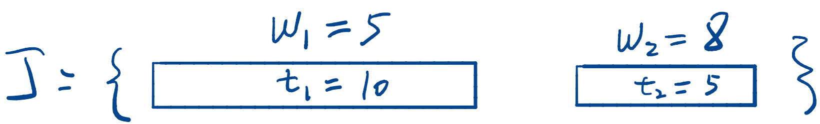
Recitation (II)

Prob 1: Weighted Completion Time on Single Machine

Given a set of n jobs $\{1, 2, 3, \cdots, n\}$, each job j with a processing time $t_j > 0$ and a weight $w_j > 0$, we need to schedule the n jobs on a machine in some order. Let C_j be the completion time of j on in the schedule. Then the goal of the problem is to find a schedule to minimize the weighted sum of the completion times, i.e, $\sum_{j=1}^{n} w_j C_j$.



Schedule: S_1 :

$w_1 t_1 + w_2(t_1 + t_2) = 170$

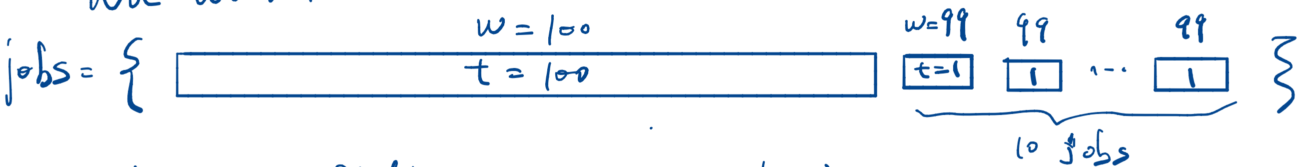
S_2 :

$w_2 t_2 + w_1(t_1 + t_2) = 115$

▷ First try:

Schedule the job with maximum weight first.

- Not work.



$$100 \times 100 + 99(101 + 102 + 103 + \cdots + 110)$$

$$99(1 + 2 + 3 + \cdots 10) + 100 \times 110$$

▷ Second try:

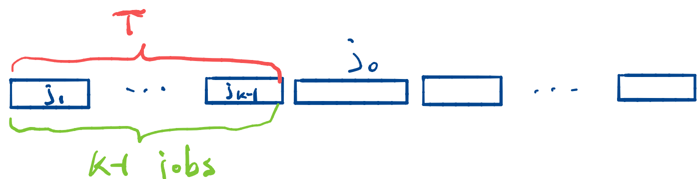
Pick the job j with largest w_j / t_j to execute first.

Proof of the safety property:

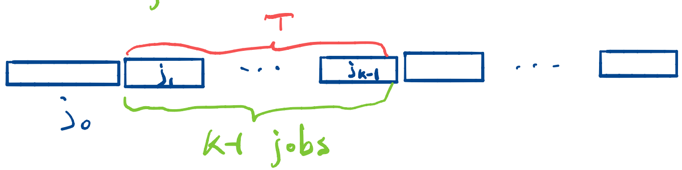
- π^* is an optimal schedule: $\pi^* : \{\text{jobs}\} \longmapsto \{1, 2, 3, \dots, n\}$.

  $\pi(j) = 2 \Rightarrow$ job j is the 2nd job to be executed.

- Let $j_0 := \underset{\text{job } i}{\operatorname{argmax}} \dfrac{w_i}{t_i}$, suppose $\pi^*(j_0) \neq 1$. $\pi^*(j_0) = k$

π^* : 

$$WCT(\pi^*) = \sum_{i=1}^{k-1} w_{j_i} C_{j_i} + w_{j_0}(T + t_{j_0})$$

π : 

$$WCT(\pi) = w_{j_0} t_{j_0} + \sum_{i=1}^{k-1} w_{j_i}(C_{j_i} + t_{j_0})$$

$$\Rightarrow WCT(\pi^*) - WCT(\pi) = w_{j_0} T - \sum_{i=1}^{k-1} w_{j_i} t_{j_0}$$

$$= \frac{w_{j_0}}{t_{j_0}} \cdot t_{j_0} \cdot T - \sum_{i=1}^{k-1} \frac{w_{j_i}}{t_{j_i}} \cdot t_{j_i} \cdot t_{j_0}$$

$$= t_{j_0}\left(\frac{w_{j_0}}{t_{j_0}}\cdot T - \sum_{i=1}^{k-1}\frac{w_{j_i}}{t_{j_i}}\cdot t_{j_i}\right)$$

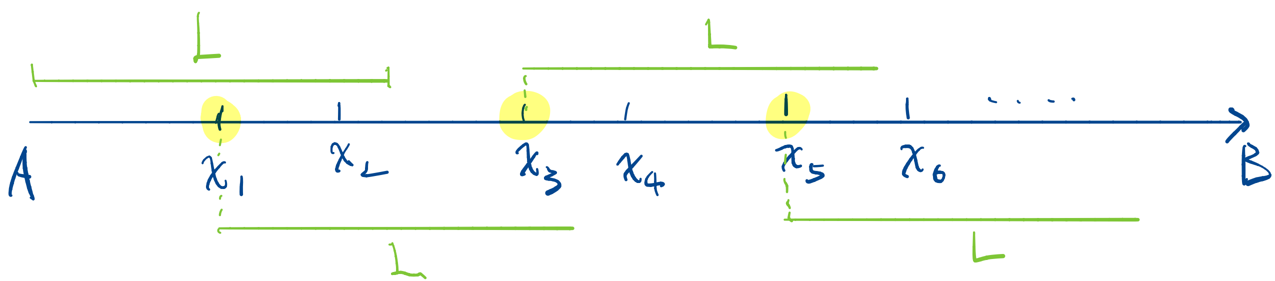
$$= t_{j_0}\cdot\sum_{i=1}^{k-1}\left(\frac{w_{j_0}}{t_{j_0}}-\frac{w_{j_i}}{t_{j_i}}\right)\cdot t_{j_i} \geq 0$$

$\Rightarrow WCT(\pi^*) = WCT(\pi) \Rightarrow \pi$ is also optimal.

$\square$

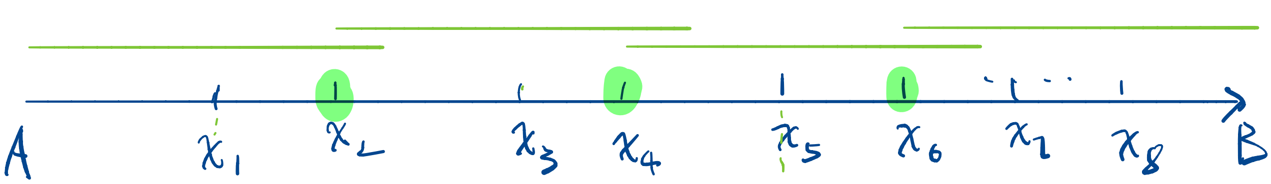
Prob 2: Least Gas Stop

You wish to drive from point A to point B along a highway minimizing the time that you are stopped for gas. You are told beforehand the capacity number L of miles you can drive when the tank is full, the locations $x_1, \cdots, x_n$ of the gas stations along the highway, where x_i indicates the distance from the i-th gas station from A. Design a greedy algorithm to compute the minimum number of times you need to fill the gas tank.



Greedy Strategy: drive as long as possible and fill gas at the last gas station encountered.

Exmp:



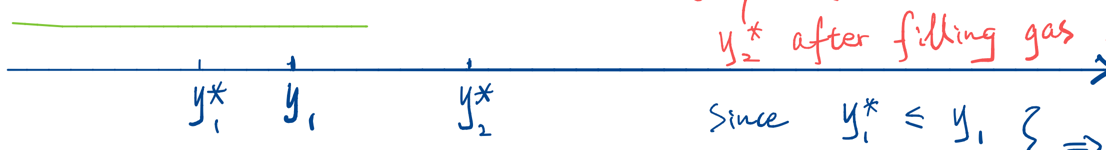
Fill 3 times.

Proof of safety property:

Let y_1 to be the first stop chosen by the greedy strategy.

- Let π^* be an optimal strategy, and π^* chooses y_1^* to be the first stop, and y_2^* to be the 2nd stop.
- Let π to stop at y_1, then follows π^* in the rest of the time

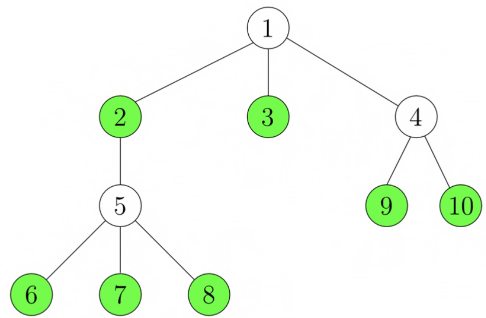
only need to show that the car can reach y_2^* after filling gas at y_1



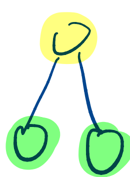
Since $\left.\begin{array}{l} y_1^* \le y_1 \\ y_2^* - y_1^* \le L \end{array}\right\} \Rightarrow y_2^* - y_1 \le L$ $\square$

Prob 3. Max-Indep. Set on Trees.

Given a tree $T = (V, E)$, find the maximum independent set of the tree. For example, maximum independent set of the tree of following tree has size 7.



Examp:

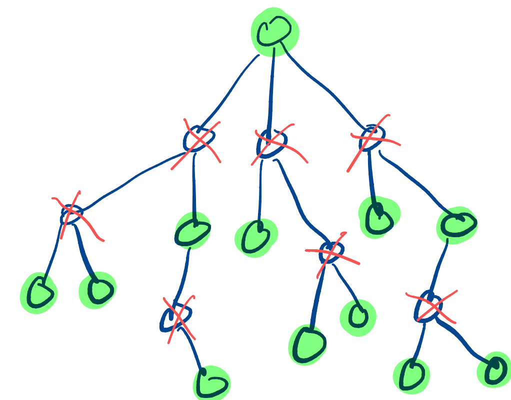


$\Rightarrow$ Observasion: It's better to pick children nodes.

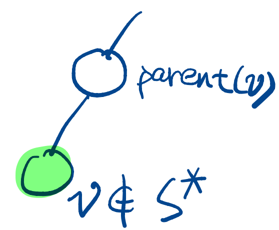
Greedy Strategy:

Add all leaves to the indep. set, then remove all the neighboring nodes (i.e. their parents), and continue.

Proof of Safety:



S^*: optimal IS, doesn't contain some leaf v



① parent(v) $\notin S^*$

$\Rightarrow$ add v to S^*

$\Rightarrow$ cannot happen

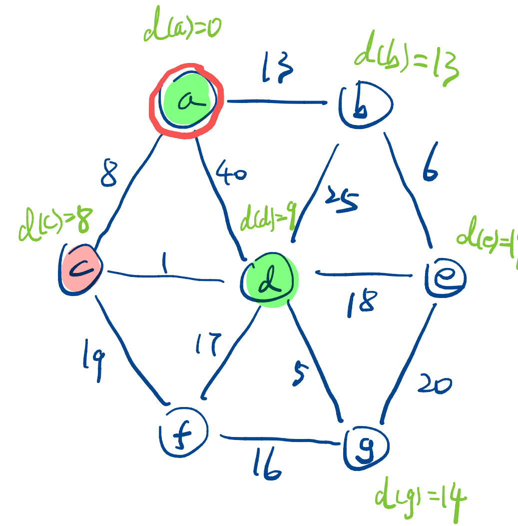
② parent(v) $\in S^*$

$\Rightarrow$ switch v with parent(v) $\Rightarrow$ get another IS of same size. $\square$

1) Shortest Path and MST

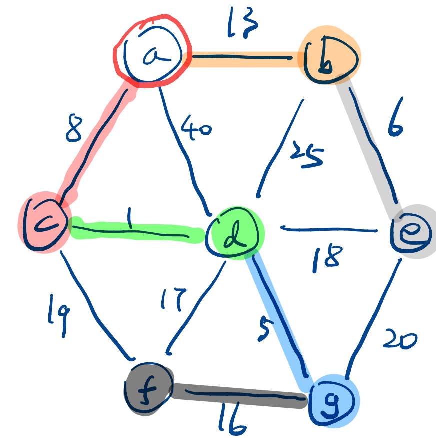
• Simulate Dijkstra Alg : Starting from a

Π	b		c		d		e		f		g	
	d(b)	π(b)	d(c)	π(c)	d	π	d	π	d	π	d	π
a	13	a	8	a	40	a	∞	/	∞	/	∞	/
c	13	a			9	c	∞	/	27	c	∞	/
d	13	a					27	d	26	d	14	d
b							19	b	26	d	14	d
g							19	b	26	d		
e									26	d		
f												

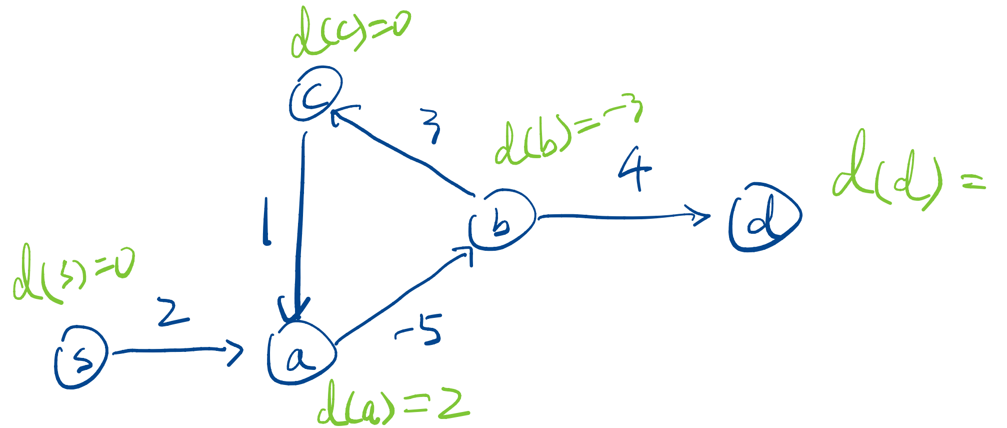


- Simulate Prim's Alg

Π	b		c		d		e		f		g	
	d(b)	π(b)	d(c)	π(c)	d	π	d	π	d	π	d	π
a	13	a	8	a	40	a	∞	/	∞	/	∞	/
c	13	a			1	c	∞	/	19	/	∞	/
d	13	a					18	d	17	d	5	d
g	13	a					18	d	16	g		
b							6	b	16	g		
e									16	g		
f												



D Dijkstra's Alg's Failure on Graphs with negative weights.



Q: Shortest path from s to d?

Ans: $-\infty$

DST paper:

Move to appendix:

Lmm 16 — 0.2 pg

Clm 10 — 0.2 pg

Section 5.4 — 1.5 pg

Lmm 23 — 0.5 pg

Fix reference to: Thm 6, Def 15, P1, P2, …, P6

OFL paper: reduce 8 pg of main body.

10 pg of full paper

To appendix

Lmm 17: 0.5 pg. Lmm 18: 0.5 pg

Lmm 21 + Lmm 22 + Lmm 23 : 1 pg.

Sec 2.1. 0.5 pg Sec 2.4 0.5 pg

Lmm 11 : 0.6 pg. Thm 7, 8. 0.5