1}{s}$

c) $F(s) = \frac{3e^2}{s+5} - \frac{9}{2s^4} + \frac{1}{2s} - \frac{s}{2(s^2+16)}$

d) $F(s) = \frac{4s}{s^2+2} - \frac{2}{3s} + \frac{24}{s^5} + \frac{12}{s^4} + \frac{2}{s^3}$

e) $F(s) = \frac{6}{5(s^2+9)} + \frac{20}{s^6} - \frac{1}{2s} - \frac{s}{2(s^2+36)}$

6.1.152:

a) $f(t) = 3 - 2e^{6t} + \frac{5}{3}\sin 3t$

b) $f(t) = \frac{1}{24}t^3 + \frac{3}{7}e^{-\pi t} + 2\cos 2t$

c) $f(t) = \frac{1}{2}e^{-3t} - \frac{3}{8}t^2 + 3\cos 3t + \frac{2}{3}\sin 3t$

d) $f(t) = 1 - \frac{2}{3}t + 5e^{\pi^2 t}$

e) $f(t) = \frac{1}{144}t^4 + 5\cos\sqrt{3}\,t - \frac{7}{\sqrt{3}}\sin\sqrt{3}\,t$

6.1.153:

a) $f(t) = -\frac{3}{2} + \frac{3}{2}e^{2t}$

b) $f(t) = \frac{1}{4} - \frac{1}{4}\cos 2t + \sin 2t$

c) $f(t) = \frac{7}{5}e^{3t} - \frac{2}{5}e^{-2t}$

d) $f(t) = -1 + 2t + 2e^{-t}$

e) $f(t) = -2\cos t + 5\cos\sqrt{2}\,t$

6.1.154:

a) $F(s) = \frac{24}{(s-3)^5}$

b) $F(s) = \frac{18}{(s+2)^4}$

c) $F(s) = \frac{s-4}{(s-4)^2+9}$

d) $F(s) = \frac{10}{(s+3)^2+25}$

e) $F(s) = \frac{3(s+\pi)}{(s+\pi)^2+4\pi^2}$

6.1.155:

a) $f(t) = \frac{1}{6}t^3e^{2t}$

b) $f(t) = \frac{1}{4}t^2 e^{-\pi t}$

c) $f(t) = e^{3t}\cos 4t$

d) $f(t) = te^{-4t}$

e) $f(t) = 4e^{-2t}\cos 3t - \frac{5}{3}e^{-2t}\sin 3t$

f) $f(t) = 3e^{t}\cos 2t + \frac{7}{2}e^{t}\sin 2t$

6.2.101: $f(t) = (t-1)\big(u(t-1) - u(t-2)\big) + u(t-2)$

6.2.102: $x(t) = (2e^{t-1} - t^2 - 1)u(t-1) - \frac{1}{2}e^{-t} + \frac{3}{2}e^{t}$

6.2.103: $H(s) = \frac{1}{s+1}$

6.2.151:

a) $x(t) = \cos 5t + \frac{2}{5}\sin 5t$

b) $x(t) = 3\cos 2t - \frac{1}{2}\sin 2t + \frac{1}{2} - \frac{1}{2}\cos 2t$

c) $x(t) = -\cos 3t + \frac{1}{3}\sin 3t + \frac{1}{8}\cos t - \frac{1}{8}\cos 3t$

d) $x(t) = \frac{1}{2} - \frac{3}{2}e^{-2t} + e^{-3t}$

e) $x(t) = 2e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t$

f) $x(t) = e^{-3t}\cos 4t + \frac{3}{2}e^{-3t}\sin 4t$

6.2.152:

a) $F(s) = \frac{(1-e^{-2\pi s})s}{s^2+1}$

b) $F(s) = \frac{e^{-2s}+e^{-5s}}{s}$

c) $F(s) = \frac{2(1-e^{-3\pi s})}{s^2+4}$

d) $F(s) = \frac{(e^{-2s}+e^{-5s})s}{s^2+\pi^2}$

e) $F(s) = \frac{-3(e^{-\pi s}+e^{-4\pi s})}{s^2+9}$

6.2.153:

a) $f(t) = u(t-3)e^{4(t-3)}$

b) $f(t) = \frac{1}{2}u(t-2)(t-2)^2$

c) $f(t) = \frac{1}{3}u(t-2\pi)\sin 3(t-2\pi)$

d) $f(t) = u(t-1)e^{-(t-1)} - u(t-2)e^{-(t-2)}$

e) $f(t) = \cos 2t + u(t-\pi)\cos 2(t-\pi)$

f) $f(t) = \frac{1}{4}u(t-2\pi)\sin 4(t-2\pi) - \frac{1}{4}u(t-3\pi)\sin 4(t-3\pi)$

6.2.154:

a) $f(t) = \frac{1}{2} - \frac{1}{2}e^{-2t}$

b) $f(t) = \frac{2}{9} - \frac{2}{9}\cos 3t$

c) $f(t) = \frac{1}{4} - \frac{1}{4}\cos 2t + \frac{1}{2}\sin 2t$

d) $f(t) = -\frac{1}{9} + \frac{1}{3}t + \frac{1}{9}e^{-3t}$

e) $f(t) = 3t - 3\sin t$

f) $f(t) = -\frac{2}{\pi^2} - \frac{2}{\pi}t + \frac{2}{\pi^2}e^{\pi t}$

6.3.101: $\frac{1}{2}(\cos t + \sin t - e^{-t})$
6.3.102: $5t - 5\sin t$
6.3.103: $\frac{1}{2}(\sin t - t\cos t)$
6.3.104: $\int_0^t f(\tau)\big(1-\cos(t-\tau)\big)\,d\tau$
6.3.151:

a) $(f*g)(t) = \frac{3}{2}\sin 2t$

b) $(f*g)(t) = \frac{2}{3}(t+3)^3 - 18$

c) $(f*g)(t) = \frac{1}{5}e^{3t} - \frac{1}{5}e^{-2t}$

d) $(f*g)(t) = t - \sin t$

e) $(f*g)(t) = -\frac{1}{4} - \frac{1}{2}t + \frac{1}{4}e^{2t}$

6.4.101: $x(t) = t$
6.4.102: $x(t) = e^{-at}$
6.4.103: $x(t) = (\cos * \sin)(t) = \frac{1}{2}t\sin(t)$
6.4.104: $\delta(t) - \sin(t)$
6.4.105: $3\delta(t-1) + 2t$
6.4.151:

a) $\cos(4\,t) + \frac{1}{4}\,u\,(t-3)\sin(4\,t-12)$

b) $x(t) = \frac{1}{9} - \frac{1}{9}\cos 3t + \frac{1}{3}u(t-4)\sin 3(t-4)$

c) $x(t) = \frac{4}{5}e^{2t} - \frac{4}{5}e^{-3t} + \frac{4}{5}u(t-2)e^{2(t-2)} - \frac{4}{5}u(t-2)e^{-3(t-2)}$

d) $x(t) = 2te^{-3t} + u(t-4)(t-4)e^{-3(t-4)}$

e) $x(t) = \frac{1}{3}u(t-\pi)e^{-(t-\pi)}\sin 3(t-\pi)$

7.1.101: Yes. Radius of convergence is 10.

7.1.102: Yes. Radius of convergence is e.

7.1.103: $\frac{1}{1-x} = -\frac{1}{1-(2-x)}$ so $\frac{1}{1-x} = \sum_{n=0}^{\infty} (-1)^{n+1}(x-2)^n$, which converges for $1 < x < 3$.

7.1.104: $\sum_{n=7}^{\infty} \frac{1}{(n-7)!}x^n$

7.1.105: $f(x) - g(x)$ is a polynomial. Hint: Use Taylor series.

7.2.101: $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, recurrence relation (for $k \geq 5$): $a_k = \frac{-2a_{k-5}}{k(k-1)}$, so:
$y(x) = a_0 + a_1x - \frac{a_0}{10}x^5 - \frac{a_1}{15}x^6 + \frac{a_0}{450}x^{10} + \frac{a_1}{825}x^{11} - \frac{a_0}{47250}x^{15} - \frac{a_1}{99000}x^{16} + \cdots$

7.2.102: a) $a_2 = \frac{1}{2}$, and for $k \geq 1$ we have $a_k = \frac{a_{k-3}+1}{k(k-1)}$, so
$y(x) = a_0 + a_1x + \frac{1}{2}x^2 + \frac{a_0+1}{6}x^3 + \frac{a_1+1}{12}x^4 + \frac{3}{40}x^5 + \frac{a_0+2}{30}x^6 + \frac{a_1+2}{42}x^7 + \frac{5}{112}x^8 + \frac{a_0+3}{72}x^9 + \frac{a_1+3}{90}x^{10} + \cdots$
b) $y(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 + \frac{1}{15}x^6 + \frac{1}{21}x^7 + \frac{5}{112}x^8 + \frac{1}{24}x^9 + \frac{1}{30}x^{10} + \cdots$

7.2.103: Applying the method of this section directly we obtain $a_k = 0$ for all k and so $y(x) = 0$ is the only solution we find.

7.3.101: a) ordinary, b) singular but not regular singular, c) regular singular, d) regular singular, e) ordinary.

7.3.102: $y = Ax^{\frac{1+\sqrt{5}}{2}} + Bx^{\frac{1-\sqrt{5}}{2}}$

7.3.103: $y = x^{3/2}\sum_{k=0}^{\infty} \frac{(-1)^{-1}}{k!\,(k+2)!}x^k$ (Note that for convenience we did not pick $a_0 = 1$)

7.3.104: $y = Ax + Bx\ln(x)$

8.1.101: a) Critical points $(0,0)$ and $(0,1)$. At $(0,0)$ using $u = x$, $v = y$ the linearization is $u' = -2u - (1/\pi)v$, $v' = -v$. At $(0,1)$ using $u = x$, $v = y - 1$ the linearization is $u' = -2u + (1/\pi)v$, $v' = v$.
b) Critical point $(0,0)$. Using $u = x$, $v = y$ the linearization is $u' = u + v$, $v' = u$.
c) Critical point $(1/2, -1/4)$. Using $u = x - 1/2$, $v = y + 1/4$ the linearization is $u' = -u + v$, $v' = u + v$.

8.1.102: 1) is c), 2) is a), 3) is b)

8.1.103: Critical points are $(0,0,0)$, and $(-1,1,-1)$. The linearization at the origin using variables $u = x$, $v = y$, $w = z$ is $u' = u$, $v' = -v$, $z' = w$. The linearization at the point $(-1,1,-1)$ using variables $u = x+1$, $v = y-1$, $w = z+1$ is $u' = u - 2w$, $v' = -v - 2w$, $w' = w - 2u$.

8.1.104: $u' = f(u,v,w)$, $v' = g(u,v,w)$, $w' = 1$.

8.2.101: a) $(0,0)$: saddle (unstable), $(1,0)$: source (unstable), b) $(0,0)$: spiral sink (asymptotically stable), $(0,1)$: saddle (unstable), c) $(1,0)$: saddle (unstable), $(0,1)$: saddle (unstable)

8.2.102: a) $\frac{1}{2}y^2 + \frac{1}{3}x^3 - 4x = C$, critical points: $(-2,0)$, an unstable saddle, and $(2,0)$, a stable center. b) $\frac{1}{2}y^2 + e^x = C$, no critical points. c) $\frac{1}{2}y^2 + xe^x = C$, critical point at $(-1,0)$ is a stable center.

8.2.103: Critical point at $(0,0)$. Trajectories are $y = \pm\sqrt{2C + (1/2)x^4}$, for $C > 0$, these give closed curves around the origin, so the critical point is a stable center.

8.2.104: A critical point x_0 is stable if $f'(x_0) < 0$ and unstable when $f'(x_0) > 0$.

8.3.101: a) Critical points are $\omega = 0$, $\theta = k\pi$ for any integer k. When k is odd, we have a saddle point. When k is even we get a sink. b) The findings mean the pendulum will simply go to one of the sinks, for example $(0,0)$ and it will not swing back and forth. The friction is too high for it to oscillate, just like an overdamped mass-spring system.

8.3.102: a) Solving for the critical points we get $(0, -h/d)$ and $(\frac{bh+ad}{ac}, \frac{a}{b})$. The Jacobian matrix at $(0, -h/d)$ is $\begin{bmatrix} a+bh/d & 0 \\ -ch/d & -d \end{bmatrix}$ whose eigenvalues are $a + bh/d$ and $-d$. So the eigenvalues are always real of opposite signs and we get a saddle (In the application however we are only looking at the positive quadrant so this critical point is not relevant). At $(\frac{bh+ad}{ac}, \frac{a}{b})$ we get Jacobian matrix $\begin{bmatrix} 0 & -\frac{b(bh+ad)}{ac} \\ \frac{ac}{b} & \frac{bh+ad}{a} - d \end{bmatrix}$. b) For the specific numbers given, the second critical point is $(\frac{550}{3}, 40)$ the matrix is $\begin{bmatrix} 0 & -11/6 \\ 3/25 & 1/4 \end{bmatrix}$, which has eigenvalues $\frac{5 \pm i\sqrt{327}}{40}$. Therefore there is a spiral source. This means the solution spirals outwards. The solution will eventually hit one of the axes, $x = 0$ or $y = 0$, so something will die out in the forest.

8.3.103: The critical points are on the line $x = 0$. In the positive quadrant the y' is always positive and so the fox population always grows. The constant of motion is $C = y^a e^{-cx-by}$, for any C this curve must hit the y-axis (why?), so the trajectory will simply approach a point on the y axis somewhere and the number of hares will go to zero.

8.4.101: Use Bendixson–Dulac Theorem. a) $f_x + g_y = 1 + 1 > 0$, so no closed trajectories. b) $f_x + g_y = -\sin^2(y) + 0 < 0$ for all x, y except the lines given by $y = k\pi$ (where we get zero), so no closed trajectories. c) $f_x + g_y = y + 0 > 0$ for all x, y except the line given by $y = 0$ (where we get zero), so no closed trajectories.

8.4.102: Using Poincaré–Bendixson Theorem, the system has a limit cycle, which is the unit circle centered at the origin as $x = \cos(t) + e^{-t}$, $y = \sin(t) + e^{-t}$ gets closer and closer to the unit circle. Thus we also have that $x = \cos(t)$, $y = \sin(t)$ is the periodic solution.

8.4.103: $f(x,y) = y$, $g(x,y) = \mu(1-x^2)y - x$. So $f_x + g_y = \mu(1-x^2)$. The Bendixson–Dulac Theorem says there is no closed trajectory lying entirely in the set $x^2 < 1$.

8.4.104: The closed trajectories are those where $\sin(r) = 0$, therefore, all the circles centered at the origin with radius that is a multiple of π are closed trajectories.

8.5.101: Critical points: $(0,0,0)$, $(3\sqrt{8}, 3\sqrt{8}, 27)$, $(-3\sqrt{8}, -3\sqrt{8}, 27)$. Linearization at $(0,0,0)$ using $u = x$, $v = y$, $w = z$ is $u' = -10u + 10v$, $v' = 28u - v$, $w' = -(8/3)w$. Linearization at $(3\sqrt{8}, 3\sqrt{8}, 27)$ using $u = x - 3\sqrt{8}$, $v = y - 3\sqrt{8}$, $w = z - 27$ is $u' = -10u + 10v$, $v' = u - v - 3\sqrt{8}w$, $w' = 3\sqrt{8}u + 3\sqrt{8}v - (8/3)w$. Linearization at $(-3\sqrt{8}, -3\sqrt{8}, 27)$ using $u = x + 3\sqrt{8}$, $v = y + 3\sqrt{8}$, $w = z - 27$ is $u' = -10u + 10v$, $v' = u - v + 3\sqrt{8}w$, $w' = -3\sqrt{8}u - 3\sqrt{8}v - (8/3)w$.