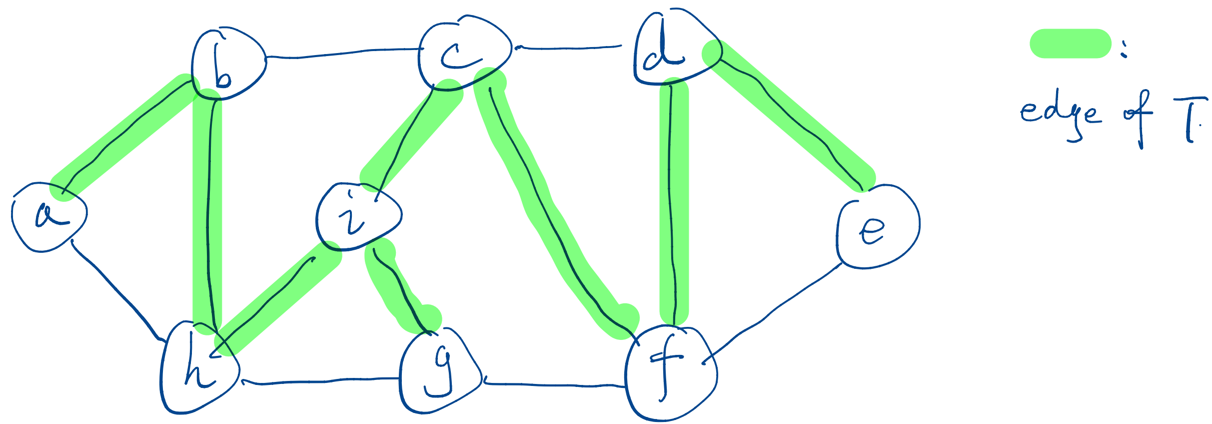
(V) Minimum Spanning Tree.

Def: Given connected graph $G=(V,E)$, a spanning tree of G is a subgraph $T=(V,F)$ that's a tree including all vertices.

Exmp:

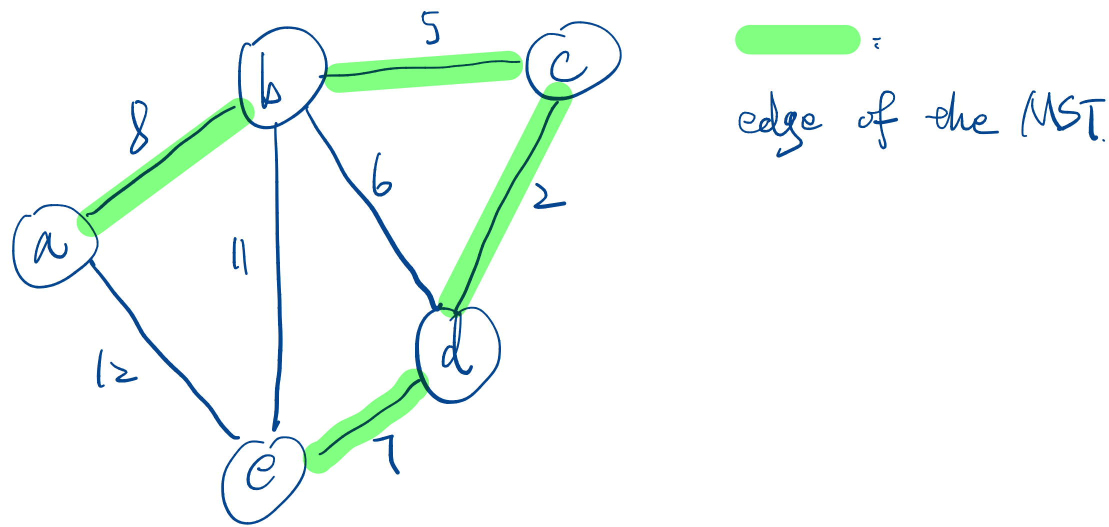


MST problem

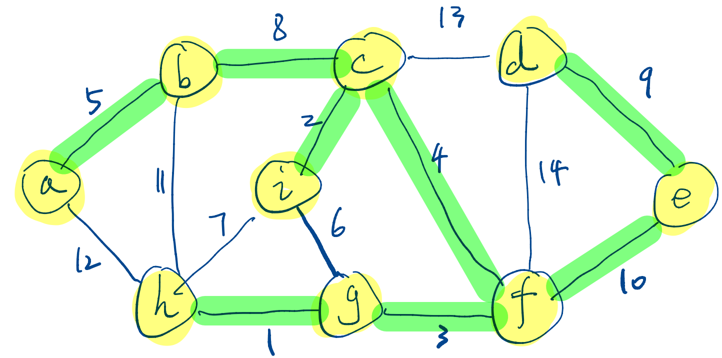
Input: connected graph $G=(V,E)$, weights $w: E \longmapsto \mathbb{R}_{\geq 0}$

Output: a spanning tree T of G with the minimum total weight.

Exmp:



▷ Alg - Prim



- Somewhat to the Dijkstra's Alg.
- Grows the MST vertex by vertex:

① $I \leftarrow \{a\}$

$F \leftarrow \phi$

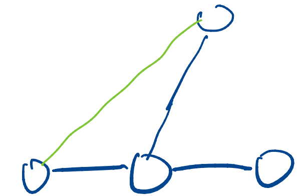
② while $I \neq V$:

$u \leftarrow$ the vertex not in I & nbring I & has the lightest edge incident to I, say this edge is (u, x) for some $x \in I$

$I \leftarrow I \cup \{u\}$.

$F \leftarrow F \cup \{(u, x)\}$

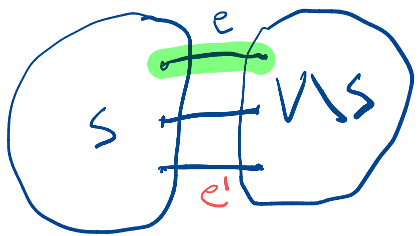
▷ Analyze Prim's Alg.

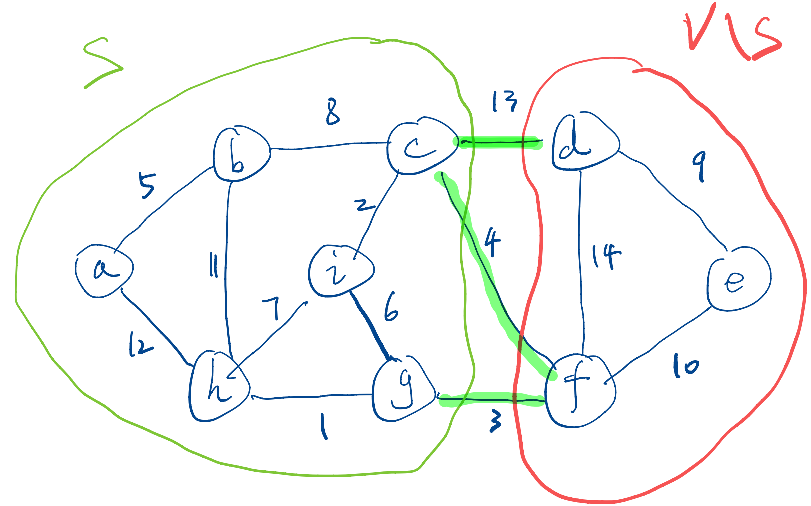


- Recap: property of (spanning) tree
  1. A tree on n vertices has $n-1$ edges.
  2. Adding an arbitrary edge to a tree creates a cycle.
  3. $\forall u, v$ in the tree, the path from u to v is unique.

- Lmm (cut property): In a connected graph $G=(V,E)$, suppose all edge weights are distinct, then $\forall S \subseteq V$, $S \neq \phi$, consider all edges that cross from S to $V \setminus S$ and let e be the minimum-weigh such edge, Then any MST must contain e.

Pf of cut property:





- Suppose there's some MST T' that doesn't contain e
- $T' \cup \{e\}$ will have a cycle C

  $\Rightarrow$ Then C must contain some other edge e' in the cut.
- $T := T' \cup \{e\} \setminus \{e'\}$ $\Rightarrow$ T is still a spanning tree.

  But weight(T) < weight(T'), contradiction.

$\square$

▷ Pf of correctness of Prim's Alg:

① $I \leftarrow \{a\}$

$F \leftarrow \phi$

② while $I \neq V$:

$u \leftarrow$ the vertex not in I & nbring I & has the lightest edge incident to I, say this edge is (u, x) for some $x \in I$

$I \leftarrow I \cup \{u\}$.

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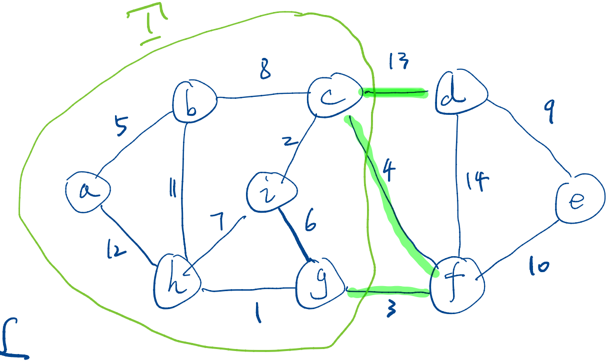

must be in some MST by the Cut Property.

$\square$

Alg-Dijkstra:

$\Gamma \leftarrow \phi$

$d(s) \leftarrow 0$

$d(v) \leftarrow \infty, \ \forall v \neq s$

$\pi(s) \leftarrow NULL$

while $\Gamma \neq V$:

  $u \leftarrow \underset{v \in V \setminus \Gamma}{\operatorname{argmin}} \ d(v)$

  $\Gamma \leftarrow \Gamma \cup \{u\}$

  for all $v \in V \setminus \Gamma$ s.t. $(u,v) \in E$:

    If $d(u) + w(u,v) < d(v)$:

      $d(v) \leftarrow d(u) + w(u,v)$

      $\pi(v) \leftarrow u$

return $d(\cdot), \pi(\cdot)$

v.s.

Alg-Prim

$O(n\log n + m\log n)$

Priority Q

$\Gamma \leftarrow \phi$

$d(s) \leftarrow 0$

$d(v) \leftarrow \infty, \ \forall v \neq s$

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while $\Gamma \neq V$:

  $u \leftarrow \underset{v \in V \setminus \Gamma}{\operatorname{argmin}} \ d(v)$

  $\Gamma \leftarrow \Gamma \cup \{u\}$

  for all $v \in V \setminus \Gamma$ s.t. $(u,v) \in E$:

    If $w(u,v) < d(v)$:

      $d(v) \leftarrow w(u,v)$

      $\pi(v) \leftarrow u$

return $d(\cdot), \pi(\cdot)$