Applied Cryptography and Computer Security
CSE 664 Spring 2017

Lecture 17: Elliptic Curves and Applications

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Lecture Outline

- **We previously looked at**
  - **discrete logarithm problem**
  - **cryptographic schemes that assume difficulty of discrete logarithm**
    - **ElGamal encryption**
    - **Digital signature algorithm**
    - **Diffie-Hellman key exchange**
- **What we are going to learn next**
  - **elliptic curves**
  - **discrete logarithm over elliptic curves**
  - **elliptic curves version of cryptographic constructions**

Discrete Logarithm

- The discrete logarithm problem
  - we are given a group $(G, \cdot)$ and $g \in G$ of order q
  - given $h \in \langle g \rangle$, find a unique integer $x \in [0, n)$ such that $g^x = h$
- Recall that the discrete logarithm problem is considered hard in
  - the multiplicative group $\mathbb{Z}_p^*$ where p is prime and $p - 1$ has at least one large factor
- It is also hard in
  - the multiplicative group of the field $\mathbb{F}_{p^n}$ where p is prime
  - the group of an elliptic curve over a finite field

Elliptic Curves

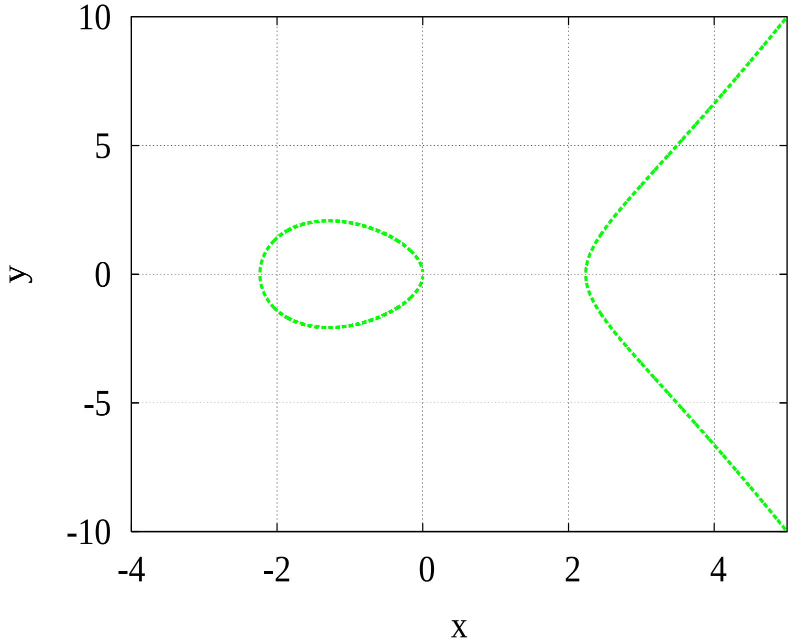
- **Elliptic curves are described by a set of solutions to certain equations in two variables x and y**
- **The curves are solutions to equations of the form $y^2 = x^3 + ax + b$**
- **They have certain properties that make them useful in cryptography**
  - **we'll be dealing with elliptic curves modulo a prime p**
  - **elliptic curve groups can be used in cryptographic algorithms in similar ways multiplicative groups of integers modulo p are used**
  - **the discrete logarithm problem is harder for elliptic curve groups than for $\mathbb{Z}_p^*$**

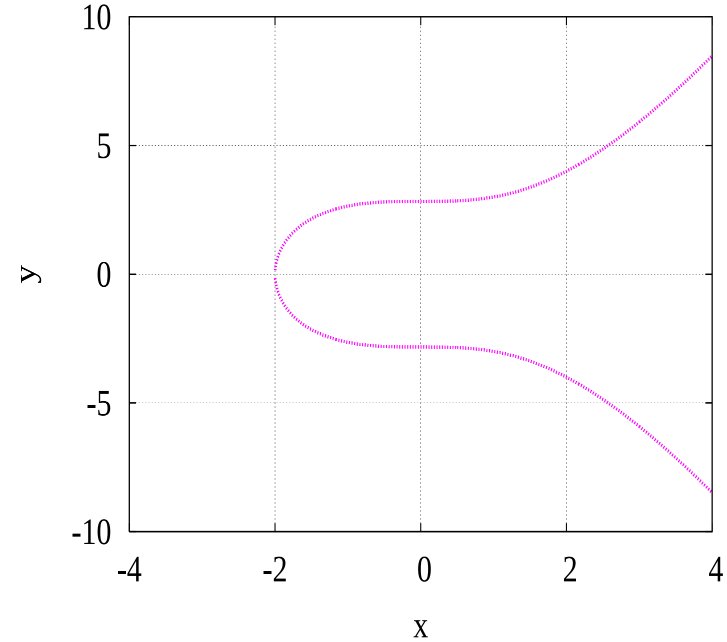
Elliptic Curves

- **Definition**
  - **an elliptic curve is the set E of solutions (x, y) to the equation $y^2 = x^3 + ax + b$**
  - **here x, y, a, and b are real numbers, rational numbers, or integers modulo $m > 1$**
  - **the set E also contains a point at infinity ∞**
- **The point ∞ is not a point on the curve $y^2 = x^3 + ax + b$**
  - **∞ is the identity of the elliptic curve group**
  - **all other points of E are on the curve**

Elliptic Curves: Examples

- **Curves** $y^2 = x^3 - 5x$ **(left) and** $y^2 = x^3 + 8$ **(right)**



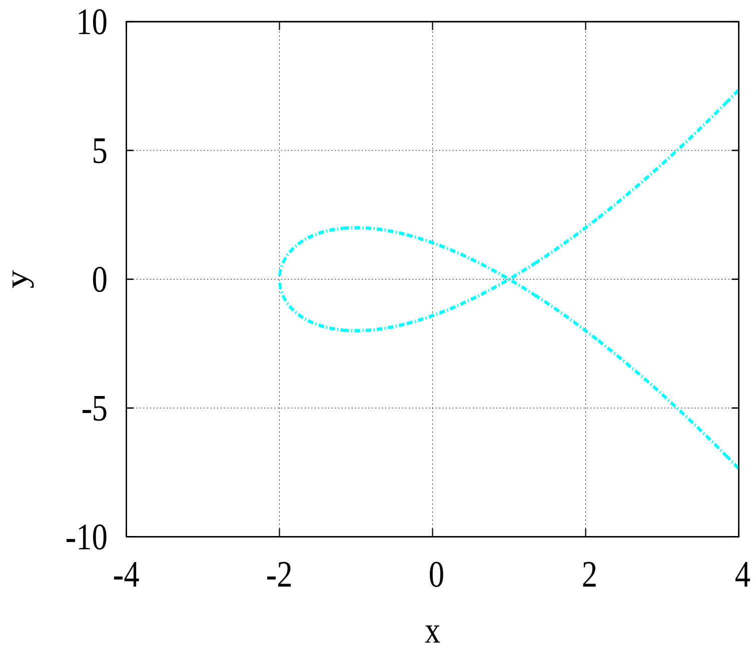


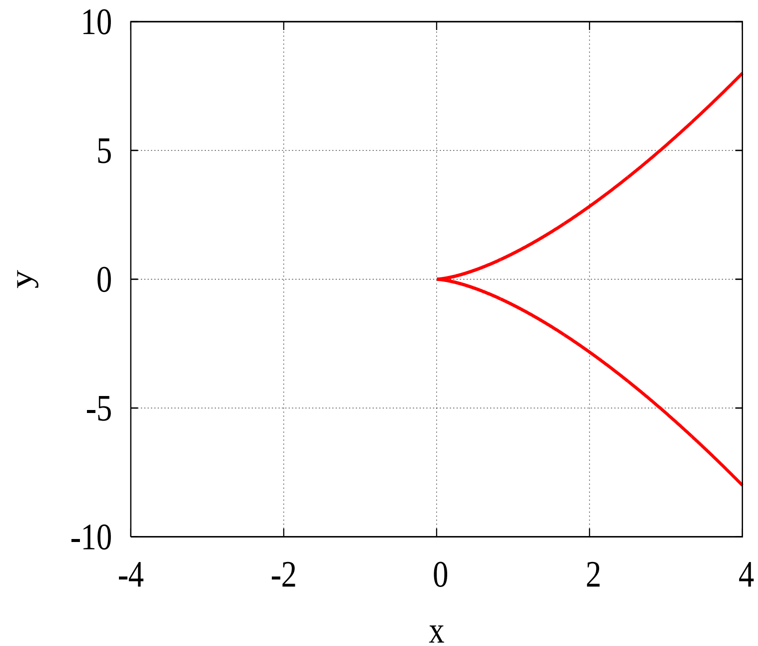
Elliptic Curves

- **Number of roots**
  - **for the cubic equation $y^2 = x^3 + ax + b$, the discriminant is $4a^3 + 27b^2$**
  - **if $4a^3 + 27b^2 = 0$, then the curve has a repeated root**
    - **such elliptic curves are called singular**
  - **if, on the other hand, $4a^3 + 27b^2 \neq 0$, then there are three distinct roots**
    - **such elliptic curves are called non-singular**
  - **we are excluding singular elliptic curves**

Elliptic Curves: Examples

- **Singular curves $y^2 = x^3 - 3x + 2$ (left) and $y^2 = x^3$ (right)**





Elliptic Curves

- **Operations on elliptic curves**
  - **we define a binary operation over E that makes it into a commutative group**
  - **this operation is normally denoted as $+$**
  - **let P and Q be two points on E such that $P = (x_1, y_1)$ and $Q = (x_2, y_2)$**
  - $P + \infty = \infty + P = P$
  - **let $P + Q = R$**
  - **such R is computed depending on the relationship between x_1 and x_2 and y_1 and y_2**

Elliptic Curves

- **Computing $P + Q = R$**
  - **there are three cases**
  - **case 1: $x_1 \neq x_2$**
    - **draw a line through P and Q and find another point R', where the line intersects the curve**
    - **reflect R' on the x-axis to obtain R**
    - **the coordinates (x_3, y_3) are computed as:**

$$x_3 = \lambda^2 - x_1 - x_2, \qquad y_3 = \lambda(x_1 - x_3) - y_1$$

where λ is the slope computed as $\lambda = (y_2 - y_1)/(x_2 - x_1)$

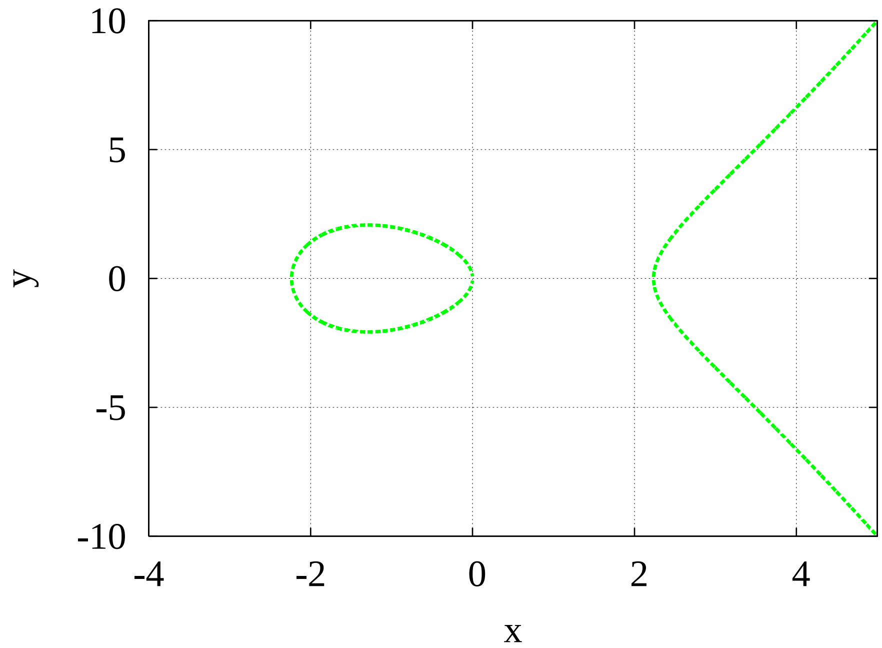
Elliptic Curves

- **Computing $P + Q = R$**
  - **case 2: $x_1 = x_2$ and $y_1 \neq y_2$**
    - **P is a reflection of Q on the x axis**
    - **in this case $P + Q = \infty$**
    - **thus Q is the inverse of P**
  - **case 3: $x_1 = x_2$ and $y_1 = y_2$**
    - **i.e., we are computing $P + P$**
    - **this case is handled similar to case 1**
    - **instead of drawing a line through P and Q, draw a tangent line to the curve at P**
    - **x_3 and y_3 are computed using the formulas from case 1**

Elliptic Curves

- **Computing** $P + Q = R$
  - **case 3**: $x_1 = x_2$ **and** $y_1 = y_2$ **(cont.)**
    - **the formula for the slope now is** $\lambda = (3x_1^2 + a)/(2y_1)$

- **Examples**



Elliptic Curves

- **Elliptic curves modulo a prime p are defined as above except that all operations are replaced by analogous operations in $\mathbb{Z}_p$**
  - **now the points are the solutions to the congruence** $y^2 \equiv x_3 + ax + b \pmod{p}$
  - $a, b \in \mathbb{Z}_p$ **are constants such that** $4a^3 + 27b^2 \not\equiv 0 \pmod{p}$
  - **given points** $P = (x_1, y_1)$ **and** $Q = (x_2, y_2)$**, as before** $P + Q = \infty$ **if** $x_1 = x_2$ **and** $y_2 = -y_1$
  - **the slope λ is computed as**

$$\lambda = \begin{cases} (y_2 - y_1)(x_2 - x_1)^{-1}, & \text{if } P \neq Q \\ (3x_1^2 + a)(2y_1)^{-1}, & \text{if } P = Q \end{cases}$$

  - **and as before** $P + \infty = \infty + P = P$

Elliptic Curves

- **Example: points on the elliptic curve $y^2 = x^3 + 3x + 4$ over $\mathbb{Z}_7$**

x	$x^3 + 3x + 4 \bmod 7$	y
0	4	2, 5
1	1	1, 6
2	4	2, 5
3	5	none
4	3	none
5	4	2, 5
6	0	0

  - **there are 10 points on this elliptic curve (including ∞)**

Elliptic Curves

- **Example:** $y^2 \equiv x^3 + 3x + 4 \pmod 7$ **(cont.)**
  - **to add points $(1, 1)$ and $(2, 5)$**
  - **to double the point $(2, 2)$**

Elliptic Curves

- **Discrete logarithms over elliptic curves**
  - **for $(G, \cdot)$ the discrete logarithm $\log_g h$ was defined as x where $g^x = h$**
  - **now $+$ is the group binary operation, so the discrete logarithm $\log_P Q$ now is a such that $aP = Q$**
- **Computing "exponentiation" aP**
  - **instead of using SQUARE-AND-MULTIPLY algorithm on g and x, we use DOUBLE-AND-ADD algorithm on P and a**

Elliptic Curves

- **Computing aP**
  - **note that additive inverses are very easy to compute**
  - **this is exploited in a generalization** DOUBLE-AND-(ADD OR SUBTRACT) **algorithm**
    - **it uses signed binary representation of integer $a = \sum_{i=0}^{\ell-1} a_i 2^i$, where each $a_i \in \{-1, 0, 1\}$**
    - **given signed binary representation of a, we compute aP by a series of doublings, additions, and subtractions**
    - **signed representation reduces the number of add/subtract operations**

Elliptic Curve Constructions

- **Let's look at elliptic curve version of cryptographic schemes**
- **Elliptic curve Diffie-Hellman key agreement**
  - **fix an elliptic curve E modulo p and a point P_0 of large order on E**
  - **Alice chooses a ($0 < a < p$) and sends aP_0 to Bob**
  - **Bob chooses b ($0 < b < p$) and sends bP_0 to Alice**
  - **Alice computes $k = a(bP_0)$ and Bob computes $k = b(aP_0)$**

Elliptic Curve Constructions

- **An elliptic curve analogue of ElGamal encryption is then:**
  - **fix an elliptic curve E modulo p and a point P_0 of large order on E**
  - **for Alice to generate a key, she chooses secret a_A $(0 < a_A < p)$ and publishes $P_A = aP_0$**
  - **when Bob wants to encrypt message m:**
    - **he first embeds it into a point P of E**
    - **he then chooses a random b $(0 < b < p)$ and sends to Alice** $c = (c_1, c_2) = (bP_0, bP_A + P)$
  - **Alice, who knows the secret key a_A, decrypts as follows:**

$$P = c_2 - a_A c_1 = bP_A + P - a_A bP_0 = ba_A P_0 + P - ba_A P_0 = P$$

Elliptic Curve Constructions

- **An elliptic curve Digital signature algorithm (ECDSA) is then**
  - **choose one of the recommended elliptic curves and curve parameters**
    - **government-recommended curves are now questioned in light of past NSA-related events**
  - **choose a point P_0 of large prime order on the curve and secret key x**
  - **set the public key to xP_0**
  - **proceed with signing similar to as before, but using elliptic curve arithmetic**
  - **see FIPS PUB 186-4 for the details and suggested implementation**

Discrete Logarithm Problem

- **How hard is the discrete logarithm problem to solve in a group over an elliptic curve E?**
  - **the powerful index calculus algorithm doesn't work for elliptic curves**
  - **the best possible algorithm is Pollard rho algorithm with $O(\sqrt{p})$ work**
- **To be secure until the year of 2030**
  - **it is suggested to choose $p \approx 2^{224}$ in case of elliptic curves**
  - **compare this with $p \approx 2^{2048}$ for groups $(\mathbb{Z}_p^*, \cdot)$**
  - **for that reason, elliptic curves have been gaining popularity, especially on constrained platforms**

Summary

- **Elliptic curves are solutions to equations of the form** $y^2 = x_3 + ax + b$
- **Groups over elliptic curves modulo a prime**
  - **often can be used in similar ways to** $(\mathbb{Z}_p^*, \cdot)$
  - **require smaller security parameters because the discrete logarithm is harder in such groups**