Geometry & Topology 24 (2020) 3219–3469

HF = HM, IV
The Seiberg–Witten Floer homology and ech correspondence

Çağatay Kutluhan
Yi-Jen Lee
Clifford Henry Taubes

This is the fourth of five papers that construct an isomorphism between the Seiberg–Witten Floer homology and the Heegaard Floer homology of a given compact, oriented 3–manifold. The isomorphism is given as a composition of three isomorphisms; the first of these relates a version of embedded contact homology on an auxiliary manifold to the Heegaard Floer homology on the original. The second isomorphism relates the relevant version of the embedded contact homology on the auxiliary manifold with a version of the Seiberg–Witten Floer homology on this same manifold. The third isomorphism relates the Seiberg–Witten Floer homology on the auxiliary manifold with the appropriate version of Seiberg–Witten Floer homology on the original manifold. The paper describes the second of these isomorphisms.

53C07; 52C15

1.	Embedded contact homology and Seiberg–Witten Floer homology	3221
2.	Solutions to the Seiberg–Witten equations on Y	3244
3.	The map Φ^{r} from $\mathcal{Z}^{L}_{\mathrm{ech},M}$ to $\mathcal{Z}_{\mathrm{SW,r}}$	3265
4.	Instantons	3280
5.	A priori bounds for the function $\underline{\mathrm{M}}$: the complement of $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^{+}\cup\hat{\gamma}_{\mathfrak{p}}^{-})$	3289
6.	A priori bounds for $\underline{\mathrm{M}}$	3298
7.	Propositions 1.1–1.4 and Theorem 1.5	3308
Appendix.	The proof of Proposition 2.6	3351
A.	The eigenvalue equation $\mathfrak{L}_{\mathfrak{c},\mathrm{r}}\mathfrak{b}=\lambda\mathfrak{b}$	3351
B.	Vortex equation solutions and (A,ψ)	3385
C.	Paths in $\mathrm{Conn}(E)\times C^{\infty}(Y;\mathbb{S})$ from vortex solutions	3430
References		3467

This is the fourth of five papers that together supply an isomorphism between Ozsváth and Szabó's Heegaard Floer homology [12; 13] of any given compact, oriented 3–manifold and a version of the Seiberg–Witten Floer homology of the same 3–manifold.

Published: 6 January 2021 DOI: 10.2140/gt.2020.24.3219

The existence of an isomorphism between these respective Floer homologies is stated as the main theorem in [8]. A particular isomorphism is described in [8], which can be written as the concatenation of three separate isomorphisms which involve an auxiliary manifold that is obtained from the original by connect summing a certain number of copies of $S^1 \times S^2$. As noted in Section 3 of [8], the middle isomorphism in this concatenation identifies a version of the Seiberg–Witten Floer homology on the connect sum manifold with a version of Michael Hutchings' embedded contact homology [3] on this same connect sum. The relevant version of this embedded contact homology is described in Section 2.2 of [8] and in the appendix of [9]. This homology is also the focus in [10]. The isomorphism between the respective Seiberg–Witten Floer and embedded contact homologies on the connect sum manifold is asserted by Theorems 3.3 and 3.4 in [8]. The latter reference deduces its Theorem 3.4 from Theorem 3.3; this paper proves [8, Theorem 3.3]. Section 1.4 explains why the latter theorem is a consequence of Theorem 1.5 in Section 1.4. Respective proofs are also given below for [8, Theorems 3.1 and 3.2]. These are seen in Section 1.3 to be consequences of Propositions 1.1–1.4.

Most of what is done in [9; 10] is not relevant for what follows. Even so, certain results and constructions from these papers are needed. In particular, the geometry needed to define the appropriate versions of the Seiberg–Witten Floer homology and the embedded contact homology is described in [9; 10]. Section 1.1 provides a summary of this geometry.

The following conventions are used throughout the remainder of this paper: Section numbers, equation numbers and other references from [8; 9; 10] are distinguished from those in this paper by the use of the Roman numerals I, II and III as a prefix. For example, "Section II.1" refers to Section 1 in [9]. Note also that the convention here as in [9; 10] is to use c_0 to denote a constant in $(1, \infty)$ whose value is independent of all relevant parameters. The value of c_0 can increase between subsequent appearances. A second convention used here and in [9; 10] concerns a function that is denoted by χ. The latter is a fixed, nonincreasing function on $\mathbb{R}$ that equals 1 on $(-\infty, 0]$ and equals 0 on $[1, \infty)$.

Acknowledgements Kutluhan was supported in part by a National Science Foundation Postdoctoral Fellowship under Award 1103795. Lee and Taubes are also supported in part by grants from the National Science Foundation. Lee was supported by Hong Kong Research Grants Council grants GRF-401913, 14316516, 14305541.

1 Embedded contact homology and Seiberg–Witten Floer homology

Let M denote the given compact, oriented 3–manifold. A self-indexing Morse function for M and certain auxiliary data are used in [9] to construct a second manifold, Y, as a connected sum of M with copies of $S^1 \times S^2$. The manifold Y has two orientations, one coming from the part it shares with M and then the opposite orientation. Theorems 3.1–3.4 in [8] compare certain Seiberg–Witten Floer homologies on the M–oriented version with a certain sort of embedded contact homology on the oppositely oriented version. These respective homologies need certain geometric data for their definition. This section starts with a description of the necessary data. It then briefly describes the relevant version of embedded contact homology and the relevant version of Seiberg–Witten Floer homology. It ends by restating Theorems 3.1–3.4 from [8].

1.1 The geometry of *Y*

The construction of Y from M is described in Section II.1. This subsection summarizes the salient features of Y.

Part 1 The construction of Y starts with the choice of a self-indexing Morse function, $f\colon M \to [0,3]$ with one index 0 critical point, one index 3 critical point and some index 1 and index 2 critical points. The number of index 1 (and thus index 2) critical points is denoted by G. The manifold Y is diffeomorphic to the connected sum of M with $\mathrm{G}+1$ copies of $S^1 \times S^2$. The manifold Y is oriented so that the part from M has the orientation opposite from M's orientation. Note that [8] uses $\overline{Y}$ to denote this orientation of the connected sum and uses Y to denote the connected sum with the orientation induced from M.

Also needed from M is the choice of a class in $H^2(M;\mathbb{Z})$, which defines a homomorphism from $H_2(M;\mathbb{Z})$ to $2\mathbb{Z}$. This class is denoted in what follows by c_{1M}. A Spin^c structure will be chosen in a moment, and its first Chern class will play the role of c_{1M}. Needed also is a chosen pairing between the set of index 1 critical points of f and the set of index 2 critical points of f. The resulting set of G pairs is denoted by Λ. An element $\mathfrak{p} \in \Lambda$ is written as an ordered pair of points with it understood that the first entry is the index 1 critical point of f and the second entry is the index 2 critical point of f. Various other Morse-theoretic items from M are needed to construct Y and its geometry, but these others play minor roles in this paper.

The definition of Y required the choice of positive numbers which are denoted by δ_* and R. This δ_* is from $(0,1)$; it is determined by the chosen function f. Meanwhile, R has the lower bound $-100\ln\delta_*$. This constant R has no a priori upper bound, and the freedom to take R as large as needed is exploited in [9; 10] and in the constructions to come in this article.

The construction of the geometry needed for the embedded contact geometry chain complex required the choice of two additional positive numbers which are denoted by δ and x_0. The trio (δ, x_0, R) are constrained by the requirements that $\delta < \delta_*/c_0$, $x_0 < \delta^3$ and $R \geq -c_0 \ln x_0$. The choice of δ determines an upper bound for x_0, and the choice of x_0 subject to this upper bound then determines a lower bound for R. Constants δ, x_0 and R that satisfy these bounds are said to be appropriate. The freedom to take δ as small as desired is also exploited in [9; 10] and in what follows.

Part 2 The manifold Y is constructed by attaching $\mathrm{G}+1$ handles to M. In particular, Y is written as the union of sets $M_\delta \cup \mathcal{H}_0 \cup \big(\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p}}\big)$, where the notation is as follows: First, M_δ is the complement in M of $2(\mathrm{G}+1)$ disjoint balls about the critical points of f. What is written as $\mathcal{H}_0$ is a 1–handle and so diffeomorphic to $[-1,1]\times S^2$. It intersects M_δ near $\{-1\}\times S^2$ as an annulus in a ball centered on the index 3 critical point of f, and it intersects M_δ near $\{1\}\times S^2$ as an annulus in a ball centered on the index 0 critical point of f. Meanwhile, the various $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ are 1–handles and so each is diffeomorphic to $[-1,1]\times S^2$ also. These are pairwise disjoint and disjoint from $\mathcal{H}_0$. Any given $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ intersects M_δ near $\{-1\}\times S^2$ as an annulus in a ball centered around $\mathfrak{p}$'s index 2 critical point and it intersects M_δ near $\{1\}\times S^2$ as an annulus in a ball centered around $\mathfrak{p}$'s index 1 critical point.

The handle $\mathcal{H}_0$ and those from the set $\{\mathcal{H}_{\mathfrak{p}}\}_{\mathfrak{p}\in\Lambda}$ have preferred coordinates, these denoted by $(u,(\theta,\phi))$ where (θ,ϕ) are spherical coordinates for the S^2 factor and where u is the Euclidean coordinate for the closed interval $[-R-\ln(7\delta_*), R+\ln(7\delta_*)]$. The function f appears in these coordinates near the $u<0$ end of $\mathcal{H}_0$ as $f=3-e^{2(u+R)}$ and near the $u>0$ end of $\mathcal{H}_0$ as $f=e^{-2(u-R)}$. Meanwhile, the function f appears near the respective negative and positive u ends of any given $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ as

$$f = 2 - e^{-2(u+R)}(1-3\cos^2\theta) \quad \text{and} \quad f = 1 + e^{2(u-R)}(1-3\cos^2\theta). \tag{1-1}$$

The 3–form $du\sin\theta\, d\theta\, d\phi$ gives the Y–orientation to each handle. Orient the cross-sectional spheres in each 1–handle using the 2–form $\sin\theta\, d\theta\, d\phi$.

Part 3 The definition of the relevant version of Hutchings' embedded contact homology uses a pair (w,a) of 2–form and 1–form on Y. The latter define a stable Hamiltonian structure, which is to say that w is closed, $a\wedge w$ is nowhere zero and defines Y's orientation, and $da\subset \mathrm{Span}(w)$. The vector field that generates the kernel of w and has pairing 1 with a is denoted by v. The salient features of w, a and v are listed in the upcoming (1-3). This equation refers to auxiliary functions x, χ_+, χ_-, f and g. These are functions on $[-R-\ln(7\delta_*), R+\ln(7\delta_*)]$ that are defined using the chosen function χ. By way of a reminder, χ is a smooth, nonincreasing function on $\mathbb{R}$ that is 1 on $(-\infty,0]$ and equals 0 on $[1,\infty)$. The aforementioned five functions are

$$x = x_0\chi(|u|-R-\ln\delta+12),$$
$$\chi_+ = \chi\bigl(-u-\tfrac{1}{4}R\bigr),\quad \mathrm{f}=x+2(\chi_+e^{2(u-R)}+\chi_-e^{-2(u+R)}),\tag{1-2}$$
$$\chi_- = \chi\bigl(u-\tfrac{1}{4}R\bigr),\qquad \mathrm{g}=(\chi_+e^{2(u-R)}-\chi_-e^{-2(u+R)}).$$

What follows is the promised list:

(1-3)
- **On M_δ** The 2–form w on M_δ is nowhere zero on the kernel of the 1–form df, and v here is a certain pseudogradient vector field for f.
- **In the handle $\mathcal{H}_0$** The 2–form w and the vector field v on $\mathcal{H}_0$ are
$$w=\sin\theta\,d\theta\wedge d\phi\quad\text{and}\quad v=\frac{1}{2(\chi_+e^{2(u-R)}+\chi_-e^{-2(u+R)})}\frac{\partial}{\partial u}.$$
- **In the handles $\{\mathcal{H}_\mathfrak{p}\}_{\mathfrak{p}\in\Lambda}$** Fix $\mathfrak{p}\in\Lambda$. The trio a, w and v on $\mathcal{H}_\mathfrak{p}$ are
$$a=(x+\mathrm{g}')(1-3\cos^2\theta)\,du-\sqrt{6}\,\mathrm{f}\cos\theta\sin^2\theta\,d\phi+6\mathrm{g}\cos\theta\sin\theta\,d\theta,$$
$$w=6x\cos\theta\sin\theta\,d\theta\wedge du-\sqrt{6}\,d\{\mathrm{f}\cos\theta\sin^2\theta\,d\phi\},$$
$$v=\hat{c}_v^{-1}\{\mathrm{f}(1-3\cos^2\theta)\,\partial_u-\sqrt{6}\,x\cos\theta\,\partial_\phi+\mathrm{f}'\cos\theta\sin\theta\,\partial_\theta\}.$$

Here, $\hat{c}_v=(x+\mathrm{g}')\mathrm{f}(1-3\cos^2\theta)^2+6(x\mathrm{f}+\mathrm{g}\mathrm{f}')\cos^2\theta\sin^2\theta$ is a positive function of (u,θ).

An additional property of w plays a central role in the story to come. To say more about this, introduce the direct sum decomposition

$$H_2(Y;\mathbb{Z})=H_2(M;\mathbb{Z})\oplus H_2(\mathcal{H}_0;\mathbb{Z})\oplus\bigoplus_{\mathfrak{p}\in\Lambda}H_2(\mathcal{H}_\mathfrak{p};\mathbb{Z})\tag{1-4}$$

that comes via Mayer–Vietoris by writing $Y=M_\delta\cup\mathcal{H}_0\cup\bigl(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_\mathfrak{p}\bigr)$. The summands in (1-4) that correspond to the various 1–handles are isomorphic to $\mathbb{Z}$, and any oriented, cross-sectional sphere is a generator.

The additional property concerns the cohomology class defined by w. This class is determined by what follows: Integration of w over closed 2–cycles defines the linear map from $H_2(Y;\mathbb{Z})$ to $\mathbb{Z}$ that has value 2 on the generator of $H_2(\mathcal{H}_0;\mathbb{Z})$; it has value zero on each $\mathfrak{p}\in\Lambda$ version of $H_2(\mathcal{H}_\mathfrak{p};\mathbb{Z})$; and it acts on the $H_2(M;\mathbb{Z})$ summand in (1-4) as the pairing with the chosen class c_{1M}.

Part 4 A particular closed integral curve of the vector field v plays a distinguished role in the embedded contact homology story. This curve is denoted by $\gamma^{(z_0)}$ here and in the other papers in this series. The curve is disjoint from $\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_\mathfrak{p}$ and it crosses $\mathcal{H}_0$ so as to have intersection number 1 with each cross-sectional sphere. Note in this regard that the convention here and in what follows is to orient the integral curves of v using v for the oriented unit tangent vector. This curve intersects $\Sigma=f^{-1}\left(\frac{3}{2}\right)$ in precisely one point. The latter is denoted by z_0.

A pair of additional 1–forms enter the story. These are denoted by $\upsilon_\diamond$ and $\hat{a}$:

(1-5) • **The 1–form $\boldsymbol{\upsilon_\diamond}$** The 1–form $\upsilon_\diamond$ is closed and is such that $\upsilon_\diamond\wedge w\geq 0$. Furthermore, $\upsilon_\diamond\wedge w=0$ only where both $u=0$ and $1-3\cos^2\theta=0$ on each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_\mathfrak{p}$. This 1–form equals df on M_δ, it is given by $\upsilon_\diamond=2(\chi_+e^{2(u-R)}+\chi_-e^{-2(u+R)})\,du$ on $\mathcal{H}_0$, and it is given by $\upsilon_\diamond=d((\chi_+e^{2(u-R)}-\chi_-e^{-2(u+R)})(1-3\cos^2\theta))$ on any given $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_\mathfrak{p}$.

The definition $\hat{a}$ refers to the function χ_δ that is defined on any given $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_\mathfrak{p}$ by the rule $\chi_\delta=\chi(|u|-R-\ln\delta+10)$.

(1-6) • **The 1–form $\hat{\boldsymbol{a}}$** The 1–form $\hat{a}$ has pairing 1 with v and is such that $\hat{a}\wedge w>0$. This 1–form is equal to $\upsilon_\diamond$ on $M_\delta\cup\mathcal{H}_0$ and it is equal to $\chi_\delta a+(1-\chi_\delta)\upsilon_\diamond$ on any given $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_\mathfrak{p}$.

The kernel of the 1–form $\hat{a}$ defines a 2–plane subbundle in TY on which w is nondegenerate. When oriented by w, the bundle $\mathrm{Ker}(\hat{a})$ has an Euler class which evaluates as 2 on the generator of the $H_2(\mathcal{H}_0;\mathbb{Z})$ summand in (1-4) and evaluates as -2 on the generator of each $\mathfrak{p}\in\Lambda$ summand $H_2(\mathcal{H}_\mathfrak{p};\mathbb{Z})$. The vector field v has pairing 1 with $\hat{a}$ also.

Various other geometric properties of Y are introduced as needed in what follows.

Part 5 The almost complex geometry of $\mathbb{R}\times Y$ is defined by an almost complex structure, this denoted by J. The latter is constrained in various ways; most of the constraints are given in Part 1 of Section II.3A and Section III.1C. The upcoming (1-7) reviews various features of J. We use s to denote the Euclidean coordinate on the $\mathbb{R}$ factor of $\mathbb{R}\times Y$.

(1-7)
- J maps the Euclidean tangent vector ∂_s to the $\mathbb{R}$ factor of $\mathbb{R}\times Y$ to v.
- J is not changed by constant translations of the coordinate s on $\mathbb{R}\times Y$.
- J preserves the kernel of the 1–form $\hat{a}$, and its restriction to this 2–plane field defines the orientation given by w.
- J on $\mathbb{R}\times\mathcal{H}_0$ and on any given $\mathfrak{p}\in\Lambda$ version of $\mathbb{R}\times\mathcal{H}_{\mathfrak{p}}$ is invariant with respect to constant translations of the $\mathbb{R}/(2\pi\mathbb{Z})$ coordinate ϕ.

It is a consequence of (1-7) that the 2–form $\hat{\omega}=ds\wedge\hat{a}+w$ on $\mathbb{R}\times Y$ is compatible with J. This is to say that the bilinear form $\hat{\omega}(\cdot\,,J(\cdot))$ on $T(\mathbb{R}\times Y)$ defines a Riemannian metric. Note in particular that this metric has the form ds^2+g_Y with g_Y being a metric on TY that makes v a unit vector that is orthogonal to the kernel of $\hat{a}$. The corresponding metric on T^*Y gives $\hat{a}$ norm 1 and is such that the Hodge star of $\hat{a}$ is w.

These respective metrics on $\mathbb{R}\times Y$ and Y are used implicitly in what follows.

1.2 Embedded contact homology on *Y*

The appendix in [9] describes the relevant version of embedded contact homology on Y. More is said about the chain complex and its homology in Sections III.1B and III.9. This subsection provides a very brief summary of what is said in these sections of [9; 10]. The summary here comprises Parts 2–4 of the five parts of this subsection. The first part of the subsection constitutes a digression that concerns Spin^c structures on M and Y. The final part summarizes some observations from [9; 10] that are particularly relevant in the subsequent sections of this paper.

Part 1 A Spin^c structure on M is chosen whose associated first Chern class is the chosen class c_{1M}. The chosen Spin^c structure is fixed for the remainder of this article. The Spin^c structure on M determines in a canonical fashion a corresponding Spin^c structure on Y. This is done using a version of Mayer–Vietoris with the decomposition of Y as $M_\delta\cup\mathcal{H}_0\cup\big(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}\big)$. The first Chern class of the resulting Spin^c structure on Y has pairing 2 with the generator of the $H_2(\mathcal{H}_0;\mathbb{Z})$ summand in (1-4) and it has pairing 0 with each of the $\mathfrak{p}\in\Lambda$ labeled summands. The pairing with the $H_2(M;\mathbb{Z})$ summand is that of the first Chern class of the Spin^c structure on M, which is to say that of c_{1M}. It follows as a consequence that the image in $H_2(Y;\mathbb{R})$ of the first Chern class of the Spin^c structure on Y is the class defined by the closed form w.

The image of $H_2(M;\mathbb{Z})$ in $\mathbb{Z}$ given by the pairing with c_{1M} is a subgroup of $\mathbb{Z}$. If c_{1M} is not torsion, use p_M to denote the largest integer that divides all of its elements. Note that p_M is in all cases even.

Part 2 The $\mathbb{Z}$–module that serves as the embedded complex homology chain complex is defined using a certain principal $\mathbb{Z}$–bundle over a set that is denoted by $\mathcal{Z}_{\mathrm{ech},M}$. The set $\mathcal{Z}_{\mathrm{ech},M}$ is described in Proposition II.2.8. The principal $\mathbb{Z}$–bundle is denoted by $\hat{\mathcal{Z}}_{\mathrm{ech},M}$ and described in Section II.1F and in Part 4 of Section III.1B. The embedded contact homology chain complex is denoted by $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{ech},M})$.

By way of a reminder, an element in $\mathcal{Z}_{\mathrm{ech},M}$ is a set, Θ, consisting of some number of closed integral curves of v that lie entirely in the union of the $f \in (1,2)$ part of M_δ and the various $\mathfrak{p} \in \Lambda$ versions of $\mathcal{H}_{\mathfrak{p}}$. In particular, the union of the curves that constitute such a set Θ intersect each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ in at most three components. There exists in all cases one component of this intersection that lies entirely in the $1-3\cos^2\theta > 0$ part of $\mathcal{H}_{\mathfrak{p}}$ as an arc that crosses $\mathcal{H}_{\mathfrak{p}}$ from the $u = -R - \ln(7\delta_*)$ end to the $u = R + \ln(7\delta_*)$ end. The locus in $\mathcal{H}_{\mathfrak{p}}$ where both $u = 0$ and $1 - 3\cos^2\theta = 0$ is a disjoint union of two closed integral curves of v, and one or both of these curves can also appear in Θ. The curve with $u = 0$ and $\cos\theta = \frac{1}{\sqrt{3}}$ is denoted by $\hat{\gamma}_{\mathfrak{p}}^{+}$ and the curve where $u = 0$ and $\cos\theta = -\frac{1}{\sqrt{3}}$ by $\hat{\gamma}_{\mathfrak{p}}^{-}$.

If γ is used to denote a closed integral curve of v, then $[\gamma]$ is used to denote both the oriented cycle defined by γ and the corresponding element in $H_1(Y;\mathbb{Z})$, where it is understood that γ is oriented by v. Meanwhile, $[\Theta] = \sum_{\gamma\in\Theta}[\gamma]$ is used to denote both a sum of oriented 1–cycles and the corresponding homology class. The latter is fixed by the chosen Spin^c–structure; this class is the same for all elements in $\mathcal{Z}_{\mathrm{ech},M}$.

The principal $\mathbb{Z}$–bundle $\hat{\mathcal{Z}}_{\mathrm{ech},M} \to \mathcal{Z}_{\mathrm{ech},M}$ is defined after choosing a fiducial element $\Theta_0 \in \mathcal{Z}_{\mathrm{ech},M}$. The fiber of $\hat{\mathcal{Z}}_{\mathrm{ech},M}$ over a given element $\Theta \in \mathcal{Z}_{\mathrm{ech},M}$ is identified with the set of equivalence classes of pairs of the form (Θ, Z) where Z is a relative cycle in $H_2(Y;[\Theta]-[\Theta_0])$. The equivalence relation is defined using the pairing with the Poincaré dual of the homology class of the closed integral curve $\gamma^{(z_0)}$. This pairing defines a homomorphism from the $\mathbb{Z}$–module of closed 2–cycles to $\mathbb{Z}$ that is denoted by $[\gamma^{(z_0)}]^{\mathrm{Pd}}(\cdot)$. The equivalence relation that defines $\hat{\mathcal{Z}}_{\mathrm{ech},M}$ has $(\Theta, Z) \sim (\Theta', Z')$ if and only if $\Theta = \Theta'$ and also $[\gamma^{(z_0)}]^{\mathrm{Pd}}(Z - Z') = 0$. The principal bundle projection map sends an equivalence class (Θ, Z) to Θ. The element $1 \in \mathbb{Z}$ acts to send (Θ, Z) to $(\Theta, Z + [S_0])$, where $[S_0]$ is the $u = 0$ sphere in $\mathcal{H}_0$.

The module $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{ech},M})$ has a relative $\mathbb{Z}$–grading when c_{1M} is torsion, and it has a relative $\mathbb{Z}/p_M\mathbb{Z}$ grading otherwise. The grading rule comes via a corresponding grading of the generating set $\hat{\mathcal{Z}}_{\mathrm{ech},M}$. The rule for assigning the relative grading of the generators is given by Hutchings in [3; 2]. The rule is described briefly in Section III.9A.

Part 3 The appendix in [9] and Sections III.1D and III.9B explain how the differential that defines the embedded contact homology is computed using J–holomorphic submanifolds in $\mathbb{R}\times Y$. Keep in mind that a J–holomorphic submanifold is properly embedded with J–invariant tangent space and such that the integral of w over the submanifold is finite. The particular J–holomorphic submanifolds that are used to define the differential form a topological space that is indexed by an ordered pair from $\hat{\mathcal{Z}}_{\mathrm{ech},M}$. Let $(\hat\Theta', \hat\Theta)$ denote such a pair. The corresponding component of this topological space is denoted by $\mathcal{M}_1(\hat\Theta', \hat\Theta)$. This space is a finite disjoint union of connected components, each being homeomorphic to $\mathbb{R}$. In fact, each component has a free $\mathbb{R}$ action that is induced by the constant translations along the $\mathbb{R}$ factor of $\mathbb{R}\times Y$.

Any given submanifold from $\mathcal{M}_1(\hat\Theta', \hat\Theta)$ is characterized in part by the behavior of its $|s|\gg 1$ part. To elaborate, suppose that C is a given submanifold from this space. There exists $s_*\gg 1$ such that the $|s|\geq s_*$ portion is a disjoint union of embedded cylinders where ds is nowhere zero. Each such cylinder is said to be an end of the given submanifold. These ends have the following properties:

(1-8)
- The $s\geq s_*$ ends are in $1{-}1$ correspondence with the integral curves from Θ. This correspondence is such that the set of constant s slices of any given end converge isotopically in Y as $s\to\infty$ to its partner in Θ.
- The $s\leq -s_*$ ends are in $1{-}1$ correspondence with the integral curves from Θ'. This correspondence is such that the set of constant s slices of any given end converge isotopically in Y as $s\to-\infty$ to its partner in Θ'.

Section III.9B associates a sign, either 1 or -1, to each component of $\mathcal{M}_1(\hat\Theta', \hat\Theta)$. This is done in accordance with the rules laid out by Hutchings in [3; 2]. These signs determine the endomorphism of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{ech},M})$ that supplies the embedded contact homology differential as follows: the relevant endomorphism of this $\mathbb{Z}$–module is given by its actions on the set of generators by the rule

$$\hat\Theta \mapsto \sum_{\hat\Theta'\in\hat{\mathcal{Z}}_{\mathrm{ech},M}} \mathrm{N}_{\hat\Theta',\hat\Theta}\,\hat\Theta', \tag{1-9}$$

where any given $\hat\Theta'\in\hat{\mathcal{Z}}_{\mathrm{ech},M}$ version of $\mathrm{N}_{\hat\Theta',\hat\Theta}$ is the sum of the $+1$'s and -1's that are associated to the components of $\mathcal{M}_1(\hat\Theta', \hat\Theta)$.

The differential on $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{ech},M})$ that defines the embedded contact homology decreases the relative grading by 1.

Part 4 The appendix in [9] and Sections III.1D and III.9C describe a certain action of $\mathbb{Z}(\mathbb{U}) \otimes \bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})$ on the embedded contact homology $\mathbb{Z}$–module. The endomorphism that generates the $\mathbb{Z}(\mathbb{U})$ factor is called the $\mathbb{U}$–map. The latter decreases the relative degree by 2 and it commutes with the generators of the $\bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})$ factor. The generators of the latter decrease relative degree by 1. The $\mathbb{U}$–map generator and those of $\bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})$ are given by corresponding endomorphisms of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{ech},M})$. Each such endomorphism is defined by a version of (1-9) with the set $\{\mathrm{N}_{\hat\Theta',\hat\Theta}\}_{\hat\Theta',\hat\Theta\in\hat{\mathcal{Z}}_{\mathrm{ech},M}}$ determined by certain sets of J–holomorphic submanifolds according to rules laid out by Hutchings in Section 12 of [5]. See also Section 2.5 of [6] for a discussion of the $\mathbb{U}$–map generator.

An endomorphism of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{ech},M})$ that defines the $\mathbb{U}$–map is defined in Sections III.1D and III.9C with the help of a chosen point in the handle $\mathcal{H}_0$. With the point chosen, the set of coefficients in the corresponding version of (1-9) is denoted by $\{\mathrm{N}^{\mathbb{U}}_{\hat\Theta',\hat\Theta}\}_{\hat\Theta',\hat\Theta\in\hat{\mathcal{Z}}_{\mathrm{ech},M}}$. These are such that any given $\mathrm{N}^{\mathbb{U}}_{\hat\Theta',\hat\Theta}$ is nonzero only when $\hat\Theta'$ and $\hat\Theta$ sit over the same element in $\mathcal{Z}_{\mathrm{ech},M}$. Moreover, there is precisely one nonzero $\mathrm{N}^{\mathbb{U}}_{\hat\Theta',\hat\Theta}$ such that both $\hat\Theta'$ and $\hat\Theta$ sit over any given element in $\mathcal{Z}_{\mathrm{ech},M}$, and the corresponding $\mathrm{N}^{\mathbb{U}}_{\hat\Theta',\hat\Theta}$ is equal to 1. A single J–holomorphic submanifold is used to compute this nonzero $\mathrm{N}^{\mathbb{U}}_{\hat\Theta',\hat\Theta}$: if $\Theta\in\mathcal{Z}_{\mathrm{ech},M}$ is the given element, then the corresponding submanifold is the union of the cylinders from the set $\{\mathbb{R}\times\gamma\}_{\gamma\in\Theta}$ and $\{0\}\times S\subset\mathbb{R}\times Y$ with S being the $u=$ constant sphere in $\mathcal{H}_0$ that contains the chosen point.

The endomorphisms of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{ech},M})$ that define a set of generators for the action of $\bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})$ on the embedded contact homology are defined with the help of a chosen set of 1–cycles that supply a basis for $H_1(Y;\mathbb{Z})/\mathrm{tors}$. Section III.1D took this set to have the form that is described in a moment. To set the background, introduce $b_1(M)$ to denote the first Betti number of M. Section II.2A describes $1+b_1(M)$ closed integral curves of v in $M_\delta\cup\mathcal{H}_0$ that have intersection number 1 with each cross-sectional sphere in $\mathcal{H}_0$. One of these curves is the aforementioned $\gamma^{(z_0)}$. The curves in this set are labeled by the intersection point with the surface $f^{-1}\bigl(\frac{3}{2}\bigr)$. This set of points is denoted by ¥ and the curve that contains a given $z\in ¥$ is denoted by $\gamma^{(z)}$. Pairing with the Poincaré duals of the homology classes of the cycles that constitute the set $\{[\gamma^{(z)}]-[\gamma^{(z_0)}]\}_{z\in ¥-z_0}$ generates the dual in $\mathrm{Hom}(H_2(Y;\mathbb{Z});\mathbb{Z})$ of the $H_2(M;\mathbb{Z})$ summand in (1-4).

The basis used in Section III.1D contains the cycle $[\gamma^{(z_0)}]$; in addition it contains a set of cycles that are labeled $\{\hat{\iota}^{(z)}\}_{z \in ¥ - z_0}$, and it is rounded out by a set of G cycles that are labeled $\{\hat{\iota}_{\mathfrak{p}}\}_{\mathfrak{p} \in \Lambda}$. A given $z \in ¥ - z_0$ version of $\hat{\iota}^{(z)}$ lies entirely in the $M_{7\delta_*}$ part of Y. It is homologous to $[\gamma^{(z)}] - [\gamma^{(z_0)}]$ and it is obtained from the latter by first truncating the $\mathcal{H}_0$ portions of the curves $\gamma^{(z)}$ and $\gamma^{(z_0)}$ and then reconnecting the respective endpoints by arcs on the boundary of the radius $7\delta_*$ coordinate balls about the index 0 and index 3 critical points of f. A given $\mathfrak{p} \in \Lambda$ version of $\hat{\iota}_{\mathfrak{p}}$ is disjoint from the $f \in [1, 2]$ part of $M_{7\delta_*}$, and it intersects the rest of $M_{7\delta_*}$ and $\mathcal{H}_0$ as a smooth curve that is transverse to the level sets of f in M_δ and the constant u spheres in $\mathcal{H}_0$; the orientation is such that it has intersection number 1 with the $u = 0$ sphere in $\mathcal{H}_0$. Meanwhile, $\hat{\iota}_{\mathfrak{p}}$ intersects $\bigcup_{\mathfrak{p}' \in \Lambda} \mathcal{H}_{\mathfrak{p}'}$ as the $\theta = 0$ arc in $\mathcal{H}_{\mathfrak{p}}$; its orientation gives it intersection number -1 with each $u = 0$ sphere in $\mathcal{H}_{\mathfrak{p}}$.

Suppose that $\hat{\iota}$ is a given cycle from the chosen basis of cycles. As noted above, the corresponding endomorphism of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{ech},M})$ has the form given in (1-9). Denote by $\{\mathrm{N}^{\hat{\iota}}_{\hat{\Theta}', \hat{\Theta}}\}_{\hat{\Theta}', \hat{\Theta} \in \hat{\mathcal{Z}}_{\mathrm{ech},M}}$ the set of coefficients. Any given $\mathrm{N}^{\hat{\iota}}_{\hat{\Theta}', \hat{\Theta}}$ is defined using the J–holomorphic submanifolds from $\mathcal{M}_1(\hat{\Theta}', \hat{\Theta})$. In particular, $\mathrm{N}^{\hat{\iota}}_{\hat{\Theta}', \hat{\Theta}}$ is the value of a sum that is indexed by the components of $\mathcal{M}_1(\hat{\Theta}', \hat{\Theta})$ whereby the component of a given submanifold C contributes either $+1$ or -1 times the algebraic intersection number between C and $\mathbb{R} \times \hat{\iota}$. This intersection number is well defined because $\hat{\iota}$ is disjoint from the integral curves of v that come from elements in $\mathcal{Z}_{\mathrm{ech},M}$. The $+1$ or -1 used here is the contribution of C's component to the version of $\mathrm{N}_{\hat{\Theta}', \hat{\Theta}}$ that defines the embedded contact homology differential.

Part 5 This last part of the subsection introduces a certain filtration of the embedded contact homology chain complex that is preserved by the differential. The filtration is depicted by (I.2-3). What follows reviews what is involved. To start, invoke Proposition II.2.8 or Theorem I.2.1 to write the set $\mathcal{Z}_{\mathrm{ech},M}$ as $\mathcal{Z}_{\mathrm{HF}} \times \big(\prod_{\mathfrak{p} \in \Lambda} (\mathbb{Z} \times \mathrm{O})\big)$. By way of a reminder, $\mathcal{Z}_{\mathrm{HF}}$ denotes a certain set that is defined using data coming strictly from M and O is the 4–element set $\{0, 1, -1, \{1, -1\}\}$. The $\mathcal{Z}_{\mathrm{HF}}$–label of any given element $\Theta \in \mathcal{Z}_{\mathrm{ech},M}$ characterizes the intersection of $\bigcup_{\gamma \in \Theta} \gamma$ with M_δ. Meanwhile, each $\mathfrak{p} \in \Lambda$ factor of $\mathbb{Z} \times \mathrm{O}$ characterizes the intersection of $\bigcup_{\gamma \in \Theta} \gamma$ with $\mathcal{H}_{\mathfrak{p}}$. The integer component of this label characterizes the segment of $\big(\bigcup_{\gamma \in \Theta} \gamma\big) \cap \mathcal{H}_{\mathfrak{p}}$ that crosses $\mathcal{H}_{\mathfrak{p}}$ from its $u = -R - \ln(7\delta_*)$ end to its $u = R + \ln(7\delta_*)$ end. The label from the set $\{0, 1, -1, \{1, -1\}\}$ signifies which, if any, integral curves from the set $\{\hat{\gamma}_{\mathfrak{p}}^+, \hat{\gamma}_{\mathfrak{p}}^-\}$ appear in Θ. The $+1$ signifies $\hat{\gamma}_{\mathfrak{p}}^+$ and the -1 signifies $\hat{\gamma}_{\mathfrak{p}}^-$. Use the identification of $\mathcal{Z}_{\mathrm{ech},M}$

with $\mathcal{Z}_{\mathrm{HF}} \times \big(\prod_{\mathfrak{p}\in\Lambda}(\mathbb{Z}\times\mathrm{O})\big)$ to write a given element Θ as $(\hat{\upsilon}, (\mathfrak{k}_{\mathfrak{p}}, \mathrm{O}_{\mathfrak{p}})_{\mathfrak{p}\in\Lambda})$. For each $\mathfrak{p}\in\Lambda$, use $|\mathrm{O}_{\mathfrak{p}}|\in\{0,1,2\}$ to denote the sum of the absolute values of the elements in $\mathrm{O}_{\mathfrak{p}}$.

Associate to each nonnegative integer L the subset $\mathcal{Z}^L_{\mathrm{ech},M} \subset \mathcal{Z}_{\mathrm{ech},M}$ whose elements are such that $\sum_{\mathfrak{p}\in\Lambda}(|\mathfrak{k}_{\mathfrak{p}}| + 2|\mathrm{O}_{\mathfrak{p}}|) < L$. These sets are such that $\mathcal{Z}^L_{\mathrm{ech},M} \subset \mathcal{Z}^{L'}_{\mathrm{ech},M}$ when $L' > L$, and their union is the whole of $\mathcal{Z}_{\mathrm{ech},M}$. Let $\hat{\mathcal{Z}}^L_{\mathrm{ech},M}$ denote the inverse image of $\mathcal{Z}^L_{\mathrm{ech},M}$ in $\hat{\mathcal{Z}}_{\mathrm{ech},M}$. It follows from Theorem I.2.3 or Theorem III.1.1 that the embedded contact homology differential maps the submodule $\mathbb{Z}(\hat{\mathcal{Z}}^L_{\mathrm{ech},M}) \subset \mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{ech},M})$ to itself and so the latter defines a subcomplex. The embedded contact homology is the direct limit of the homology for the filtered sequence of chain subcomplexes

$$\cdots \subset \mathbb{Z}(\hat{\mathcal{Z}}^L_{\mathrm{ech},M}) \subset \mathbb{Z}(\hat{\mathcal{Z}}^{L+1}_{\mathrm{ech},M}) \subset \cdots \tag{1-10}$$

of the chain complex $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{ech},M})$.

1.3 The Seiberg–Witten Floer homology on *Y*

This subsection describes various versions of the Seiberg–Witten Floer homology on the manifold Y. The presentation that follows takes for granted the basic constructions and theorems about Seiberg–Witten Floer homology and focuses almost exclusively on those parts of the story that are specific to the geometry at hand. The book by Kronheimer and Mrowka [7] is the recommended textbook for those who are not familiar with the foundational background. There are nine parts to what follows.

Parts 1–5 introduce various geometric notions that are used in Part 6 to define the chain complex and differential whose homology groups constitute the desired versions of Seiberg–Witten Floer homology. These groups are introduced in Part 8. The intervening Part 7 describes certain canonical endomorphisms of the chain complex that are used to generate an action of $\mathbb{Z}[\mathbb{U}] \otimes \big(\bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})\big)$ on the homology. Part 9 explains why Theorems I.3.1 and I.3.2 are direct consequences of what is said in Parts 6–8.

Part 1 Part 5 in Section 1.1 defined a Riemannian metric on Y, this being a metric with $*w = \hat{a}$ and $|\hat{a}| = 1$. Use this metric to define the bundle of oriented, orthonormal frames for Y. The given Spin^c–structure on Y determines a lift of this bundle to a principal $U(1)$–bundle. The defining action of $U(2)$ on $\mathbb{C}^2$ supplies an associated Hermitian $\mathbb{C}^2$–bundle. The latter is denoted by $\mathbb{S}$. Use $\det(\mathbb{S})$ to denote the complex line bundle $\bigwedge^2\mathbb{S}$.

There is a canonical homomorphism from T^*Y into $\mathrm{End}(\mathbb{S})$, this being the Clifford multiplication. The homomorphism is denoted by cl; it is characterized as follows: Let

a and b denote a pair of covectors in a given fiber of T^*Y. Then

$$\mathrm{cl}(a)^\dagger = -\mathrm{cl}(a) \quad \text{and} \quad \mathrm{cl}(a)\,\mathrm{cl}(b) = -\langle a, b\rangle - \mathrm{cl}(*(a \wedge b)), \tag{1-11}$$

where $\langle\cdot,\cdot\rangle$ here denotes the metric inner product and $*$ denotes the metric's Hodge dual. This Clifford multiplication map induces two other useful endomorphisms. The first, denoted by $\hat{c}$, maps $\mathbb{S} \otimes T^*Y$ to $\mathbb{S}$. It is defined so as to send a reducible element $\psi \otimes a$ to $\mathrm{cl}(a)\psi$. The second is the $\mathbb{R}$–linear homomorphism from $\mathbb{S}\otimes\mathbb{S}$ to $T^*Y \otimes_{\mathbb{R}} \mathbb{C}$ that is written as $\eta \otimes \psi \mapsto \eta^\dagger \tau \psi$ and defined by the rule whereby $\langle a, \eta^\dagger \tau\psi\rangle = \eta^\dagger\,\mathrm{cl}(a)\psi$.

Clifford multiplication by $\hat{a}$ splits $\mathbb{S}$ as a direct sum of complex line bundles, this written as

$$\mathbb{S} = E \oplus (E \otimes K^{-1}). \tag{1-12}$$

Here, K^{-1} is isomorphic as a real 2–plane bundle to the oriented bundle $\mathrm{Ker}(\hat{a})$, this being the kernel of $\hat{a}$ with the orientation defined by w. The convention in what follows takes the left-most summand as the $+i$ eigenspace of $\mathrm{cl}(\hat{a})$. The corresponding component decomposition of a given section of $\mathbb{S}$ is written as (α, β) with α being a section of E and β being a section of $E \otimes K^{-1}$.

A unitary connection on $\det(\mathbb{S})$ with the Levi-Civita connection on TY jointly define a unitary connection on $\mathbb{S}$ and thus a covariant derivative, this being a map from $C^\infty(Y;\mathbb{S})$ to $C^\infty(Y;\mathbb{S}\otimes T^*Y)$. Meanwhile, the Clifford multiplication endomorphism defines the endomorphism $\widehat{\mathrm{cl}}\colon \mathbb{S}\otimes T^*Y \to \mathbb{S}$. The composition of the covariant derivative and $\widehat{\mathrm{cl}}$ then defines a first-order, elliptic operator from $C^\infty(Y;\mathbb{S})$ to itself, this being the Dirac operator.

The Dirac operator is used in a moment to define a canonical connection on K^{-1}. To do this, introduce $I_{\mathbb{C}}$ to denote a topologically trivial complex line bundle over Y, and let $\mathbb{S}_I$ denote the Spin^c structure given by the $E = I_{\mathbb{C}}$ version of (1-12). Fix a unit norm section, $1_{\mathbb{C}}$, of $I_{\mathbb{C}}$ and view the pair $(1_{\mathbb{C}}, 0)$ as a section of $\mathbb{S}$ using the splitting in (1-12). Since $\det(\mathbb{S}_I) = K^{-1}$, a unitary connection on K^{-1} defines an associated Dirac operator. The canonical connection on K^{-1} is characterized by the fact that $(1_{\mathbb{C}}, 0)$ is annihilated by its associated Dirac operator. This connection on K^{-1} is denoted by A_K.

Let $\mathbb{S}$ be the $\mathbb{C}^2$–bundle that comes from the given Spin^c–structure. With (1-12) understood, $\det(\mathbb{S}) = E^2K^{-1}$ and thus any given unitary connection on $\det(\mathbb{S})$ can be written as $A_K + 2A$, where A is a connection on E. With A a given connection on E, the symbol D_A is used in what follows to denote the Dirac operator on sections of $\mathbb{S}$ that is defined using $A_K + 2A$ for the connection on $\det(\mathbb{S})$. Use $\mathrm{Conn}(E)$ to denote the Fréchet space of smooth, unitary connections on E.

The Fréchet Lie group $C^\infty(Y; S^1)$ acts smoothly on $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ by the rule whereby any given map u sends any given pair (A, ψ) to $(A - u^{-1}du, u\psi)$.

Part 2 The $\mathbb{Z}$–module that serves as the chain complex for the Seiberg–Witten Floer homology is constructed using solutions to certain versions of the Seiberg–Witten equations. These are equations for pairs (A, ψ) with ψ a section of $\mathbb{S}$ and $A \in \mathrm{Conn}(E)$. The simplest of the relevant versions constitutes a family of equations whose members are labeled by a real number greater than π and a smooth 1–form on Y. The version defined by a given $\mathrm{r} \in [1, \infty)$ and 1–form μ asks that (A, ψ) obey

$$\begin{cases} B_A - \mathrm{r}(\psi^\dagger \tau \psi - i\hat{a}) + \frac{1}{2} B_{A_K} - i *d\mu = 0, \\ D_A \psi = 0, \end{cases} \tag{1-13}$$

where the notation is such that $B_A = *F_A$ with F_A being the curvature 2–form of the connection A. Likewise, B_{A_K} denotes the Hodge dual of the curvature 2–form of A_K.

Certain perturbed versions of (1-13) are needed to guarantee that the solutions to the associated equation and its instanton counterpart are suitably generic. A given perturbed version of (1-13) is defined using a chosen element in a certain Banach space of $C^\infty(Y; S^1)$–invariant functions on $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ (see Chapter 11 in [7]). This Banach space is denoted by $\mathcal{P}$ and its norm is called the $\mathcal{P}$–norm. The Banach space is such that the differential of a given $\mathfrak{g} \in \mathcal{P}$ is a smooth map from $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ to $C^\infty(Y; iT^*Y) \oplus C^\infty(Y; \mathbb{S})$. With $\mathfrak{g}$ chosen, write its differential at a given (A, ψ) as $(\mathfrak{T}_{(A,\psi)}, \mathfrak{S}_{(A,\psi)})$, this being a pair consisting of an $i\mathbb{R}$–valued 1–form on Y and a section of $\mathbb{S}$. The 1–form $\mathfrak{T}_{(A,\psi)}$ is in the image of the operator $*d$. The $\mathfrak{g}$–perturbed version of the Seiberg–Witten equations are

$$\begin{cases} B_A - \mathrm{r}(\psi^\dagger \tau \psi - i\hat{a}) + \frac{1}{2} B_{A_K} - \mathfrak{T}_{(A,\psi)} = 0, \\ D_A \psi - \mathfrak{S}_{(A,\psi)} = 0. \end{cases} \tag{1-14}$$

The simplest but nontrivial perturbation has $\mathfrak{T} = i *d\mu$ and $\mathfrak{S} = 0$ with μ a smooth 1–form on Y taken from a certain Banach space of such forms; this perturbation gives the equation in (1-13). The Banach space is denoted by Ω. The norm on this space is also called the $\mathcal{P}$–norm. The latter is such that the inclusion $\Omega \to C^\infty(Y; iT^*Y)$ defines a bounded, compact mapping. The convention in this paper is to use 1–forms

μ from Ω with $\mathcal{P}$–norm less than 1. All of the assertions hold (with the same proofs) given any a priori upper bound on the $\mathcal{P}$–norm.

When $\mu \in \Omega$, then the corresponding version of $\mathfrak{g}$ is denoted by $\mathfrak{e}_\mu$; it is the function that assigns to any given (A, ψ) the integral over Y of the 3–form $-iF_A \wedge \mu$.

If (A, ψ) is a solution to (1-14), then so is $(A - u^{-1}du, u\psi)$ for any $u \in C^\infty(Y; S^1)$. Use $\mathcal{Z}_{\mathrm{SW,r}}$ in what follows to denote the $C^\infty(Y; S^1)$–quotient of the space of solutions to a given $\mathfrak{g} \in \mathcal{P}$ version of (1-14). Note in this regard that the group $C^\infty(Y; S^1)$ acts freely on the space of solutions to any given $\mathrm{r} > \pi$ and $\mathfrak{g} \in \mathcal{P}$ version of (1-14). This is so because the group acts freely on $\mathrm{Conn}(E) \times (C^\infty(Y; S^1) - 0)$ and no $\psi = 0$ pair can solve (1-14) because both $\frac{i}{2\pi}\big(F_A + \frac{1}{2} * B_{A_K}\big)$ and w represent the first Chern class of $\det(\mathbb{S})$. Note also that $\mathcal{Z}_{\mathrm{SW,r}}$ is in all cases compact (see Chapter 29 in [7]). By way of a warning, the notation does not indicate that $\mathcal{Z}_{\mathrm{SW,r}}$ depends on the chosen perturbation $\mathfrak{g}$.

Part 3 The definition of the $\mathbb{Z}$–module for the Seiberg–Witten Floer homology requires the introduction of a subgroup of $C^\infty(Y; S^1)$ which is denoted by $\mathcal{G}_{M_\Lambda}$. A given map u sits in this subgroup when $-\frac{i}{2\pi}\int_{\gamma^{(z_0)}} u^{-1}\,du = 0$. Note in this regard that $-\frac{i}{2\pi}u^{-1}\,du$ has integer periods.

Use $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ to denote the space of $\mathcal{G}_{M_\Lambda}$–orbits of solutions to a given $\mathrm{r} \in [1, \infty)$ and $\mathfrak{g} \in \mathcal{P}$ version of (1-14). The space is a principal $\mathbb{Z} = H_2(\mathcal{H}_0; \mathbb{Z})$–bundle over $\mathcal{Z}_{\mathrm{SW,r}}$. The action of $k \in \mathbb{Z}$ sends the $\mathcal{G}_{M_\Lambda}$–equivalence class of (A, ψ) to that of $(A - u^{-1}du, u\psi)$ with $u \in C^\infty(Y; S^1)$ any map with $-\frac{i}{2\pi}\int_{\gamma^{(z_0)}} u^{-1}\,du = k$.

The geometry of Y supplies certain $\mathbb{Z}$–equivariant maps from $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ to $\mathbb{R}$. The definition requires the choice of a fiducial connection on E, this denoted by A_E. This choice is constrained by the requirement that A_E be flat on $\mathcal{H}_0$ and have holonomy 1 around the curve $\gamma^{(z_0)}$. The definition of the $\mathbb{Z}$–equivariant map from $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ to $\mathbb{R}$ also requires the choice of a smooth function $\wp\colon [0, \infty) \to [0, \infty)$ which is nondecreasing, obeys $\wp(x) = 0$ for $x < \frac{7}{16}$ and $\wp(x) = 1$ for $x \geq \frac{9}{16}$. It proves convenient to choose $\wp$ so that its derivative, $\wp'$, is bounded by $2^{10}(1-\wp)^{3/4}$. A function with these properties can be readily constructed from the function on $\mathbb{R}$ that is set equal to 0 for $t < 0$ and set equal to $e^{-1/t}$ for $t > 0$. Fix such a function $\wp$.

Given $\mathfrak{c} = (A, \psi)$ from $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$, write $\psi = (\alpha, \beta)$ and define

(1-15) $$\hat{A} = A - \tfrac{1}{2}\wp(|\alpha|^2)|\alpha|^{-2}(\bar{\alpha}\nabla_A\alpha - \alpha\nabla_A\bar{\alpha}),$$

this being a connection on E. The corresponding equivariant map to $\mathbb{R}$ is the map from $(\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S}))/\mathcal{G}_{M_\Lambda}$ obtained from the $\mathcal{G}_{M_\Lambda}$–invariant map from $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ to $\mathbb{R}$ that sends any given pair $\mathfrak{c} = (A, \psi)$ to

$$\mathrm{x}(\mathfrak{c}) = \frac{i}{2\pi} \int_{\gamma^{(z_0)}} (\hat{A} - A_E). \tag{1-16}$$

A given element $\mathfrak{c} \in \mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ is deemed to be *holonomy nondegenerate* when $\mathrm{x}(\mathfrak{c}) - \frac{1}{2}$ is not an integer. The locus where $\mathrm{x}(\mathfrak{c}) - \frac{1}{2} \in \mathbb{Z}$ is a codimension 1 submanifold in $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$. The element $\mathfrak{c}$ is holonomy nondegenerate if and only if all pairs in its $C^\infty(Y;S^1)$ orbit are holonomy nondegenerate.

Part 4 A certain symmetric, first-order elliptic operator on $C^\infty(Y; iT^*Y \oplus \mathbb{S} \oplus i\mathbb{R})$ is associated to each pair in $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$. Fix $\mathfrak{c} = (A, \psi) \in \mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$. The corresponding operator in the case when $\mathfrak{g} = \mathfrak{e}_\mu$ sends a section $\mathfrak{b} = (b, \eta, \phi)$ to the section with respective iT^*Y, $\mathbb{S}$ and $i\mathbb{R}$ components

$$\begin{cases} *db - d\phi - 2^{-1/2}\mathrm{r}^{1/2}(\psi^\dagger \tau \eta + \eta^\dagger \tau \psi), \\ D_A \eta + 2^{1/2}\mathrm{r}^{1/2}(\mathrm{cl}(b)\psi + \phi\psi), \\ *d *b - 2^{-1/2}\mathrm{r}^{1/2}(\eta^\dagger \psi - \psi^\dagger \eta). \end{cases} \tag{1-17}$$

If $\mathfrak{g}$ is generic, the operator is obtained from (1-17) by adding $(2/\mathrm{r})^{1/2}\big(\frac{d}{d\tau}\mathfrak{T}_{\mathfrak{c}+\tau\hat{\mathfrak{b}}}\big)_{\tau=0}$ to the top term and $\big(\frac{d}{d\tau}\mathfrak{S}_{\mathfrak{c}+\tau\hat{\mathfrak{b}}}\big)_{\tau=0}$ to the middle term with $\hat{\mathfrak{b}} = ((2/\mathrm{r})^{1/2}b, \eta)$. The operator in all cases is denoted by $\mathfrak{L}_{\mathfrak{c},\mathrm{r}}$. The pair $\mathfrak{c}$ is said to be nondegenerate when $\mathfrak{L}_{\mathfrak{c},\mathrm{r}}$ has trivial kernel. A given pair $\mathfrak{c}$ is nondegenerate if and only if all pairs in its $C^\infty(Y;S^1)$ orbit are likewise nondegenerate.

The following statements are analogs of what is asserted in Lemma 3.6 and Proposition 3.11 of [17] for the case when $\hat{a}$ is a contact 1–form and $w = d\hat{a}$. The proofs differ only slightly from those given in [17].

(1-18)
- Fix $\mathrm{r} \geq 1$. Then there is an open, dense set in Ω which is characterized as follows: if μ is from this set, then the corresponding $\mathfrak{g} = \mathfrak{e}_\mu$ version of $\mathcal{Z}_{\mathrm{SW},\mathrm{r}}$ is a finite set of orbits of pairs in $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ and each such pair is nondegenerate and holonomy nondegenerate.
- There exists a residual set in Ω that is characterized as follows: Fix μ from this set. There is a countable, nonaccumulating set in (π, ∞) such that if r is from its complement, then the corresponding $(\mathrm{r}, \mathfrak{g} = \mathfrak{e}_\mu)$ version of $\mathcal{Z}_{\mathrm{SW},\mathrm{r}}$ is a finite set of $C^\infty(Y;S^1)$ orbits in $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ and each such orbit contains only nondegenerate and holonomy nondegenerate pairs.

Suppose now that $(\mathrm{r}, \mathfrak{g})$ is such that the corresponding $\mathcal{Z}_{\mathrm{SW,r}}$ is a finite set of orbits and that each orbit contains only nondegenerate and holonomy nondegenerate pairs. The principal bundle in this case has a canonical $\mathbb{Z}$–equivariant isomorphism

$$\hat{\mathcal{Z}}_{\mathrm{SW,r}} = \mathcal{Z}_{\mathrm{SW,r}} \times \mathbb{Z} \tag{1-19}$$

that is characterized as follows: the section $\mathcal{Z}_{\mathrm{SW,r}} \times \{0\}$ of the product bundle corresponds to the set of the $\mathcal{G}_{M_\Lambda}$–orbits of solutions to (1-14) with $\mathrm{x}(\cdot) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$. Granted this identification, use $\hat{\mathcal{Z}}^{\geq}_{\mathrm{SW,r}} \subset \hat{\mathcal{Z}}_{\mathrm{SW,r}}$ to denote the subset that is identified via (1-19) to $\mathcal{Z}_{\mathrm{SW,r}} \times \{0, 1, 2, \dots\} \subset \mathcal{Z}_{\mathrm{SW,r}} \times \mathbb{Z}$.

Part 5 Certain versions of the Seiberg–Witten equations on $\mathbb{R} \times Y$ play a central role in the story. As in the case of (1-14), the equations require the choice of $\mathrm{r} \geq 1$ and a perturbation $\mathfrak{g}$ from $\mathcal{P}$. The corresponding equations is viewed here as a system of differential equations for a map from $\mathbb{R}$ to $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$. The equations demand that $s \mapsto \mathfrak{d}(s) = (A, \psi)|_s$ obey

$$\begin{cases} \frac{\partial}{\partial s} A + B_A - \mathrm{r}(\psi^\dagger \tau \psi - i\hat{a}) + \frac{1}{2} B_{A_K} - \mathfrak{T}_{(A,\psi)} = 0, \\ \frac{\partial}{\partial s}\psi + D_A\psi - \mathfrak{S}_{(A,\psi)} = 0. \end{cases} \tag{1-20}$$

A solution to (1-19) is said to be an *instanton* if it has respective $s \to \pm\infty$ limits that obey (1-14). Any constant $\mathbb{R}$ translate of an instanton solution to (1-20) is also an instanton solution.

An instanton is said to be *nondegenerate* if a certain operator is Fredholm and has trivial cokernel. The relevant operator maps an L^2_1–Hilbert space completion of the space of compactly supported elements in $C^\infty(\mathbb{R} \times Y; iT^*Y \oplus \mathbb{S} \oplus i\mathbb{R})$ to an L^2 completion. This operator in the case $\mathfrak{g} = \mathfrak{e}_\mu$ sends a section (b, η, ϕ) to the section whose respective iT^*Y, $\mathbb{S}$ and $i\mathbb{R}$ components are

$$\begin{cases} \frac{\partial}{\partial s} b + {*db} - d\phi - 2^{-1/2}\mathrm{r}^{1/2}(\psi^\dagger\tau\eta + \eta^\dagger\tau\psi), \\ \frac{\partial}{\partial s}\eta + D_A\eta + 2^{1/2}\mathrm{r}^{1/2}(\mathrm{cl}(b)\psi + \phi\psi), \\ \frac{\partial}{\partial s}\phi + {*d*b} - 2^{-1/2}\mathrm{r}^{1/2}(\eta^\dagger\psi - \psi^\dagger\eta). \end{cases} \tag{1-21}$$

If $\mathfrak{g}$ is generic, the operator is obtained from (1-21) by adding $(2/\mathrm{r})^{1/2}\left(\frac{\partial}{\partial\tau}\mathfrak{T}_{\mathfrak{d}+\tau\hat{\mathfrak{b}}}\right)_{\tau=0}$ to the top term and $\left(\frac{\partial}{\partial\tau}\mathfrak{S}_{\mathfrak{d}+\tau\hat{\mathfrak{b}}}\right)_{\tau=0}$ to the middle term. The operator in any case is denoted by $\mathfrak{D}_{\mathfrak{d}}$.

Let $\mathfrak{d}\colon \mathbb{R} \to \mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ denote an instanton solution to some $(\mathrm{r}, \mathfrak{g})$ version of (1-20). The corresponding version of (1-21) is a Fredholm operator if and only if the $s \to \pm\infty$ limits of $\mathfrak{d}$ are nondegenerate pairs in $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$. If this is the case, then $\mathfrak{D}_\mathfrak{d}$ has a corresponding Fredholm index, this denoted by $\iota_\mathfrak{d}$.

The assertion made by the upcoming (1-22) is used when considering perturbations. This fact has an almost verbatim analog stated in Part 5 from Section 3b of [19] for the case when $\hat{a}$ is a contact 1–form on a given 3–manifold. The argument that proves (1-22) differs only cosmetically from that given for their [19] analog.

To set the notation for (1-22), suppose that $\mathfrak{p} \in \mathcal{P}$ and that $\mathfrak{c}$ is a given pair from $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$. Then $\mathfrak{p}$ is said to vanishes to second order at $\mathfrak{c}$ if $\mathfrak{p}(\mathfrak{c}) = 0$, and if both the first and second derivatives at $\tau = 0$ of the function $\tau \mapsto \mathfrak{p}(\mathfrak{c} + \tau\mathfrak{b})$ are zero for all pairs $\mathfrak{b} \in C^\infty(Y; iT^*Y \oplus \mathbb{S})$. With this term understood, introduce $\mathcal{P}_\mu \subset \mathcal{P}$ to denote the subset whose members have the following property: if $\mathfrak{p} \in \mathcal{P}_\mu$, then $\mathfrak{p} = 0$ to second order on any solution to (1-14) and $\mathfrak{p} = 0$ to second order at all points on any path $s \mapsto \mathfrak{d}(s)$ with $\mathfrak{d}$ an $\iota_\mathfrak{d} \leq 2$, nondegenerate instanton solution to the $(\mathrm{r}, \mathfrak{g} = \mathfrak{e}_\mu)$ version of (1-20).

(1-22) Fix $\mathrm{r} \geq 1$ and μ such that all solutions to the corresponding $\mathfrak{g} = \mathfrak{e}_\mu$ version of (1-14) are nondegenerate. There is a residual set in $\mathcal{P}_\mu$ characterized as follows: if $\mathfrak{p}$ is a member, then all instanton solutions to the $(\mathrm{r}, \mathfrak{g} = \mathfrak{e}_\mu + \mathfrak{p})$ version of (1-20) are nondegenerate.

Suppose that $(\mathrm{r}, \mathfrak{g})$ is such that all solutions to (1-14) are nondegenerate and such that all $\iota_\mathfrak{d} \leq 2$ instanton solutions to (1-20) are nondegenerate. If this is the case, then the set of $\iota_\mathfrak{d} = 0$ instanton solutions is the set of constant maps from $\mathbb{R}$ to the set of solutions to (1-14). To say something about the set of $\iota_\mathfrak{d} = 1$ instanton solutions to (1-20), suppose that $\mathfrak{c}_-$ and $\mathfrak{c}_+$ are given solutions to (1-14). Introduce $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$ to denote the set of instanton solutions to (1-20) with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit a pair on the orbit of $\mathfrak{c}_+$. This set $\mathcal{M}_1(\mathfrak{c}_-, \mathfrak{c}_+)$ has the structure of a smooth 1–dimensional manifold with a finite set of components, each being a copy of $\mathbb{R}$. Moreover, the group of constant translations of $\mathbb{R}$ induces a smooth, free $\mathbb{R}$ action on each component.

Part 6 This part defines chain complexes whose corresponding homology groups are of central concern here. To this end, fix $\mathrm{r} \geq 1$ and fix $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1 and such that all solutions to the $(\mathrm{r}, \mathfrak{e}_\mu)$ version of (1-14) are nondegenerate and also holonomy nondegenerate. A $\mathbb{Z}$–module that serves for the chain complex of interest is the free module generated by the elements in the $(\mathrm{r}, \mathfrak{e}_\mu)$ version of $\widehat{\mathcal{Z}}_{\mathrm{SW,r}}$.

This module is denoted in what follows by $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$. The action of the generator of $H_2(\mathcal{H}_0;\mathbb{Z})$ gives this module a $\mathbb{Z}[t,t^{-1}]$ structure.

The $\mathbb{Z}$–module $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ has a relative $\mathbb{Z}/p_M\mathbb{Z}$–grading which is defined as follows: Let $\mathfrak{c}_0$ and $\mathfrak{c}_1$ denote two solutions to the relevant version of (1-14). Introduce $\mathrm{gr}_{\mathrm{SW}}(\mathfrak{c}_0)-\mathrm{gr}_{\mathrm{SW}}(\mathfrak{c}_1)$ to denote the difference between the grading degrees of their respective $\mathcal{G}_{M_\Lambda}$–orbits in $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$. This number is -1 times the spectral flow for the $[0,1]$–parametrized family of operators $\tau\mapsto\mathfrak{L}_{\mathfrak{c}(\tau)}$ with $\mathfrak{c}(0)=\mathfrak{c}_0$ and $\mathfrak{c}(1)=\mathfrak{c}_1$. Those unfamiliar with the notion of spectral flow can read about it in Chapter 14.2 of [7] or in [16]. The generator of the $\mathbb{Z}[t,t^{-1}]$ action on $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ acts as a degree -2 endomorphism.

The relevant differential is a certain square-zero endomorphism of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$. This endomorphism is defined by its action on the generators. To say more, introduce by way of notation $[\mathfrak{c}]$ to denote the $\mathcal{G}_{M_\Lambda}$–equivalence class of a given pair $\mathfrak{c}=(A,\psi)$ from $\mathrm{Conn}(E)\times C^\infty(Y;\mathbb{S})$. Any given endomorphism of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ is defined by a rule

$$[\mathfrak{c}]\mapsto \mathrm{w}_{[\mathfrak{c}],[\mathfrak{c}']}[\mathfrak{c}'] \tag{1-23}$$

where each $[\mathfrak{c}']$ version of $\mathrm{w}_{[\mathfrak{c}'],[\mathfrak{c}]}$ is an integer and only finitely many are nonzero.

In the case of the differential, the specification of the coefficient set $\{\mathrm{w}_{[\mathfrak{c}'],[\mathfrak{c}]}\}$ requires first the choice of an element $\mathfrak{p}\in\mathcal{P}$ with norm much less than 1 such that the conclusions of (1-22) hold. Any given $[\mathfrak{c}],[\mathfrak{c}']\in\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ version of $\mathrm{w}_{[\mathfrak{c}'],[\mathfrak{c}]}$ is a sum that is indexed by the components of the $(\mathrm{r},\mathfrak{g}=\mathfrak{e}_\mu+\mathfrak{p})$ version of $\mathcal{M}_1(\mathfrak{c}',\mathfrak{c})$ with each component contributing either $+1$ or -1 to the sum. The sign is obtained by comparing two orientations for the component, one given by the generator of the $\mathbb{R}$ action and the other using Quillen's notion of a determinant line bundle for a family of Fredholm operators. This is done according to the rules given in Chapters 20–22 of [7]; see also Section 3 of [21].

Proposition 1.1 *There exists $\kappa\ge 1$ with the following significance: Fix $\mathrm{r}\ge\kappa$ and an element $\mu\in\Omega$ with $\mathcal{P}$–norm less than 1 such that all solutions to the (r,μ) version of (1-13) are nondegenerate. Suppose that $\mathfrak{p}\in\mathcal{P}_\mu$ has small $\mathcal{P}$–norm and is described by (1-22).*

- *The rules given in [7] for specifying the various $[\mathfrak{c}],[\mathfrak{c}']\in\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ versions of $\mathrm{w}_{[\mathfrak{c}],[\mathfrak{c}']}$ define a square-zero endomorphism of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$.*

- *Each solution to* (1-13) *is holonomy nondegenerate and so* $\mathbb{Z}(\hat{\mathcal{Z}}^{\geq}_{\mathrm{SW,r}})$ *is well defined.*
- *The endomorphism given by the first bullet maps the submodule* $\mathbb{Z}(\hat{\mathcal{Z}}^{\geq}_{\mathrm{SW,r}})$ *to itself.*

This proposition is proved in Section 7.1. Take it on faith for now and use $\partial_{\mathrm{SW},Y}$ to denote the resulting endomorphism of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$. It follows from the definition that $\partial_{\mathrm{SW},Y}$ decreases the $\mathbb{Z}/p_M\mathbb{Z}$–grading by 1.

Part 7 This part of the subsection describes endomorphisms of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ that generate an action of $\mathbb{Z}[\mathbb{U}] \otimes \left(\bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})\right)$ on the $\partial_{\mathrm{SW},Y}$ homology. Each such endomorphism is defined by the coefficients that appear in the relevant version of (1-23).

Consider first the endomorphism that generates the $\mathbb{Z}[\mathbb{U}]$ factor. Fix $[\mathfrak{c}]$ and $[\mathfrak{c}']$ so as to specify the corresponding version of $\mathrm{w}_{[\mathfrak{c}],[\mathfrak{c}']}$. The specification of these coefficients requires the choice of a point $p \in \mathcal{H}_0$. Reintroduce $\mathfrak{p} \in \mathcal{P}_\mu$ from Part 6 and use $\mathcal{M}_2(\mathfrak{c},\mathfrak{c}')$ to denote the set of instanton solutions to the r and $\mathfrak{g} = \mathfrak{e}_\mu + \mathfrak{p}$ version of (1-20) with the corresponding Fredholm index $\iota_{(\cdot)}$ equal to 2, with $s \to -\infty$ limit equal to $\mathfrak{c}$ and with $s \to \infty$ limit in the $\mathcal{G}_{M_\Lambda}$–orbit of $\mathfrak{c}'$. Use $\mathcal{M}_{2,p}(\mathfrak{c},\mathfrak{c}')$ to denote the subset of $\mathcal{M}_2(\mathfrak{c},\mathfrak{c}')$ that is characterized as follows: a given instanton $\mathfrak{d} = (A, \psi = (\alpha,\beta))$ is a member if and only if $\alpha|_{s=0}$ vanishes at p. The upcoming Proposition 1.2 asserts in part that $\mathcal{M}_{2,p}(\mathfrak{c},\mathfrak{c}')$ is a finite set if r is large. Granted that this is so, the coefficient $\mathrm{w}_{[\mathfrak{c}],[\mathfrak{c}']}$ is given as a sum that is indexed by the instantons from $\mathcal{M}_{2,p}(\mathfrak{c},\mathfrak{c}')$. The contribution from each such instanton is specified using the rules in Chapter 23 of [7]. Parts 3 and 4 of Section 1b in [23] describe these same rules in the case when $\hat{a}$ is replaced by a contact 1–form and w is replaced by the latter's exterior derivative.

The specification of the various endomorphisms that are meant to generate the action of $\bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})$ on the ∂_{SW} homology requires the reintroduction of the set of 1–cycles $\{[\gamma^{(z)}]\}_{z\in¥}$, $\{\hat{\iota}_\mathfrak{p}\}_{\mathfrak{p}\in\Lambda}$ from Part 4 of Section 1.2. Each cycle from this set labels a corresponding endomorphism. Let $\hat{\iota}$ denote such a cycle. Use $\mathrm{w}^{\hat{\iota}}_{[\mathfrak{c}],[\mathfrak{c}']}$ to denote any given $[\mathfrak{c}],[\mathfrak{c}']$ coefficient in $\hat{\iota}$'s version of (1-23). This coefficient is a weighted sum of intersection numbers that are defined using the elements in $\mathcal{M}_1(\mathfrak{c},\mathfrak{c}')$ whereby the contribution of a given instanton $(A, \psi = (\alpha,\beta))$ to the sum is either $+1$ or -1 times the algebraic intersection number between $\alpha^{-1}(0)$ and the locus $\mathbb{R}\times\hat{\iota}$ in $\mathbb{R}\times Y$. The rules for assigning a $+1$ or -1 weight to the intersection number are laid out in Chapter 23 of [7]. Part 3 of Section 1b in [23] describe these same rules in the case when $\hat{a}$ is replaced by a contact 1–form and w is replaced by the latter's exterior derivative.

Proposition 1.2 *There exists $\kappa \geq 1$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 such that all solutions to the (r, μ) version of (1-13) are nondegenerate and holonomy nondegenerate. If $\mathfrak{p} \in \mathcal{P}$ has small $\mathcal{P}$–norm and is described by (1-22), then the rules given in [7] for specifying the coefficients for the just-described endomorphisms of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ define an action of $\mathbb{Z}[\mathbb{U}] \otimes \big(\bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})\big)$ on the ∂_{SW} homology. The generator of the action of the $\mathbb{Z}[\mathbb{U}]$ factor decreases the relative grading by 2 and those that generate the action of $H_1(Y;\mathbb{Z})/\mathrm{tors}$ decrease the relative grading by 1. In addition, all of these endomorphisms map the submodule $\mathbb{Z}(\hat{\mathcal{Z}}^{\geq}_{\mathrm{SW,r}})$ to itself.*

This proposition is also proved in Section 7.1.

Part 8 The formal adjoint of ∂_{SW} on the $\mathbb{Z}$–module $\mathrm{Hom}(\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}}), \mathbb{Z})$ defines the differential for what is formally a version of Seiberg–Witten Floer cohomology. This formal adjoint of ∂_{SW} is denoted by ∂^*_{SW}. The endomorphism ∂^*_{SW} sends a given basis element $[\mathfrak{c}]$ in $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ to

$$\partial^*_{\mathrm{SW}}[\mathfrak{c}] = \sum_{[\mathfrak{c}'] \in \hat{\mathcal{Z}}_{\mathrm{SW,r}}} \mathrm{w}_{[\mathfrak{c}'],[\mathfrak{c}]}[\mathfrak{c}']. \tag{1-24}$$

This endomorphism increases the relative $\mathbb{Z}/p_M\mathbb{Z}$–grading by 1 and has square zero. The resulting ∂^*_{SW} homology groups enjoy an action of $\mathbb{Z}[\mathbb{U}] \otimes \big(\bigwedge^*(H_1(M;\mathbb{Z})/\mathrm{tors})\big)$ with the generator of $\mathbb{Z}[\mathbb{U}]$ now increasing the grading by 2 and the generators of $H_1(M;\mathbb{Z})/\mathrm{tors}$ increasing the grading by 1. The generators of this action come from the adjoints of the endomorphisms that are defined in Part 7.

Proposition 1.3 *There exists $\kappa \geq 1$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 such that all solutions to the (r, μ) version of (1-13) are nondegenerate. Suppose that $\mathfrak{p} \in \mathcal{P}_\mu$ has small $\mathcal{P}$–norm and is described by (1-22). Then the expression on the right-hand side in (1-24) defines ∂^*_{SW} as a square-zero endomorphism of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$. The adjoints of the endomorphism of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ that are defined in Part 7 likewise map $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ to itself and so define an action of $\mathbb{Z}[\mathbb{U}] \otimes \bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})$ on the homology groups of ∂^*_{SW}.*

Proposition 1.3 is likewise proved in Section 7.1.

Let $\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}} \subset \hat{\mathcal{Z}}_{\mathrm{SW,r}}$ denote the subset that corresponds via the identification in (1-19) to $\mathcal{Z}_{\mathrm{SW,r}} \times \{\dots, -2, -1\}$. The endomorphism ∂^*_{SW} preserves the submodule $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ as do those that give the generators of the $\mathbb{Z}[\mathbb{U}] \otimes \bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})$ action.

Granted what was just said, introduce $H^{\infty}_{\mathrm{SW,r}}$, $H^{-}_{\mathrm{SW,r}}$ and $H^{+}_{\mathrm{SW,r}}$ to denote the respective ∂^*_{SW} homology on the chain complexes $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$, $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ and $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})/\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$. Each of these homology groups has a relative $\mathbb{Z}/p_M\mathbb{Z}$–grading, and each admits an action of $\mathbb{Z}[\mathbb{U}] \otimes \bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})$. Moreover, the latter are intertwined by the long exact sequence that is induced by the evident short exact sequence.

The next proposition speaks to the r–dependence of these ∂^*_{SW} homology groups.

Proposition 1.4 *The versions of* κ *that appear in Propositions 1.2 and 1.3 can be chosen so that the following is true: Suppose that* r_1, $\mathrm{r}_2 \geq \kappa$, *and that* $(\mu_1, \mathfrak{p}_1)$ *and* $(\mu_2, \mathfrak{p}_2)$ *are pairs in* $\Omega \times \mathcal{P}$ *such that* μ_1 *and* μ_2 *have* $\mathcal{P}$*–norm less than* 1, *such that* $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *have* $\mathcal{P}$*–norm much less than* 1, *and such that the conclusions of Propositions 1.1 and 1.2 hold for the data sets* $(\mathrm{r}_1, \mu_1, \mathfrak{p}_1)$ *and* $(\mathrm{r}_2, \mu_2, \mathfrak{p}_2)$. *Use these respective data sets to define the corresponding* $\mathrm{r} = \mathrm{r}_1$ *and* $\mathrm{r} = \mathrm{r}_2$ *versions of the groups* $H^{\infty}_{\mathrm{SW,r}}$, $H^{-}_{\mathrm{SW,r}}$ *and* $H^{+}_{\mathrm{SW,r}}$.

- *There is a canonical isomorphism between the* $(\mathrm{r}_1, \mu_1, \mathfrak{p}_1)$ *and* $(\mathrm{r}_2, \mu_2, \mathfrak{p}_2)$ *versions of* $H^{\infty}_{\mathrm{SW,r}}$ *that preserves the relative* $\mathbb{Z}/p_M\mathbb{Z}$*–gradings and intertwines the respective actions of* $\mathbb{Z}[\mathbb{U}] \otimes \bigwedge^*(H_1(Y;\mathbb{Z})/\mathrm{tors})$.
- *This canonical isomorphism maps the* $(\mathrm{r}_1, \mu_1, \mathfrak{p}_1)$ *version of* $H^{-}_{\mathrm{SW,r}}$ *isomorphically to the* $(\mathrm{r}_2, \mu_2, \mathfrak{p}_2)$ *of* $H^{-}_{\mathrm{SW,r}}$ *version, it induces an isomorphism between the two versions of* $H^{+}_{\mathrm{SW,r}}$ *and it intertwines the respective long exact sequence homomorphisms.*
- *This canonical isomorphism is induced by a chain complex homomorphism from the* $(\mathrm{r}_1, \mu_1, \mathfrak{p}_1)$ *version of* $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ *to the* $(\mathrm{r}_2, \mu_2, \mathfrak{p}_2)$ *version of* $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ *that maps the* $(\mathrm{r}_1, \mu_1, \mathfrak{p}_1)$ *version of* $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ *to the* $(\mathrm{r}_2, \mu_2, \mathfrak{p}_2)$ *version.*

This proposition is proved in Section 7.3.

The canonical isomorphisms described by Proposition 1.4 are henceforth used to identify distinct $(\mathrm{r}, \mu, \mathfrak{p})$ versions of $H^{\infty}_{\mathrm{SW,r}}$, $H^{-}_{\mathrm{SW,r}}$ and $H^{+}_{\mathrm{SW,r}}$ and so write these groups respectively as H^{∞}_{SW}, H^{-}_{SW} and H^{+}_{SW}.

Part 9 This last part of the subsection brings the orientation-reversed version of Y into the story so as to connect with what is said in [8]. What is said here explains why Theorems I.3.1 and I.3.2 in [8] follow directly from Propositions 1.1–1.4.

The orientation-reversed twin of Y is denoted here by $\overline{Y}$. So as to be clear, the orientation on $\overline{Y}$ is defined so that the inclusion map $M_\delta \to M$ is orientation-preserving and that of M_δ into Y is orientation-reversing. The orientation is such that both of the inclusion maps $M_\delta \to M$ and $M_\delta \to \overline{Y}$ are orientation-*preserving*. As noted in the introduction, the convention used here for which orientation signifies Y and which signifies $\overline{Y}$ is opposite the convention used in [8].

As explained below, the groups H^∞_{SW}, H^-_{SW} and H^+_{SW} are canonically isomorphic to certain Seiberg–Witten Floer homology groups on Y, these being the respective groups H^∞_*, H^-_* and H^+_* that are defined at the end of Section I.3.2. To see the connection, write the first line of (1-13) as

$$F_A - \mathrm{r}(*\psi^\dagger\tau\psi - iw) + \tfrac{1}{2}F_{A_K} = 0. \tag{1-25}$$

Now introduce $\bar{*}$ to denote the Hodge star as defined by the orientation for $\overline{Y}$. The latter is equal to $-*$. Likewise, introduce $\overline{\mathrm{cl}}$ to denote the $\overline{Y}$ version of the Clifford multiplication map. The latter is equal to $-\mathrm{cl}$ and, as a consequence, the version of $\psi^\dagger\tau\psi$ is equal to -1 times the Y version. Granted these last two observations, what is written in (1-13) is the equation that results when $\bar{*}$ is applied to both sides of the top line in (I.3-1). Meanwhile the lower line in (I.3-1) is -1 times the lower line in (1-13).

What was just said canonically identifies $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ with a corresponding equivalence class of solutions to (I.3-1), this denoted in [8] by $\hat{\mathcal{Z}}_{\mathrm{SW},Y,\mathrm{r}}$. To see about the relation between ∂^*_{SW} and the differential on $\hat{\mathcal{Z}}_{\mathrm{SW},Y,\mathrm{r}}$, a look at (1-20) leads to the following observation: Let $\mathfrak{c}_-$ and $\mathfrak{c}_+$ denote solutions to (1-13) and suppose that $s \mapsto \mathfrak{d}(s)$ is an instanton solution on Y with $s \to -\infty$ limit equal to $\mathfrak{c}_-$ and $s \to \infty$ limit equal to $\mathfrak{c}_+$. Then the map $s \mapsto \mathfrak{d}(-s)$ is an instanton solution to (I.3-3) with $s \to -\infty$ limit $\mathfrak{c}_+$ and $s \to \infty$ limit $\mathfrak{c}_-$. This last observation implies that the identification just described between $\hat{\mathcal{Z}}_{\mathrm{SW},Y,\mathrm{r}}$ and $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ extends in a linear fashion to give an isomorphism between the chain complex on $\overline{Y}$ that is used to define the aforementioned groups H^∞_*, H^-_* and H^+_* from Section I.3.2 and the chain complex $\mathbb{Z}(\hat{\mathcal{Z}}^<_{\mathrm{SW,r}})$ with the differential ∂^*_{SW}.

The conclusions of the preceding two paragraphs make Theorems I.3.1 and I.3.2 immediate consequences of Propositions 1.1–1.4.

1.4 Seiberg–Witten Floer homology and embedded contact homology

This subsection describes the relationship between the Seiberg–Witten Floer chain complex from Section 1.3 and its ∂^*_{SW} homology and the embedded contact homology chain complex from Section 1.2. This is the content of Theorem 1.5. This relationship is the analog of that described by Theorem 4.5 in [19].

Theorem I.3.3 from [8] follows directly from what is said in Theorem 1.5 and what is said in Part 9 of the previous subsection.

The upcoming Theorem 1.5 refers to the filtration of $\hat{\mathcal{Z}}_{\mathrm{ech},M}$ given in (1-10). The theorem also refers to a certain subset in the various $L \geq 1$ versions of what is denoted in Part 5 of Section 1.2 by $\hat{\mathcal{Z}}^{L}_{\mathrm{ech},M}$. The subset in question is denoted in the theorem and in what follows by $\hat{\mathcal{Z}}^{L,<}_{\mathrm{ech},M}$ and it is defined as follows: Part 4 in Section II.1B defines a principal $\mathbb{Z}$–bundle isomorphism $\hat{\mathcal{Z}}_{\mathrm{ech},M} = \mathcal{Z}_{\mathrm{ech},M} \times \mathbb{Z}$. This isomorphism sends the equivalence class (Θ, Z) to the pair (Θ, k) when Z has intersection number k with $\gamma^{(z_0)}$. The subset $\hat{\mathcal{Z}}^{L}_{\mathrm{ech},M}$ corresponds via this isomorphism to $\mathcal{Z}^{L}_{\mathrm{ech},M} \times \{-\infty, \dots, -1\}$.

Theorem 1.5 *Let $\mathbb{H}^{\infty}$, $\mathbb{H}^{-}$ and $\mathbb{H}^{+}$ denote finitely generated subgroups of the respective groups H^{∞}_{SW}, H^{-}_{SW} and H^{+}_{SW}. Given these subgroups, there exists $L^{\mathbb{H}}$ and given $L \geq L^{\mathbb{H}}$ there exists $L' \geq L$ with the following significance: Fix r sufficiently large, and then fix a pair $(\mu, \mathfrak{p}) \in \Omega \times \mathcal{P}$ such that μ has $\mathcal{P}$–norm less than 1, such that $\mathfrak{p}$ has sufficiently small $\mathcal{P}$–norm, and such that Propositions 1.1–1.3 can be invoked to define the chain complex $(\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}}), \partial^{*}_{\mathrm{SW}})$, the subcomplex $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ and the $\mathbb{Z}[\mathbb{U}] \otimes \bigwedge^{*}(H_1(Y;\mathbb{Z})/\mathrm{tors})$ action on the homology. There exists an injective principal $\mathbb{Z}$–bundle map $\hat{\Phi}^{\mathrm{r}} \colon \hat{\mathcal{Z}}^{L'}_{\mathrm{ech},M} \to \hat{\mathcal{Z}}_{\mathrm{SW,r}}$ that defines a $\mathbb{Z}$–module homomorphism*

$$\mathbb{L}^{\mathrm{r}} \colon \mathbb{Z}(\hat{\mathcal{Z}}^{L'}_{\mathrm{ech},M}) \to \mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$$

with the properties listed below:

- *$\mathbb{L}^{\mathrm{r}}$ reverses the sign of relative grading degrees.*
- *$\mathbb{L}^{\mathrm{r}}$ induces a monomorphism from $\mathbb{Z}(\hat{\mathcal{Z}}^{L',<}_{\mathrm{ech},M})$ to $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ and another from $\mathbb{Z}(\hat{\mathcal{Z}}^{L'}_{\mathrm{ech},M})/\mathbb{Z}(\hat{\mathcal{Z}}^{L',<}_{\mathrm{ech},M})$ to $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})/\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$.*
- *$\mathbb{L}^{\mathrm{r}}$ intertwines ∂_{ech} with $\partial^{*}_{\mathrm{SW}}$ and it also intertwines the endomorphisms that define the generators of the respective $\mathbb{Z}[\mathbb{U}] \otimes \bigwedge^{*}(H_1(Y;\mathbb{Z})/\mathrm{tors})$ actions on the ∂_{ech} homology and $\partial^{*}_{\mathrm{SW}}$ homology.*
- *Let $\mathbb{Q}^{L}_{\mathrm{ech}}$ denote either $\mathbb{Z}(\hat{\mathcal{Z}}^{L}_{\mathrm{ech},M})$, $\mathbb{Z}(\hat{\mathcal{Z}}^{L,<}_{\mathrm{ech},M})$ or $\mathbb{Z}(\hat{\mathcal{Z}}^{L}_{\mathrm{ech},M})/\mathbb{Z}(\hat{\mathcal{Z}}^{L,<}_{\mathrm{ech},M})$ and let $\mathbb{Q}^{L'}_{\mathrm{ech}}$ denote the L' version. Use $\mathbb{Q}_{\mathrm{SW}}$ to denote the corresponding $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$, $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ or $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})/\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ as the case may be. If $\varsigma \in \mathbb{Q}^{L}_{\mathrm{ech}}$ is such that $\mathbb{L}^{\mathrm{r}}(\varsigma) = \partial^{*}_{\mathrm{SW}} z$ for some $z \in \mathbb{Q}_{\mathrm{SW}}$, then $\varsigma = \partial_{\mathrm{ech}} \varsigma'$ for some $\varsigma' \in \mathbb{Q}^{L'}_{\mathrm{ech}}$.*

- *The subgroups* $\mathbb{H}^\infty$, $\mathbb{H}^-$ *and* $\mathbb{H}^+$ *are represented by elements in the respective* $\mathbb{L}^{\mathrm{r}}$ *images of* $\mathbb{Z}(\hat{\mathcal{Z}}^L_{\mathrm{ech},M})$, $\mathbb{Z}(\hat{\mathcal{Z}}^{L,<}_{\mathrm{ech},M})$ *and* $\mathbb{Z}(\hat{\mathcal{Z}}^L_{\mathrm{ech},M})/\mathbb{Z}(\hat{\mathcal{Z}}^{L,<}_{\mathrm{ech},M})$.

Suppose that distinct choices for $(\mathrm{r}, \mu, \mathfrak{p})$ *are suitable for defining* $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$, *the differential* ∂^*_{SW} *and the subcomplex* $\mathbb{Z}(\hat{\mathcal{Z}}^<_{\mathrm{SW,r}})$. *If the respective values of* r *are large enough to define* $\mathbb{L}^{\mathrm{r}}$ *on* $\mathbb{Z}(\hat{\mathcal{Z}}^{L'}_{\mathrm{ech},M})$, *and, in any event, sufficiently large, then the homomorphism from the third bullet of Proposition 1.4 can be chosen to intertwine the resulting versions of* $\mathbb{L}^{\mathrm{r}}$.

This theorem is proved in Sections 7.4 and 7.6.

1.5 Functions on $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$

This subsection introduces functions on $\mathrm{Conn}(E)\times C^\infty(Y;\mathbb{S})$ that play essential roles in the story. In what follows, $\mathfrak{c} = (A, \psi = (\alpha, \beta)) \in \mathrm{Conn}(E)\times C^\infty(Y;\mathbb{S})$ is a given element.

The first function is the Chern–Simons function. Reintroduce the chosen fiducial connection A_E from Part 3 of Section 1.3 and write $A = A_E + \hat{a}_A$ with $\hat{a}_A$ an $i\mathbb{R}$–valued 1–form. The Chern–Simons function sends A to

$$\mathfrak{cs}(A) = -\int_Y \hat{a}_A \wedge d\hat{a}_A - 2\int_Y \hat{a}_A \wedge \big(F_{A_E} + \tfrac{1}{2}F_{A_K}\big). \tag{1-26}$$

Note that $\mathfrak{cs}$ is invariant only under the action on $\mathrm{Conn}(E)$ of the subgroup in $C^\infty(Y;S^1)$ of maps u that define classes in $H_1(Y;S^1)$ that have cup product pairing zero with the first Chern class of $\det(\mathbb{S})$. This subgroup is denoted by $\mathcal{G}_{\mathbb{S}}$.

The second function is

$$\mathrm{w}(A) = i\int_Y \hat{a}_A \wedge w. \tag{1-27}$$

This function is invariant only under the action on $\mathrm{Conn}(E)$ of $\mathcal{G}_{\mathbb{S}}$.

The next function is denoted by $\mathfrak{a}$. The critical points of $\mathfrak{a}$ are the solutions to (1-13). This function is given by

$$\mathfrak{a} = \mathfrak{cs} - \mathrm{r}\mathrm{w} + \mathfrak{e}_\mu + \mathrm{r}\int_Y \psi^\dagger D_A\psi. \tag{1-28}$$

The *spectral flow* function is denoted by $\mathfrak{f}_s$. This function is constant on the components of the complement in $\mathrm{Conn}(E)\times C^\infty(Y;\mathbb{S})$ of the codimension 1 subvariety where the operator $\mathfrak{L}_{(\cdot)}$ in (1-17) has nontrivial kernel. It is discontinuous across this subvariety, but in any event it is locally bounded. A precise definition can be found in [16]. What follows defines $\mathfrak{f}_s$ where $\mathfrak{L}_{(\cdot)}$ has trivial kernel. The definition of $\mathfrak{f}_s$ requires the choice of a section of $\mathbb{S}$, this denoted by ψ_E. This section must be chosen so that the $\mathfrak{c}_E = (A_E, \psi_E)$ and the $\mathrm{r} = 1$ version of the operator $\mathfrak{L}_{(\cdot),\mathrm{r}}$ has trivial kernel. The existence of such a section can be established using the Bochner–Weitzenböck formula in (A-12) for the square of the operator $\mathfrak{L}_{(\cdot),\mathrm{r}}$. Now suppose that $\mathfrak{c}$ is a given pair in $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ with the kernel of $\mathfrak{L}_{\mathfrak{c},\mathrm{r}} = \{0\}$. Select a smooth map $\mathfrak{c}(\cdot)$ from $[0,1]$ to $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ with $\mathfrak{c}(0) = \mathfrak{c}_E$ and $\mathfrak{c}(1) = \mathfrak{c}$ and a smooth map $r(\cdot)$ from $[0,1]$ to $[1,\mathrm{r}]$ with $r(0) = 1$ and $r(1) = \mathrm{r}$. The function $\mathfrak{f}_s$ assigns to $\mathfrak{c}$ the spectral flow for the $[0,1]$–parametrized path of operators $\{\mathfrak{L}_{\mathfrak{c}(\tau),r(\tau)}\}_{\tau\in[0,1]}$. Note that $\mathfrak{f}_s(\mathfrak{c})$ is independent of the chosen maps $\mathfrak{c}(\cdot)$ and $r(\cdot)$. So defined, the function $\mathfrak{f}_s$ is also constant on the $\mathcal{G}_{\mathbb{S}}$ orbit of $\mathfrak{c}$.

Neither $\mathfrak{a}$, $\mathfrak{cs}$, W nor $\mathfrak{f}_s$ are invariant with respect to the action of $C^\infty(Y;S^1)$ on $\mathrm{Conn}(E)$ although all are invariant with respect to the action of the subgroup $\mathcal{G}_{\mathbb{S}}$. However, the following functions are invariant under the full action of $C^\infty(Y;S^1)$:

$$\mathfrak{cs}^{\mathfrak{f}} = \mathfrak{cs} - 4\pi^2\mathfrak{f}_s, \quad \mathrm{W}^{\mathfrak{f}} = \mathrm{W} - 2\pi\mathfrak{f}_s \quad \text{and} \quad \mathfrak{a}^{\mathfrak{f}} = \mathfrak{a} + 2\pi(\mathrm{r}-\pi)\mathfrak{f}_s. \tag{1-29}$$

The last of the functions of interest is denoted by M and it is given by

$$\mathrm{M} = \mathrm{r}\int_Y (1 - |\alpha|^2). \tag{1-30}$$

The question of bounding M on a solution to (1-14) or along the path of an instanton is a central concern in what follows.

2 Solutions to the Seiberg–Witten equations on Y

The solutions to the large r versions of (1-13) have certain properties that play central roles in many of the subsequent arguments that supply input to the proofs of the propositions and theorems in Section 1. These properties are given by the various lemmas and propositions in the first two subsections that follow. The third subsection contains the proof of a proposition in the first subsection.

2.1 A priori properties of solutions to (1-13)

The lemmas in the first parts of this subsection consider the pointwise behavior of solutions to (1-13). The second part of the subsection concerns the locus in Y where the curvature 2–form is large. This second part also talks about the function $\textsc{m}$ in (1-30). The third part of what follows discusses the spectral flow function $\mathfrak{f}_s$ and the final part discusses the functions W, $\mathfrak{cs}$ and $\mathfrak{a}$ from Section 1.5.

Part 1 The upcoming Lemmas 2.1–2.3 have close analogs in Section 2a of [22], in Section 6 of [17] and in Section 3 of [18]. When the proof of a given lemma here differs only slightly from its partner in one of these references, then only the salient differences (if any) are noted.

The first lemma speaks to the size of the $C^\infty(Y;\mathbb{S})$ component of a solution.

Lemma 2.1 *There exists $\kappa > 1$ with the following significance: Fix $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 and $\mathrm{r} \geq \kappa$. Let $(A, \psi = (\alpha, \beta))$ denote a solution to the corresponding (r, μ) version of (1-13). Then:*

- $|\alpha| \leq 1 + \kappa \mathrm{r}^{-1}$.
- $|\beta|^2 \leq \kappa \mathrm{r}^{-1}(1 - |\alpha|^2) + \kappa^3 \mathrm{r}^{-2}$.
- $|\nabla_A \alpha|^2 \leq \kappa \mathrm{r}(1 - |\alpha|^2) + \kappa^3$.
- $|\nabla_A \beta|^2 \leq \kappa(1 - |\alpha|^2) + \kappa^3 \mathrm{r}^{-1}$.

In addition, for each $q \geq 1$, there exists $\kappa_q \in (1, \infty)$ which is independent of (A, ψ), r and μ, and is such that

- $|\nabla_A^q \alpha| + \mathrm{r}^{1/2} |\nabla_A^q \beta| \leq \kappa_q \mathrm{r}^{q/2}$.

Proof The lemma and its proof differ only in notation from Lemma 2.3 in [22] and the latter's proof. □

Given the equation in the top line of (1-13), what follows is an immediate consequence of the first two bullets in Lemma 2.1:

$$|B_A| = i \ast(\hat{a} \wedge \ast B_A) + \mathfrak{e} \tag{2-1}$$

where $|\mathfrak{e}| \leq c_0$.

The second lemma addresses the size of the connection A.

Lemma 2.2 *There exists $\kappa > 1$ with the following significance: Fix $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 and $\mathrm{r} \geq \kappa$. Let $(A, \psi = (\alpha, \beta))$ denote a solution to the corresponding (r, μ) version of (1-13). There is a map $u \in C^\infty(Y; \mathbb{S})$ which is homotopic to the* *identity and is such that* $A - u^{-1}du$ *can be written as* $A_E + \hat{a}^{\perp} + \mathfrak{p}_A$, *where* $\mathfrak{p}_A$ *is a harmonic* 1*–form and* $\hat{a}^{\perp}$ *is coclosed*, L^2*–orthogonal to the space of harmonic* 1*–forms, and such that* $|\hat{a}^{\perp}| \leq \kappa(|\mathrm{M}|^{1/3}\mathrm{r}^{2/3} + 1)$.

Proof Given (2-1), the proof is identical to that of Lemma 2.4 in [17]. □

The third lemma in this series extends what is said in Lemma 2.1 with some precise bounds for the size of $1 - |\alpha|^2$ and the covariant derivatives of α and β.

Lemma 2.3 *There exists* $\kappa > 1$ *with the following significance*: *Fix* $\mu \in \Omega$ *with* $\mathcal{P}$*–norm less than* 1 *and* $\mathrm{r} \geq 1$. *Let* $(A, \psi = (\alpha, \beta))$ *denote a solution to the* (r, μ) *version of* (1-13). *Let* $Y_* \subset Y$ *denote the subset of points where* $1 - |\alpha|^2 \geq \kappa^{-1}$. *Then*

$$\left|1 - |\alpha|^2\right| \leq (e^{-\sqrt{\mathrm{r}}\,\mathrm{dist}(\cdot, Y_*)/\kappa} + \kappa\mathrm{r}^{-1}) \quad \textit{where } 1 - |\alpha|^2 \geq \kappa^{-1}.$$

Proof The manipulations done to prove Proposition 4.4 of [14] can be repeated here to obtain the desired inequality. □

Part 2 The upcoming Proposition 2.4 describes the zero locus of the α part of a solution to a given large r version of (1-13) at the points in Y with distances greater than $c_0\mathrm{r}^{-1/2}$ from the curves in the set $\bigcup_{\mathfrak{p}\in\Lambda}\{\hat{\gamma}_{\mathfrak{p}}^{+}, \hat{\gamma}_{\mathfrak{p}}^{-}\}$. The proposition refers to the connection $\hat{A}$ that is defined from any given pair of connection on E and section of E in (1-16).

Proposition 2.4 *There exists* $\kappa > 1$ *with the following significance*: *Fix* $\mathrm{r} \geq \kappa$ *and* $\mu \in \Omega$ *with* $\mathcal{P}$*–norm bounded by* 1 *and suppose that* $(A, \psi = (\alpha, \beta))$ *is a solution to the corresponding* (r, μ) *version of* (1-13). *Let* $Y_{\mathrm{r}} \subset Y$ *denote the set of points with distance greater than* $\kappa^2\mathrm{r}^{-1/2}$ *from the curves in* $\bigcup_{\mathfrak{p}\in\Lambda}\{\hat{\gamma}_{\mathfrak{p}}^{+}, \hat{\gamma}_{\mathfrak{p}}^{-}\}$. *The zero locus of* α *in the closure of* Y_{r} *is transversal and it consists of the disjoint union of at most* G *components with each a properly embedded arc or circle. The zero locus of* α *has the following additional properties*:

- *The tangent line to each component has distance at most* $\kappa\mathrm{r}^{-1/2}$ *from* v.
- *Each component lies where* $1 - 3\cos^2\theta > 0$.
- *The intersection of the zero set with* M_δ *consists of* G *properly embedded segments that pair the index* 1 *and index* 2 *critical points of the incarnation of* f *as a function on* M *in the sense that distinct segments start on the boundary of*

the radius δ coordinate balls about distinct index 1 critical points of f and end on the boundary of the radius δ coordinate balls about distinct index 2 critical points.

- *The absolute value of $1-|\alpha|^2$ is less than κ^{-1} at all points with distance greater than $\kappa \mathrm{r}^{-1/2}$ from the zero locus of α in Y, and less than $\kappa \mathrm{r}^{-1}$ at all points with distance $\kappa(\ln \mathrm{r})\mathrm{r}^{-1/2}$ or more from the zero locus of α in Y.*
- *The 2–form $\frac{i}{2\pi}F_{\hat{A}}$ has compact support and integral 1 on any disk in Y_{r} that intersects $\alpha^{-1}(0)$ transversally at its center point, is otherwise disjoint from $\alpha^{-1}(0)$, and has closure with all boundary points at distance at least $\kappa \mathrm{r}^{-1/2}$ from $\alpha^{-1}(0)$.*

The proof of Proposition 2.4 is given in Section 2.3. The first assertion of the next lemma is little more than a corollary to Proposition 2.4. The second assertion refers to the 1–form $\upsilon_\diamond$ from (1-5).

Lemma 2.5 *There exists $\kappa > 1$ with the following significance: Fix $\mathrm{r} \ge \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1 and suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the corresponding (r, μ) version of (1-13).*

- *Set $\mathrm{M} = \mathrm{r}\int_Y (1-|\alpha|^2)$. Then $-\kappa \le \mathrm{M} \le \kappa \ln \mathrm{r}$.*
- $\mathrm{r}\int_Y |\upsilon_\diamond|^2 \big|1-|\alpha|^2\big| \le \kappa$.

Proof The lower bound on M follows directly from Lemma 2.1. To obtain the asserted upper bound, use Proposition 2.4 to characterize the zero locus of α in Y_{r}. In particular, Lemma 2.3 with the third bullet of Proposition 2.4 bound $1-|\alpha|^2$ at distance ρ in Y_{r} from $\alpha^{-1}(0) \cap Y_{\mathrm{r}}$ by c_0. It follows from the first bullet of Proposition 2.4 and the formula for v in (1-3) that the length $\alpha^{-1}(0) \cap Y_{\mathrm{r}}$ is at most $c_0 \ln \mathrm{r}$. These bounds together imply that the Y_{r} contribution to the integral that defines M is at most $c_0 \ln \mathrm{r}$. Meanwhile, the volume of $Y - Y_{\mathrm{r}}$ is at most $c_0 \mathrm{r}^{-1}$ and so the $Y - Y_{\mathrm{r}}$ contribution to the integral that defines M is at most $c_0 \ln \mathrm{r}$.

To prove the assertion of the second bullet, note that the integral over Y of $\upsilon_\diamond \wedge \frac{i}{2\pi}F_A$ is equal to the pairing between $c_1(E)$ and the class in $H_2(Y;\mathbb{R})$ that is Poincaré dual to the class that is defined by the de Rham cohomology by the closed form $\upsilon_\diamond$. With this fact in mind, write $\upsilon_\diamond$ as $q_\diamond \hat{a} + \mathfrak{b}$, where $\mathfrak{b}$ annihilates the vector field v. Note in particular that what is said in Part 4 of Section 1.1 can be used to see that $|\upsilon_\diamond|^2 \le c_0 q_\diamond$. Granted this last point, use the top equation in (1-13) to see that

$$i \ast(\upsilon_\diamond \wedge F_A) = \mathfrak{r} q_\diamond (1-|\alpha|^2) + \mathfrak{r}, \tag{2-2}$$

where $|\mathfrak{r}| \le c_0 \mathrm{r} |\alpha||\beta||\upsilon_\diamond|$. Given that $|\upsilon_\diamond| \le c_0 q_\diamond^{1/2}$, the first and second bullets in Lemma 2.1 with (2-2) find

$$i \ast(\upsilon_\diamond \wedge F_A) \ge \tfrac{1}{2} \mathrm{r} q_\diamond \big|1-|\alpha|^2\big| - c_0. \tag{2-3}$$

The lemma's assertion follows from (2-3) with a second use of the bound $c_0^{-1}|\upsilon_\diamond|^2 \le q_\diamond$. □

Part 3 The spectral flow function $\mathfrak{f}_s$ plays a central role in the proofs of Proposition 1.4 and Theorem 1.5. The upcoming Proposition 2.6 supplies a crucial bound for its absolute value. To set the stage for this proposition, reintroduce from Part 4 of Section 1.2 the set $\{\gamma^{(z)}\}_{z\in ¥}$ of closed integral curves of v. This set has $1+b_1(M)$ elements. Each curve in this set lies in $M_\delta \cap \mathcal{H}_0$ and it has distance c_0^{-1} or more from any segment of an integral curve of v in the $f^{-1}(1,2)$ part of M_δ that starts on the boundary of the radius δ coordinate ball about an index 1 critical point of f in M and ends on the boundary of the radius δ coordinate ball about an index 2 critical point of f. Associate to each $z \in ¥$ the map $\mathrm{x}^{(z)}$ from $\mathrm{Conn}(E)$ to $\mathbb{R}$ given by following rule: Let A denote any given connection on E, write A as $A = A_E + \hat{a}_A$ and set

$$\mathrm{x}^{(z)}(A) = \frac{i}{2\pi} \int_{\gamma^{(z)}} \hat{a}_A. \tag{2-4}$$

Use $[\gamma^{(z)}]$ in what follows to denote the class in $H_1(M;\mathbb{Z})/\mathrm{tors}$ that is defined by a given $z \in ¥$ loop $\gamma^{(z)}$. The set of such cycles generates the image of the Poincaré dual of the classes in $H^2(Y;\mathbb{Z})$ that annihilate the $\bigoplus_{\mathrm{p}\in\Lambda} H_2(\mathcal{H}_\mathrm{p};\mathbb{Z})$ summand in (1-4)'s depiction of $H_2(Y;\mathbb{Z})$. As the first Chern class of $\det(\mathbb{S})$ annihilates this summand, the image of its Poincaré dual in $H_1(M;\mathbb{Z})/\mathrm{tors}$ can be written as $\sum_{z\in ¥} \mathrm{c}_{\mathbb{S},z}[\gamma^{(z)}]$ with coefficients $\{\mathrm{c}_{\mathbb{S},z}\}_{z\in ¥} \in \mathbb{Z}$. Use $\mathrm{x}_\mathbb{S}$ to denote the corresponding map $\sum_{z\in ¥} \mathrm{c}_{\mathbb{S},z}\mathrm{x}^{(z)}$.

What follows is a consequence of the fact that the classes from the set $\{[\gamma^{(z)}]\}_{z\in ¥}$ are linearly independent in $H_1(Y;\mathbb{Z})/\mathrm{tors}$: Let A denote a connection on E. There is a smooth map $u\colon Y \to S^1$ such that $A - u^{-1}du$ obeys $0 \le \mathrm{x}^{(z)}(A-u^{-1}du) < 1$ for each $z \in ¥$. Note that u can be chosen so that $A - u^{-1}du - A_E = \hat{a}_A - u^{-1}du$ is a coclosed 1–form.

Proposition 2.6 *There exists $\kappa > 1$ with the following significance: Suppose that $\mathrm{r} > \kappa$ and that $\mu \in \Omega$ has $\mathcal{P}$–norm less than 1. Let (A, ψ) denote a nondegenerate solution to the (r, μ) version of (1-13). Then $|\mathfrak{f}_s(A, \psi) - \mathrm{X}_{\mathbb{S}}(A, \psi)| \leq \kappa$.*

Section Bc extends the function $|\mathfrak{f}_s|$ as a piecewise constant function on the whole of $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$. This understood, the assertion in Proposition 2.6 also holds in the case when the (A, ψ) version of (1-17) has nontrivial kernel.

The proof of Proposition 2.6 is in the appendix. The placement of the proof in an appendix is not a reflection of the importance of Proposition 2.6; this proposition is absolutely crucial with regards to what is said subsequently about instanton solutions to (1-20). The proof is in the appendix as it is long and as the notions that enter are not used elsewhere.

Part 4 The proposition that follows supplies a priori bounds for the functions $\mathfrak{cs}$ in (1-26), the function W in (1-27) and the function $\mathfrak{a}$ in (1-28).

Proposition 2.7 *There exists $\kappa > 1$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1 and suppose that (A, ψ) is a solution to the (r, μ) version of (1-13). Then*

- $|\mathfrak{cs}^{\mathfrak{f}}| \leq \kappa(\mathrm{r}^{2/3}\mathrm{M}^{4/3} + \mathrm{M} + 1)$,
- $|\mathrm{W}^{\mathfrak{f}} - \mathrm{M}| < \kappa$,
- $-\kappa\mathrm{r}(\mathrm{M} + 1) \leq \mathfrak{a}^{\mathfrak{f}} \leq \kappa\mathrm{r}$.

Proof The assertion in the third bullet about $\mathfrak{a}^{\mathfrak{f}}$ follows from the assertions about $\mathfrak{cs}^{\mathfrak{f}}$ and $\mathrm{W}^{\mathfrak{f}}$. To prove the asserted bound for $\mathfrak{cs}^{\mathfrak{f}}$, use the Green's function for the operator $d + d^*$ to construct a smooth, coclosed 1–form on $Y - \big(\bigcup_{z \in \yen} \gamma^{(z)}\big)$ with the following properties: Let $\mathrm{B}_{\mathbb{S}}$ denote this 1–form. Then $|\mathrm{B}_{\mathbb{S}}| \leq c_0 \sum_{z \in \yen} \mathrm{dist}(\,\cdot\,, \gamma^{(z)})^{-1}$ and

$$\frac{i}{2\pi}\int_Y \hat{a} \wedge \big(F_{A_E} + \tfrac{1}{2}F_{A_K}\big) = \sum_{z \in \yen} \mathrm{C}_{\mathbb{S},z} \int_{\gamma^{(z)}} \hat{a} + \frac{i}{2\pi}\int_Y \mathrm{B}_{\mathbb{S}} \wedge d\hat{a} \tag{2-5}$$

with $\hat{a}$ being any given 1–form on Y. Granted (2-5), write $\mathfrak{cs}(A)$ as

$$-\int_Y \hat{a}_A \wedge d\hat{a}_A - 2\int_Y \mathrm{B}_{\mathbb{S}} \wedge d\hat{a}_A + 4\pi \mathrm{X}_{\mathbb{S}}. \tag{2-6}$$

To bound the integral of $\mathrm{B}_{\mathbb{S}} \wedge d\hat{a}_A$, use the top equation in (1-13) to see that the 2–form $d\hat{a}_A$ differs from $*B_A$ by a smooth, bounded form. This understood, use this same equation with (2-1) and Lemma 2.1 to bound $|d\hat{a}_A|$ at distances c_0^{-1} or less from any curve in the set $\{\gamma^{(z)}\}_{z\in ¥}$ by $c_0\mathrm{r}(1-|\alpha|^2)+c_0$, and use Lemma 2.3 to bound the latter by c_0. The absolute value of the contribution to the integral of $\mathrm{B}_{\mathbb{S}}\wedge d\hat{a}_A$ from the radius c_0^{-1} tubular neighborhood of any curve from $\{\gamma^{(z)}\}_{z\in ¥}$ is therefore bounded by c_0. Meanwhile, the absolute value of the contribution to the integral of $\mathrm{B}_{\mathbb{S}}\wedge d\hat{a}_A$ from the complement in Y of the union of these neighborhoods is less than $c_0\mathrm{M}$.

To bound the left-most integral in (2-6), note first that both of the integrals over Y in (2-6) do not change when A is replaced by $A-u^{-1}du$ with u being any map from Y to S^1. This understood, choose a map u so that the L^2–orthogonal projection of $\hat{a}_A-u^{-1}du$ has L^2–norm bounded by c_0. Having done this, use Lemma 2.2 to bound the left-most integral in (2-6) by $c_0\mathrm{r}^{2/3}\mathrm{M}^{4/3}$. Granted these bounds, Proposition 2.7's bound of $\mathfrak{cs}^{\mathfrak{f}}$ follows from (2-6) and Proposition 2.6.

Consider next the assertion made by the second bullet of the proposition. Look at (1-3) and (1-6) to see that w on the $|u|\le R+c_0\ln\delta$ part of any $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ can be written as $d\hat{a}$. As a consequence, the function χ can be used with the Green's function for the operator $*d+d*$ to construct a smooth 1–form on $Y-\big(\bigcup_{z\in ¥}\gamma^{(z)}\big)$ with the properties listed next. Use B_w to denote this 1–form. The form B_w is zero on the $|u|\le R+c_0\ln\delta$ part of each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$. In addition, $|\mathrm{B}_w|\le c_0\sum_{z\in ¥}\operatorname{dist}(\cdot,\gamma^{(z)})^{-1}$ and

$$\text{(2-7)}\qquad i\int_Y \hat{a}\wedge w = i\sum_{z\in ¥}\mathrm{c}_{\mathbb{S},z}\int_{\gamma^{(z)}}\hat{a}+i\int_Y\hat{a}\wedge d\hat{a}+i\int_Y \mathrm{B}_w\wedge d\hat{a},$$

with $\hat{a}$ being any given 1–form on Y.

Take $\hat{a}$ in (2-7) to be the 1–form $\hat{a}_A$. The left-hand side of the $\hat{a}_A$ version of (2-7) is $\mathrm{W}(A)$. The term on the right-hand side with the sum indexed by ¥ is $2\pi\mathrm{X}_{\mathbb{S}}$. Use the top equation in (1-13) with (1-30) to see that the integral of $\hat{a}\wedge d\hat{a}_A$ can be written as $-i\mathrm{M}+\mathfrak{r}_A$ with $|\mathfrak{r}_A|\le c_0$. Meanwhile, $\mathrm{B}_w\wedge d\hat{a}_A$ can be written as $\mathrm{B}_w\wedge F_A+\mathfrak{q}_A$, where F_A denotes A's curvature 2–form and $\mathfrak{q}_A$ is a 2–form with $|\mathfrak{q}_A|\le c_0$. Granted the preceding, the second bullet of the proposition follows with a bound by c_0 on the absolute value of the integral over Y of the form $\mathrm{B}_w\wedge F_A$. Such a bound follows from the second bullet of Lemma 2.5 and Lemma 2.1.

Given what was said in the preceding two paragraphs, the bound for $|\mathrm{W}^{\mathfrak{f}}-\mathrm{M}|$ given in Proposition 2.7 follows from (2-7) and Proposition 2.6. □

2.2 The vortex equations, I

The proof of Proposition 2.4 in Section 2.3 invokes various properties of the *vortex equations on* $\mathbb{C}$. Properties of these equations are also used to prove Proposition 2.6 and are used elsewhere as well. This section introduces these equations and supplies what is needed for the proof of Proposition 2.4. More is said about these equations in Sections 3 and 4.

The *vortex equations* ask that a pair (A_0, α_0) of connection on a complex line bundle over $\mathbb{C}$ and section of this bundle obey

$$\begin{cases} *F_{A_0} = -i(1-|\alpha_0|^2), \\ \bar{\partial}_{A_0}\alpha_0 = 0, \\ |\alpha_0| \le 1. \end{cases} \tag{2-8}$$

The notation here is such that $*$ denotes the Euclidean Hodge dual on $\mathbb{C}$, while F_{A_0} and $\bar{\partial}_{A_0}$ denote the respective curvature 2–form of A_0 and the d–bar operator defined by A_0 on the space of sections of the given complex line bundle. The solutions with $1-|\alpha_0|^2$ integrable are discussed at length in Sections 1 and 2 of [20]. Solutions to (2-8) are also solutions to (4.1) in [22], so what is said in Proposition 4.2 in [22] applies as well.

Two properties of the solutions to (2-8) are needed for the proof of Proposition 2.4 that are not stated explicitly in Sections 1 and 2 of [20] or by Proposition 4.1 in [22]. These are given by:

Lemma 2.8 *Let (A_0, α_0) denote a solution to the vortex equations. Then $|\alpha_0|$ cannot have a local, nonzero minimum. Given $\varepsilon > 0$, there exists $\kappa > 1$ with the following significance: Suppose that (A_0, α_0) is a solution to the vortex equations and $|\alpha_0| < 1-\varepsilon$ at the origin in $\mathbb{C}$. Then $|\alpha_0| < \varepsilon$ at a point with distance at most κ from the origin.*

Proof The function $|\alpha_0|$ can be written as e^u on a set where it is nowhere zero with $u < 0$ a smooth function. The top equation in (2-8) requires that $-\Delta u = (1-e^{2u})$, where Δ here denotes the Laplacian on $\mathbb{R}^2$. This understood, the first assertion of the lemma follows from the maximum principle. To prove the second assertion, suppose to the contrary that it is false for some $\varepsilon > 0$. The equations in (2-8) are uniformly elliptic, and thus taking limits with counterexamples for the successive cases $\kappa = 1, 2, \dots$ finds a solution (A_0, α_0) with $|\alpha_0| < 1-\varepsilon$ at the origin and with $|\alpha_0| > \varepsilon$ on $\mathbb{C}$. Introduce the function $\mathfrak{t}$ on $[0, \infty)$ whose value at any given $s \in [0, \infty)$ is the average value of $-u$ on the circle in $\mathbb{C}$ of radius s. The equation $-\Delta u = 1 - e^{2u}$ implies the equation $s\partial_s t = \mathfrak{h}$, where $\mathfrak{h}(s)$ is the integral of $1-|\alpha_0|^2$ over the radius s disk in $\mathbb{C}$ centered at the origin. The fact that $1-|\alpha_0|^2 < \varepsilon$ at the origin implies that $\mathfrak{h} \geq c_0^{-1}\varepsilon$ on $[1,\infty)$ and so $s\partial_s \mathfrak{t} \geq c_0^{-1}\varepsilon$ on $[1,\infty)$. This being the case, then $\mathfrak{t} \geq c_0\varepsilon(\ln s) - c_0^2$. On the other hand, $\mathfrak{t} \leq |\ln\varepsilon|$ if $|\alpha_0| > \varepsilon$ and this bound is violated when $\ln s \geq c_0^{-1}\varepsilon^{-1}|\ln\varepsilon| + c_0^2$. □

The vortex equations enter Proposition 2.4's proof via the upcoming Lemma 2.9. The lemma refers to a *transverse disk* with a given radius through a given point in Y. Such a disk is the image via the metric's exponential map of the centered disk of the given radius in the 2–plane bundle $\mathrm{Ker}(\hat{a})$ at the given point. There exists $c_0 > 100$ such that any transverse disk with radius c_0^{-1} is embedded with a priori bounds on the derivatives to any given order of its extrinsic curvature. In addition, the vector field v along D_0 is everywhere c_0^{-1} close to the normal vector to D_0. All transverse disks are assumed implicitly to have radius less than c_0^{-1} so as to invoke these two properties.

Lemma 2.9 uses J to view $\mathrm{Ker}(\hat{a})$ as a complex line bundle and it uses the Riemannian metric to define a compatible Hermitian structure on $\mathrm{Ker}(\hat{a})$. Fix $p \in Y$ and an isometric isomorphism from $\mathrm{Ker}(\hat{a})|_p$ to $\mathbb{C}$. Use φ in what follows to denote the map from $\mathbb{C}$ to Y that is obtained by composing first the isomorphism with $\mathrm{Ker}(\hat{a})|_p$ and then the metric's exponential map. With $\mathrm{r} \geq 1$ given, Lemma 2.9 uses φ_r to denote the composition of first multiplication by $\mathrm{r}^{-1/2}$ on $\mathbb{C}$ and then φ. To finish the notational preliminaries, suppose next that $(A, \psi = (\alpha,\beta))$ is a given solution to some $\mathrm{r} \geq 1$ and $\mu \in \Omega$ version of (1-13). Use $(A_\mathrm{r}, \psi_\mathrm{r})$ to denote $\varphi_\mathrm{r}^*(A,\psi)$. Lemma 2.9 writes ψ_r as $(\alpha_\mathrm{r}, \beta_\mathrm{r})$.

Lemma 2.9 *Fix an integer $k \geq 1$; there exists $\kappa > 1$ with the following properties: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1 and suppose that (A,ψ) is a solution to the corresponding (r,μ) version of (1-13). Fix a point in Y and use the associated map φ_r to define the pair $(A_\mathrm{r},\alpha_\mathrm{r})$ of connection on and section of a complex line bundle over $\mathbb{C}$. There exists a solution to the vortex equation on $\mathbb{C}$ whose restriction to the radius k disk about the origin in $\mathbb{C}$ has C^k–distance less than $\frac{1}{k}$ from $(A_\mathrm{r},\alpha_\mathrm{r})$ on this same disk.*

Proof The argument is essentially identical to that used to prove Lemma 6.1 in [17]. □

2.3 Proof of Proposition 2.4

The proof of the proposition has seven parts. By way of a look ahead, the arguments are much like those in Section 6.4 of [17].

Part 1 Let D_0 denote a transverse disk in Y with the following properties: First, the disk has radius $\rho > c_0 \mathrm{r}^{-1/2}$. Second, all points in the disk have distance at least $(c_0 + 10^8)\rho$ from $\bigcup_{\mathfrak{p}\in\Lambda}\{\hat{\gamma}_{\mathfrak{p}}^+, \hat{\gamma}_{\mathfrak{p}}^-\}$. Lie transport by v moves D_0 to a new disk. For $t \in \mathbb{R}$, use D_t to denote the new disk that is obtained by moving the points in D_0 a distance t along the integral curves of v. The formula for v in (1-3) can be used to see that $t = t_1$ and $t = t_2$ versions of D_t are disjoint unless $t_1 = t_2$.

Fix a compactly supported function on D_0 which is equal to 1 on the radius $\frac{1}{2}\rho$ concentric subdisk in D_0 and with the absolute value of its derivative bounded by $c_0\rho^{-1}$. Use χ_0 to denote this function and use χ_t to denote the time-t Lie transport of χ_0 by v.

Part 2 Fix $\kappa_0 \geq 1$. Fix $\mathrm{r} \geq c_0$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1 and suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of (1-13). Let D_0 denote a disk as described in Part 1 with $1 - |\alpha|^2 \geq \kappa_0^{-1}$ at the center point of D_0.

Use $\mathfrak{f}$ to denote the function on $[0, \infty)$ that is given by the rule

$$t \mapsto \mathfrak{f}(t) = \mathrm{r}\int_{D_t} \chi_t(1 - |\alpha|^2). \tag{2-9}$$

Note that $\mathfrak{f}(0) \geq c_0^{-1}$. This lower bound follows from the upper bound on $|\nabla_A\alpha|$ given in Lemma 2.1 with the fact that $1 - |\alpha|^2 \geq \kappa_0^{-1}$ at the center point of D_0.

The derivative of $\mathfrak{f}$ is denoted in what follows by $\mathfrak{f}'$; it is given by

$$\mathfrak{f}' = \mathrm{r}\int_{D_t} \chi_t(\bar{\alpha}(\nabla_A\alpha)_v + (\nabla_A\bar{\alpha})_v\alpha), \tag{2-10}$$

where $(\nabla_A\alpha)_v$ is used here and subsequently to denote the section of E that is obtained by pairing $\nabla_A\alpha$ with the vector field v. As explained in the next paragraph, the norm of the derivative of $\mathfrak{f}$ is such that

$$|\mathfrak{f}'| \leq c_0\mathrm{r}\int_{D_t} \big(|\partial_t\chi_t||\beta|^2 + (|v^\perp| + |d^\perp\chi_t|)|\alpha||\beta|\big), \tag{2-11}$$

where the notation uses $v^\perp$ and $d^\perp\chi_t$ to denote the orthogonal projections of v and $d\chi_t$ to the respective tangent and cotangent bundles of D_t. Granted (2-11), use Lemma 2.1 to see that

$$|\mathfrak{f}(t) - \mathfrak{f}(0)| \leq c_0 t. \tag{2-12}$$

This last inequality implies that

$$\mathfrak{f}(t) \geq c_0^{-1}\kappa_0^{-1} \quad \text{when } |t| \leq c_0^{-2}\kappa_0^{-1}. \tag{2-13}$$

To prove (2-11), note first that J defines an almost complex structure on the kernel of $\hat{a}$. Equation (1-13) identifies $(\nabla_A\alpha)_v$ with a constant multiple of the part of $\nabla_A\beta$ that comes from the $(1,0)$ part of the dual to the kernel of $\hat{a}$. Meanwhile, it identifies $(\nabla_A\beta)_v$ with a constant multiple of the part of $\nabla_A\alpha$ that comes from the dual to the $(0,1)$ part of the kernel of $\hat{a}$. Equation (2-11) follows from these observations with an integration by parts. By way of a warning, these same identifications are used later in the proof without further comment.

Part 3 This part constitutes a digression of sorts to draw attention to some consequences of the bounds given by Lemma 2.1. The remarks that follow here concern the integral over disks in Y of the curvature of the connection $\hat{A}$ given in (1-16) and the curvatures of analogs of $\hat{A}$ that are defined using (1-16) with a different version of the function $\wp$. In particular, allow in (1-16) any function $\wp$ on $[0,\infty)$ that is nondecreasing and such that $\wp(x) \le c_0 x$ for x near 0.

Given $\mathrm{r} \ge c_0$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1, let $(A, \psi = (\alpha,\beta))$ denote a solution to the (r,μ) version of (1-13). Use the pair (A,α) to define $\hat{A}$. The corresponding curvature 2–form is denoted by $F_{\hat{A}}$; it is given by the formula

$$F_{\hat{A}} = (1-\wp)F_A - \wp'(\nabla_A\bar{\alpha} \wedge \nabla_A\alpha), \tag{2-14}$$

where F_A here denotes the curvature 2–form of the connection A. In the context at hand, $F_A = *B_A$. What is said in the last paragraph of Part 2 with the bounds provided by Lemma 2.1 can be used to write

$$F_{\hat{A}} = ((1-\wp)\mathrm{r}(1-|\alpha|^2) + \wp'|\nabla_A\alpha|^2 + \mathfrak{e}_v)w + \hat{a} \wedge \mathfrak{e}^{\perp}, \tag{2-15}$$

where

$$|\mathfrak{e}_v| \le c_0((1-\wp)+\wp')\big(\big|1-|\alpha|^2\big| + \mathrm{r}^{-1}\big),$$
$$|\mathfrak{e}^{\perp}| \le c_0((1-\wp)+\wp')\big(\mathrm{r}^{1/2}\big|1-|\alpha|^2\big|^{1/2} + 1\big).$$

This depiction of $F_{\hat{A}}$ plays an important role in subsequent arguments.

An additional fact about $\hat{A}$ is used extensively, this concerning the case when $\wp$ is chosen to equal 1 on a neighborhood of $[1,\infty)$ in $[0,\infty)$: if $\kappa_0 > 1$ is such that $\wp = 1$ on $[1-\kappa_0^{-1},\infty)$, then $\hat{A}$ is flat and $\alpha|\alpha|^{-1}$ is $\hat{A}$–covariantly constant where $1-|\alpha|^2 < \kappa_0^{-1}$.

Part 4 Fix $\kappa_0 > 4$ and a function $\wp$ on $[0,\infty)$ that is zero on $[0, 1-2\kappa_0^{-1}]$ and is equal to 1 on $[1-\kappa_0^{-1},\infty)$. Use this version of $\wp$ to define the connection $\hat{A}$. The 2–form $\frac{i}{2\pi}F_{\hat{A}}$ represents the first Chern class of E in the de Rham cohomology of Y and so it has integral G on the $f \in [1+\delta, 2-\delta]$ level sets in M_δ. It follows as a consequence that there are points on any such surface where $1-|\alpha|^2 > \kappa_0^{-1}$. Meanwhile, it has integral zero on any level set of f with f not in this range, and it has integral zero on the $u=$ constant 2–spheres in $\mathcal{H}_0$. This last observation implies that $1-|\alpha|^2$ must be $\mathcal{O}(\mathrm{r}^{-1})$ on much of Y. The next lemma describes this region.

To set the stage for the lemma, fix $q \geq 1$ and let $\mathcal{Y}_q$ denote the set of points in Y with the following property: a point is in $\mathcal{Y}_q$ if it lies on a segment of an integral curve of v with length q or less and with one endpoint in $\mathcal{H}_0$. Note that $\mathcal{Y}_q$ contains $\mathcal{H}_0$, and it contains both the $f \leq 1$ and $f \geq 2$ parts of M_δ if $q > c_0$. If $q > c_0$, then it also contains a small radius tubular neighborhood of the integral curve segments of v in M_δ that start on the boundary of the radius δ coordinate ball about an index 1 critical point of f and end on the boundary of the radius δ coordinate ball about an index 2 critical point of f. It also contains much of the $1-3\cos^2\theta \leq 0$ portion of any given $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$; the missing part is a small radius tubular neighborhood of $\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-$.

To say more about these last parts of $\mathcal{Y}_q$, fix $\varepsilon > 0$ and fix $\mathfrak{p} \in \Lambda$. Let $\mathcal{H}_{\mathfrak{p},\varepsilon} \subset \mathcal{H}_\mathfrak{p}$ denote the subset of points with distance greater than ε from $\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-$ and with (u,θ) coordinates such that either $1-3\cos^2\theta \leq 0$ or $\mathrm{f}(u)|\cos\theta|\sin^2\theta > \frac{2}{3\sqrt{3}}(x_0 + 4e^{-2R}) + \varepsilon$. By way of a reminder, the function f is given in (1-2). Lemma II.2.2 finds $q_\varepsilon > 1$ such that each point in $\mathcal{H}_{\mathfrak{p},\varepsilon}$ has distance q_ε or less along an integral curve of v from $\mathcal{H}_0$. For example, $\mathcal{H}_\mathfrak{p} \cap M_\delta$ is the part of $\mathcal{H}_\mathfrak{p}$ where $|u| > R + \ln\delta$ and so a given point in $\mathcal{H}_\mathfrak{p} \cap M_\delta$ is in $\mathcal{H}_{\mathfrak{p},\varepsilon}$ unless both $1-3\cos^2\theta > 0$ and $|\cos\theta| < c_0\delta^{-2}(x_0+\varepsilon)$. This has the following consequence when $\varepsilon \leq x_0$. The complement of the radius $c_0 x_0 \delta^{-2}$ tubular neighborhood of the M_δ part of the union of the ascending disks from the index 1 critical points of f and the descending disks from the index 2 critical points of f is in $\mathcal{Y}_q$ if $q > q_\varepsilon$.

Lemma 2.10 *Fix $q \geq 1$ and there exists $\kappa > 1$ with the following significance: Suppose that $\mathrm{r} \geq \kappa$ and that $\mu \in \Omega$ has $\mathcal{P}$–norm less than 1. Let $(A, \psi = (\alpha,\beta))$ denote a solution to the (r,μ) version of (1-13). Then $1-|\alpha|^2 \leq \kappa \mathrm{r}^{-1}$ at all points in $\mathcal{Y}_q$.*

The proof is given in a moment. The lemma that follows directly plays a central role in the proof of Lemma 2.10.

Lemma 2.11 *There exists $\kappa > 1$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Let $(A, \psi = (\alpha,\beta))$ denote a solution to the (r,μ) version of (1-13). Then $1-|\alpha|^2 \leq \kappa \mathrm{r}^{-1}$ on $\mathcal{H}_0$ and on the part of M_δ where $f \leq 1 - 2\delta^2 - \kappa(\ln \mathrm{r})\mathrm{r}^{-1/2}$ or where $f \geq 2 + 2\delta^2 + \kappa(\ln \mathrm{r})\mathrm{r}^{-1/2}$.*

The proof of Lemma 2.11 and subsequent parts of the proof of Proposition 2.4 use κ_* to denote the constant that appears in Lemma 2.3.

Proof of Lemma 2.11 To prove the assertion, let S denote either a constant u sphere in $\mathcal{H}_0$ or a compact, level set of f in M_δ with f either less than $1-2\delta^2$ or greater than $2+2\delta^2$. Suppose that $p \in S$ is a point where $1-|\alpha|^2 > \frac{1}{4}\kappa_*^{-1}$. It follows from Lemma 2.1's bound on $|\nabla_A \alpha|$ that the integral of $\frac{i}{2\pi}F_A$ over the disk in S centered at this point with radius $\mathrm{r}^{-1/2}$ is greater than $c_0^{-1}\kappa_*^{-1}$. This understood, use the formula for B_A in (1-13) with the bounds on $|\beta|$ supplied by Lemma 2.1 to see that the integral of $\frac{i}{2\pi}F_A$ over S is larger than $c_0^{-1}\kappa_*^{-1}$. But this is impossible given that the 2–form $\frac{i}{2\pi}F_A$ represents the first Chern class of E. Granted this conclusion, use Lemma 2.3 to conclude that $1-|\alpha|^2 \le c_0\mathrm{r}^{-1}$ on $\mathcal{H}_0$ and on the parts of M_δ where $f \le 1-2\delta^2 - c_0(\ln \mathrm{r})\mathrm{r}^{-1/2}$ and where $f \ge 2+2\delta^2 + c_0(\ln \mathrm{r})\mathrm{r}^{-1/2}$. □

Proof of Lemma 2.10 Fix $z > 1$ to be specified shortly. It is enough to consider the cases when $q = nz^{-1}$ with $n \in \{0,1,2,\dots\}$. Since Lemma 2.11 gives the case for $n = 0$, an induction argument will prove the lemma for the general case for a suitable $z = c_0$. This understood, suppose that the lemma holds for a given integer $n \ge 0$ and suppose for the sake of argument that there is a point in Y where $1-|\alpha|^2 \ge \frac{1}{2}\kappa_*$ and with distance less than $(n+2)z$ along a segment of an integral curve of v with an endpoint from the part of Y that is described in Lemma 2.10. Let D_0 denote the transverse disk centered at this point with radius $c_0\mathrm{r}^{-1/2}$. The function $\mathfrak{f}$ given in Part 2 is such that $\mathfrak{f}(0) \ge c_0^{-1}\kappa_* - 1$. It follows from (2-11) that the function $\mathfrak{f}(t)$ is greater than $\frac{1}{4}c_0^{-1}\kappa_*$ if $|t| \le c_0^{-1}$. If $z < c_0^{-1}$, this last conclusion violates the induction hypothesis that $1-|\alpha|^2 \le c_0\mathrm{r}^{-1}$ on integral curve segments of length nz or less with one endpoint in the set described by Lemma 2.11. Thus, $1-|\alpha|^2$ must be less than $\frac{1}{2}\kappa_*$ at all points along an integral curve segment of length less than $(n+2)z^{-1}$ with one endpoint in this same set from Lemma 2.11. Granted this fact, then what is said in Lemma 2.3 completes the proof. □

Part 5 Let $S \subset M_\delta$ denote a level set of f with $f \in (1+\delta, 2-\delta)$. As noted in the first paragraph of Part 4, there are points on S where $1-|\alpha|^2$ is greater than $\frac{1}{2}\kappa_*^{-1}$. Let p denote such a point. It follows from Lemmas 2.8 and 2.9 that there is a point in S with distance at most $c_0\mathrm{r}^{-1/2}$ from p where $|\alpha| < \frac{1}{100}$. In fact, there is a point in S with distance less than $c_0\mathrm{r}^{-1/2}$ from p where $\alpha = 0$. To prove the existence of such an $\alpha = 0$ point, define $\hat{A}$ as in the first paragraph of Part 4 using $\kappa_0 = 2\kappa_*$ and a suitable function $\wp$. It follows from Lemmas 2.8 and 2.9 that a disk of radius $c_0 \mathrm{r}^{-1/2}$ centered on any point in S where $\hat{A}$ is not flat must contribute at least c_0^{-1} to the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over S. Use this last observation with (2-15) to see that there can be at most c_0 pairwise disjoint disks in S where $\hat{A}$ is not flat. It follows that $1-|\alpha|^2 < \frac{1}{2}\kappa_*^{-1}$ on the boundary of the disk centered at p with radius at most $c_0 \mathrm{r}^{-1/2}$. In particular, the connection $\hat{A}$ is flat near the boundary of the latter disk and $\alpha/|\alpha|$ is $\hat{A}$–covariantly constant. As the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over this disk is nonzero, it is a positive integer. These last two facts require a zero of α in this disk because the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over the disk is the sum of the local Euler numbers at the zeros of α in the disk. What follows summarizes this. There exists $z \in [1, c_0]$ and at most G pairwise disjoint disks in S of radius $z\mathrm{r}^{-1/2}$ with the following properties:

(2-16)
- $1-|\alpha|^2$ is less than $\frac{1}{2}\kappa_*^{-1}$ on the complement of the union of these disks.
- The integral of $\frac{i}{2\pi} F_{\hat{A}}$ over each disk is a positive integer, and this integer for any given disk is the sum of the local Euler numbers of the zeros of α in the disk.

Use the fact that there are at most G such disks to see that there is a set of at most G disks of radius at most $(z+c_0)\mathrm{r}^{-1/2}$ such that each disk in the set obeys (2-16) and such that the distance between pairwise distinct disks from the set is greater than c_0^{-1}. Let N denote the number of elements in this set. Enumerate this set of disks as $\{D_S^{(i)}\}_{1 \le i \le \mathrm{N}}$. For each index $i \in \{1, \dots, \mathrm{N}\}$ and for each $t \in \mathbb{R}$, use $D_{S,t}^{(i)}$ to denote the disk in Y that is obtained from $D_S^{(i)}$ by moving its points for time t along the integral curves of v. Let t_S denote the value of f on S. If $t_S + t \in (1+\delta, 2-\delta)$, then each $D_{S,t}^{(i)}$ is a disk in the $t_S + t$ level set of f. It follows from (2-15) and from the comment in the final paragraph of Part 2 that there exists $c_0 \in (1, c_0)$ with the following property: If $t_S + t \in (1+\delta, 2-\delta)$ and if $|t| < c_0^{-1}$, then $1-|\alpha|^2 < \kappa_*^{-1}$ on the complement of $\bigcup_{1 \le i \le \mathrm{N}} D_{S,t}^{(i)}$ and the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over any given $D_t^{(i)}$ is the same as its integral over $D_S^{(i)}$. Meanwhile, the diameter of $D_{S,t}^{(i)}$ is bounded by $c_0 \mathrm{r}^{-1/2}$ and the pairwise separation between $D_{S,t}^{(i)}$ and $D_{S,t}^{(i')}$ when $i \ne i'$ is at least c_0^{-1}.

Granted the preceding observations, let t be such that $t_S + t \in [1+\delta, 2-\delta]$ and such that $|t| < c_0^{-1}$. Let S' denote the $t_S + t$ level set. Define N' and $\{D_{S'}^{(i)}\}_{1 \le i \le \mathrm{N}'}$ as done above for the case of S. It follows from what was said in the preceding paragraph that $\mathrm{N}' = \mathrm{N}$ and that the set $\{D_{S'}^{(i)}\}_{1 \le i \le \mathrm{N}}$ can be labeled so that any given $D_{S'}^{(i)}$ has nonempty intersection with $D_{S,t}^{(i)}$ and has distance at most c_0^{-1} from any $i' \ne i$ version of $D_{S,t}^{(i)}$. If $\mathrm{r} \ge c_0$, then these facts when applied sequentially to some $2\mathrm{N} \le c_0$ level sets of f starting with Σ, and then with f value $\frac{3}{2} \pm c_0^{-1}$, then with f value $\frac{3}{2} \pm 2c_0^{-1}$, and so on lead to the following:

(2-17)
- The integer N is independent of the value of $f \in [1+\delta, 2-\delta]$.
- There is a set of N segments of integral curves of v with the following properties:
  (a) Each segment starts on the $f = 1+\delta$ level set and ends on the $f = 2-\delta$ level set.
  (b) Distinct segments from the set have pairwise separation no less than c_0^{-1}.
  (c) The intersection of each segment with a level set of f has distance at most $c_0 \mathrm{r}^{-1/2}$ from a zero of α.
  (d) $1 - |\alpha|^2 < \frac{1}{2}\kappa_*$ at the points in the $f \in (1+\delta, 2-\delta)$ part of $M_\delta \cup \mathcal{H}_0$ with distance greater than $c_0^2 \mathrm{r}^{-1/2}$ from the union of the segments in this set.

Part 6 explains why $\mathrm{N} = \mathrm{G}$ and it says more about the start- and endpoints of (2-17)'s integral curve segments.

Part 6 Fix $\mathfrak{p} \in \Lambda$ and $\varepsilon > 0$ so as to reintroduce from Part 4 the subset $\mathcal{H}_{\mathfrak{p},\varepsilon} \subset \mathcal{H}_{\mathfrak{p}}$. By way of a reminder, this is the subset of points with distance greater than ε from $\hat{\gamma}_{\mathfrak{p}}^{+} \cup \hat{\gamma}_{\mathfrak{p}}^{-}$ and such that either $1 - 3\cos^2\theta \le 0$ or $f(u)|\cos\theta|\sin^2\theta > \frac{2}{3\sqrt{3}}(x_0 + 4e^{-2R}) + \varepsilon$. Lemma 2.9 finds $c_\varepsilon > 1$ such that if $\mathrm{r} > c_\varepsilon$, then $1 - |\alpha|^2 \le c_0 \mathrm{r}^{-1/2}$ on $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p},\varepsilon}$. Fix $\varepsilon \le x_0$ and assume henceforth that $\mathrm{r} > c_\varepsilon$.

Given what was just said, the segments of integral curves of v that arise in (2-17) intersect the $f \le 1 + \delta_*^2$ part of M_δ in the union of the radius δ_* coordinate balls about the index 1 critical points of f and they likewise intersect the $f > 2 - \delta_*^2$ part of M_δ in the union of the radius δ_*^2 coordinate balls about the index 2 critical points of f. Moreover, each such intersection lies where $1 - 3\cos^2\theta > 0$ and $|\cos\theta| \le c_0 x_0 \delta^{-2}$. This understood, Lemma 2.10 implies that $1 - |\alpha|^2 < c_0 \mathrm{r}^{-1}$ on the $|\cos\theta| > c_0 x_0 \delta^{-2}$ portion of the $|u| \ge R + \ln\delta$ part of $\mathcal{H}_{\mathfrak{p}}$.

Define $\hat{A}$ as in Part 5. As the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over any given constant u sphere in $\mathcal{H}_{\mathfrak{p}}$ is equal to 1, and as $\hat{A}$ is flat where $1 - |\alpha|^2 < \frac{1}{2}\kappa_*$, it follows from what was just said that the radius δ_* coordinate ball about any given index 1 critical point of f must contain the starting point of one of (2-17)'s integral curve segments. By the same token, the radius δ_* coordinate ball about any given index 2 critical point of f must contain the ending point of one of (2-17)'s integral curve segments. This can happen only if $\mathrm{N}=\mathrm{G}$.

Granted that $\mathrm{N}=\mathrm{G}$, then the following must also hold: Let D denote an embedded disk in M_δ that intersects just one of (2-17)'s integral curve segments and is such that its boundary has distance $c_0\mathrm{r}^{-1/2}$ or greater from all of (2-17)'s integral curve segments. Then the integral of $\frac{i}{2\pi}F_{\hat{A}}$ over D is equal to 1.

Part 7 Let T_ε denote the set of points with distance ε or less from $\alpha^{-1}(0)$. What is said by Lemma 2.10, by (2-17) and in Part 6 verify all of Proposition 2.4 on $M_\delta\cup\mathcal{H}_0$ but for the assertion that $\alpha^{-1}(0)$ is transversal in $Y-T_\varepsilon$ with G components and with tangent line very close to v. As explained next, these as yet unproved assertions are all consequences of Lemmas 2.9 and 2.10.

To see about a proof, fix $R>c_0$ for the moment and let $D\subset M_\delta\cup\mathcal{H}_0$ denote a smoothly embedded, transverse disk of radius $2R\mathrm{r}^{-1/2}$ and center on one of (2-17)'s integral curve segments. Assume that v is normal to D at its center point. Define $(A_\mathrm{r},\alpha_\mathrm{r})$ as in Lemma 2.10 and fix $\varepsilon'>0$. According to Lemma 2.10, there is a constant $\mathrm{r}_{\varepsilon'}$ that depends only on ε' and is such that if $\mathrm{r}\ge\mathrm{r}_{\varepsilon'}$, then there is a solution to (2-8) with C^2–distance at most ε' from $(A_\mathrm{r},\alpha_\mathrm{r})$ on the disk of radius R centered at the origin in $\mathbb{C}$. Let (A_0,α_0) denote such a solution.

To say more about (A_0,α_0), keep in mind that $\hat{A}$ is flat and $\alpha/|\alpha|$ is $\hat{A}$–covariantly constant along D at points with distance greater than some fixed multiple of $\mathrm{r}^{-1/2}$ that is independent of R. Denote this multiple as r and assume that $R\gg r$. If $\varepsilon'<c_0^{-1}$, it then follows that $1-|\alpha_0|^2<\frac{3}{4}\kappa_*^{-1}$ on the radius r disk about the origin in $\mathbb{C}$. It also follows that the (A_0,α_0) version of the connection $\hat{A}$ is flat on the annulus in $\mathbb{C}$ with inner radius r and outer radius R, and is such that $\alpha_0/|\alpha_0|$ is $\hat{A}_0$–covariantly constant on this same annulus Moreover, the integral of the (A_0,α_0) version of $\frac{i}{2\pi}F_{\hat{A}}$ over any centered disk in $\mathbb{C}$ with radius between r and R must equal 1. This understood, then α_0 must have at least one zero in the centered, radius r disk. Given that α_0 is $\bar\partial_{A_0}$–holomorphic, there can be at most one such zero and it is necessarily nondegenerate with local Euler number 1.

Use $z_0\in\mathbb{C}$ to denote this zero. Given the aforementioned holomorphicity, α_0 must appear near z_0 as $\alpha_0=\varsigma(z-z_0)+\mathfrak{e}$, where $\varsigma\in\mathbb{C}$ and with $\mathfrak{e}$ such that $|\mathfrak{e}|\le\varsigma'|z-z_0|^2$. Note in this regard $|\varsigma|\ge c_0^{-1}$ and $|\varsigma'|<c_0$, this being a consequence of the fact that the equations in (2-8) are elliptic modulo the action of $C^\infty(\mathbb{C};S^1)$ and so any sequence of

solutions has a subsequence that converges up to this group action in the C^∞ Fréchet topology on compact subsets of $\mathbb{C}$. This same sequential compactness property implies that $|\alpha_0| \geq c_0^{-1}|z-z_0|/(1+|z-z_0|)$ in the radius R disk about $0 \in \mathbb{C}$.

These last facts about α_0 have the implications that follow with regards to α. First, if $\varepsilon' < c_0^{-1}$, then α has a single, transverse zero in D with distance at most $c_0\varepsilon'\mathrm{r}^{-1/2}$ from D's center point. To give the second implication, use J as before to define the $(1,0)$ and $(0,1)$ parts of the complexification of the 2–plane bundle $\mathrm{Ker}(\hat{a})$. Introduce by way of notation $\partial_A\alpha$ to denote the $(1,0)$ part of $\nabla_A\alpha$, this being the homomorphism from the $(1,0)$ part of this complexification to E that is defined by restricting the domain of $\nabla_A\alpha$. It must also be the case that $|\partial_A\alpha| \geq c_0\mathrm{r}^{1/2}$ at this zero of α. Note in this regard that the corresponding restriction $\bar{\partial}_A\alpha$ of $\nabla_A\alpha$ to the $(0,1)$ part of $\mathrm{Ker}(\hat{a}) \otimes_{\mathbb{R}} \mathbb{C}$ is equal to the directional covariant derivative of β along v and so has norm bounded by c_0. By way of a reminder, the directional covariant derivative of α along v was denoted by $(\nabla_A\alpha)_v$ and, being a linear combination of covariant derivatives of β, it too has norm bounded by c_0.

What was just said as applied to transverse disks along the various components of (2-17)'s integral curve segments verifies the claim that $\alpha^{-1}(0)$ is transverse and it verifies the claim that each component of the radius $c_0\mathrm{r}^{-1/2}$ tubular neighborhood of (2-17)'s integral curve segments contains precisely one component. To see about the tangent line to a component, parametrize a neighborhood of a given point in a segment at unit speed by a map from a small interval about the origin in $\mathbb{R}$ to Y. Use x to denote this map. Let ∂_t denote the Euclidean vector field on $\mathbb{R}$. Then $x_*\partial_t$ pairs with $\nabla_A\alpha$ to give zero. With this in mind, write $x_*\partial_t$ at the origin in $\mathbb{R}$ as $x_v v + x_{(1,0)} + x_{(0,1)}$, where $x_{(1,0)}$ is the projection of $x_*\partial_t$ to the $(1,0)$ part of $\mathrm{Ker}(\hat{a}) \otimes_{\mathbb{R}} \mathbb{C}$, and $x_{(0,1)}$ is the complex conjugate of $x_{(1,0)}$. The fact that $\nabla_A\alpha$ annihilates $x_*\partial_t$ means that

$$x_v(\nabla_A\alpha)_v + x_{(1,0)}\partial_A\alpha + x_{(0,1)}\bar{\partial}_A\alpha = 0. \tag{2-18}$$

Given what is said in the preceding paragraph about the relative sizes of the various projections of the covariant derivative of α, the equality in (2-18) cannot hold unless

$$\mathrm{r}^{1/2}|x_{(1,0)}| \leq c_0|x_v|. \tag{2-19}$$

This last inequality implies the claim about the tangent vector to $\alpha^{-1}(0)$.

Part 8 What is said by Lemma 2.10 and by (2-17) with what is said in Parts 6 and 7 verify the assertions of Proposition 2.4 for the $M_\delta \cup \mathcal{H}_0$ part of Y. The upcoming Lemma 2.12 is used in a moment to extend the domain where these assertions hold into $\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}$. To set the stage for this lemma, fix $\mathfrak{p}\in\Lambda$. Given $\mathrm{D}>0$, use $\mathcal{H}_{\mathfrak{p},\mathrm{D}}$ to denote the set of points in $\mathcal{H}_{\mathfrak{p}}$ with distance at least D from all points in $\hat{\gamma}_{\mathfrak{p}}^{+}\cup\hat{\gamma}_{\mathfrak{p}}^{-}$.

Lemma 2.12 *There exists $\kappa>1$ with the following significance: Fix $\mathrm{r}\geq\kappa$ and fix $\mu\in\Omega$ with $\mathcal{P}$–norm less than 1. Let $(A,\psi=(\alpha,\beta))$ denote a solution to the (r,μ) version of* (1-13)*. If n is a given positive integer, set $\mathrm{D}(n)=(1+\kappa^{-1})^n\kappa\mathrm{r}^{-1/2}$. Assume that n is such that the assertions of Proposition 2.4 hold in $M_\delta\cup\mathcal{H}_0\cup\big(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p},\mathrm{D}(n+1)}\big)$. Then the assertions of Proposition 2.4 also hold in $M_\delta\cup\mathcal{H}_0\cup\big(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p},\mathrm{D}(n)}\big)$.*

This lemma is proved in a moment.

To finish the proof of Proposition 2.4, introduce κ from Lemma 2.12 and let N denote the least integer such that $\mathcal{H}_{\mathfrak{p},\mathrm{D}(N+1)}\subset\mathcal{H}_{\mathfrak{p}}\cap M_\delta$ for all $\mathfrak{p}\in\Lambda$. The assertions of Proposition 2.4 have been verified on $M_\delta\cup\mathcal{H}_0\cup\big(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p},\mathrm{D}(N+1)}\big)$. This understood, invoke Lemma 2.12 a total of N times to prove sequentially that the assertions of Proposition 2.4 hold on $M_\delta\cup\mathcal{H}_0\cup\big(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p},\mathrm{D}(N)}\big)$, then on $M_\delta\cup\mathcal{H}_0\cup\big(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p},\mathrm{D}(N-1)}\big)$, etc. □

Proof of Lemma 2.12 Fix $L>1$ and set $\mathrm{D}(n)=(1+L^{-1})^nL\mathrm{r}^{-1/2}$. Suppose that the assertions of Proposition 2.4 hold on $M_\delta\cup\mathcal{H}_0\cup\big(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p},\mathrm{D}(n+1)}\big)$. The proof that they also hold on $M_\delta\cup\mathcal{H}_0\cup\big(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p},\mathrm{D}(n)}\big)$ for a suitable, r–independent choice of L is given in seven steps.

Step 1 Only the third bullet of Proposition 2.4 needs comment when $1-|\alpha|^2<\kappa_*^{-1}$ on $\mathcal{H}_{\mathfrak{p},\mathrm{D}(n)}-\mathcal{H}_{\mathfrak{p},\mathrm{D}(n+1)}$. In any event, the third bullet restates part of Lemma 2.3 and so it holds whether or not $1-|\alpha|^2<\kappa_*^{-1}$ on the whole of $\mathcal{H}_{\mathfrak{p},\mathrm{D}(n)}-\mathcal{H}_{\mathfrak{p},\mathrm{D}(n+1)}$.

Step 2 Fix $\mathfrak{p}\in\Lambda$. Given $c\geq c_0$, suppose that D_0 denotes an embedded disk in $\mathcal{H}_{\mathfrak{p}}$ with radius $c\mathrm{r}^{-1/2}$ whose points have distance at least $(c_0+10^8)c\mathrm{r}^{-1/2}$ from both $\hat{\gamma}_{\mathfrak{p}}^{+}$ and $\hat{\gamma}_{\mathfrak{p}}^{-}$. Assume in addition that the vector field v along D_0 is at all points c_0^{-1} close to the normal vector. For example, a transverse disk has this last property.

Use $d_*(t)$ to denote the distance from $\hat{\gamma}_{\mathfrak{p}}^{+}\cup\hat{\gamma}_{\mathfrak{p}}^{-}$ to the point at time t along the integral curve of v that starts at the center point of D_0. Let D_t denote the time-t flow of D_0 under v. It follows from (1-3)'s formula for v (see equation (II.2-3)) that there exists $\lambda\geq c_0^{-1}$ with the following property: either one or both of the inequalities $d_*(t)\geq d_*(0)e^{\lambda t}$ and $d_*(-t)\geq d_*(0)e^{\lambda|t|}$ hold if t is such that the relevant point at time t on the integral curve is in $\mathcal{H}_{\mathfrak{p}}$. The discussion that follows assumes the first of these conditions and t is assumed implicitly to be nonnegative.

The rest of this step contains observations on the geometry of D_0 and the $t > 0$ versions of D_t. Assume for all of these that the center point of D_t is in $\mathcal{H}_{\mathfrak{p}}$. Granted this assumption, then $d_*(t)$ can serve as a proxy of sorts for the distance between any given point in D_t and $\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-$. In particular, all points in D_t have distance at least $(1 - c_0^{-1}) d_*(t)$ from $\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-$ and distance at most $(1 + c_0^{-1}) d_*(t)$ from $\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-$ because the points in D_0 have distance at least $10^8 c\mathrm{r}^{-1/2}$ from $\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-$ and at most $c\mathrm{r}^{-1/2}$ from each other.

It is also the case that the image in D_t of two points in D_0 with a given distance ρ from each other are distance at most $\rho e^{\lambda t}$ apart in D_t. This is to say that the disk D_t is not seriously distorted if $t < c_0^{-1}$. Moreover, if $t \le c_0^{-1}$, then the normal vector to D_t at all points will be close to v.

There is one other point to keep in mind about D_0, this concerning the number of intersection points between D_0 and a given segment in $\mathcal{H}_{\mathfrak{p}}$ of an integral curve of v. In particular, D_0 has at most one intersection with any such segment. This is proved using the formula for v in (1-3) given the assumption that $d_*(0)$ is at least c_0 times D_0's diameter.

Step 3 Assume in this step that the function $\mathfrak{f}$ from (2-9) is such that $\mathfrak{f}(0)$ is greater than $c^{1/3}$. Given that $d_*(t) \ge d_*(0) e^{\lambda t}$, so $d_*(t) \ge (1 + L^{-1})^3 d_*(0)$ if $t \ge 3\lambda^{-1} \ln(1 + L^{-1})$. Set $t_* = 100\lambda^{-1} \ln(1 + L^{-1})$ and use (2-20) to see that

$$\mathfrak{f}(t_*) \ge 10^{-2} c^{1/3} - c_0 \ln(1 + L^{-1}). \tag{2-20}$$

Suppose that $D_0 \subset \mathcal{H}_{\mathfrak{p},\mathrm{D}(n-1)}$ and that $c > c_0(1 + \ln(1 + L^{-1}))^3$. If such is the case, then the inequality in (2-20) is not compatible with the assumption that the assertions of Proposition 2.4 hold on $M_\delta \cup \mathcal{H}_0 \cup \big(\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p},\mathrm{D}(n+1)}\big)$. It follows as a consequence that $\mathfrak{f}(0)$ can be no greater than $c^{1/3}$ if $c > c_0(1 + \ln(1 + L^{-1}))^3$. Assume this bound for c in what follows and likewise assume that $D_0 \subset \mathcal{H}_{\mathfrak{p},\mathrm{D}(n-1)}$ so as to guarantee that $\mathfrak{f}(0) < c^{1/3}$.

Step 4 Fix $R \ge 2$ but less than $c_0^{-1} c^{2/3}$. Since $\mathfrak{f}(0) < c^{1/3}$, the bounds from Lemma 2.1 for $|\nabla_A \alpha|$ requires a point $x \in [1, Rc^{1/3}]$ such that $1 - |\alpha|^2 \le c_0 R^{-1}$ on the concentric annulus in D_0 with inner radius $x\mathrm{r}^{-1/2}$ and outer radius $(x + 1)\mathrm{r}^{-1/2}$. With this understood, use Lemma 2.3 to deduce the following: if $R \ge c_0 \kappa_*$, then $1 - |\alpha|^2 \le \frac{1}{16} \kappa_*^{-1}$ on such an annulus. Assume in what follows that $c > c_0$ is such that R can be chosen greater than $c_0 \kappa_*$. Reintroduce the connection $\hat{A}$ from Part 4 as defined with $\kappa_0 = 2\kappa_*$. This connection is flat and $\alpha|\alpha|^{-1}$ is $\hat{A}$–covariantly constant on this annuli in D_0.

Step 5 Take D_0 to be a transverse disk. With κ_* denoting as before the constant from Lemma 2.3, assume in addition that $1-|\alpha|^2 \geq \kappa_*^{-1}$ at the center point of D_0. This assumption with Lemmas 2.8 and 2.9 guarantee a point with distance at most $c_0 \mathrm{r}^{-1/2}$ from the origin in D_0 where $1-|\alpha|^2 > \frac{9}{10}$. If $c > c_0$, then the existence of such a point implies that the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over the subdisk in D_0 of radius $(x+1)\mathrm{r}^{-1/2}$ is nonzero and positive. Moreover, this integral must be a positive integer because $\hat{A}$ is flat with a covariantly constant section near the boundary of this subdisk. Use n_1 in what follows to denote this integer. Keep in mind that α has a zero in this subdisk with positive local Euler number because the sum of the local Euler numbers of the zeros of α in the subdisk is equal to this same n_1.

Use (1-13) with Lemma 2.1's bound on $|\nabla_A \beta|$ to draw the following conclusion: there exists $t_0 \geq c_0^{-1}$ such that $1-|\alpha|^2 < \frac{1}{2}\kappa_*^{-1}$ at all times $t \leq t_0$ on the image in D_t of the annulus with inner radius $x\mathrm{r}^{-1/2}$ and outer radius $(x+1)\mathrm{r}^{-1/2}$. This being the case, the integral of $\frac{i}{2\pi} F_{\hat{A}_0}$ over the image in D_t of the radius $(x+1)\mathrm{r}^{-1/2}$ subdisk in D_0 is still equal to n_1 and α still has at least one zero with positive local Euler number in the image in D_t of the radius $(x+1)\mathrm{r}^{-1/2}$ subdisk of D_0.

With the preceding understood, remark that if $L \geq c_0$, then $t_0 > t_*$ with t_* as defined in Step 3. Assume that this is the case. Then $D_{t_0} \subset \mathcal{H}_{\mathfrak{p},\mathrm{D}(n+1)}$ and, as a consequence, n_1 must equal 1 because Proposition 2.4's assertions hold on $M_\delta \cup \mathcal{H}_0 \cup \big(\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p},\mathrm{D}(n+1)}\big)$.

Step 6 Let $x \in [1, Rc^{1/3}]$ denote the constant from Step 4. It follows from what is said in Step 3 that there is a zero of α in D_0 with local Euler number 1 with distance at most $c_0\mathrm{r}^{-1/2}$ from the center of D_0. Use D_0 now to denote the transverse disk through this point with radius $(1-c_0^{-1})c\mathrm{r}^{-1/2}$. The conclusions of the preceding steps hold for this new version of D_0 also. In particular, there exists $x \in [1, Rc^{1/3}]$ such that the connection $\hat{A}$ is flat with covariantly constant section $\alpha|\alpha|^{-1}$ on the concentric annulus with inner and outer radii $x\mathrm{r}^{-1/2}$ and $(x+1)\mathrm{r}^{-1/2}$. Let $D_\diamond$ denote the concentric subdisk in D_0 with radius $(x+1)\mathrm{r}^{-1/2}$. As before, the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over $D_\diamond$ is equal to 1. This value of 1 for the integral of $\frac{i}{2\pi} F_{\hat{A}}$ demands the following:

(2-21) There exists $z \geq 1$ that is independent of (A,α), μ and r such that if $\mathrm{r} \geq z$, then $1-|\alpha|^2 \leq \frac{1}{32}\kappa_*^{-1}$ on the part of $D_\diamond$ with distance greater than $z\mathrm{r}^{-1/2}$ from the origin.

What follows explains why (2-21) is true. To start, let $p \in D_\diamond$ denote a given point where $1-|\alpha|^2 \geq \frac{1}{32}\kappa_*^{-1}$. Use Lemmas 2.8 and 2.9 to find a point p' in the transverse disk of radius $c_0 \mathrm{r}^{-1/2}$ through p where $|\alpha|$ is less than 10^{-5}. Since v is almost normal to this transverse disk at p' and also to $D_\diamond$, there is a point in $D_\diamond$ on the integral curve of v through p' with distance at most $c_0 \mathrm{r}^{-1/2}$ from p'. Let p'' denote the latter point. As noted previously, the Dirac equation in (1-13) identifies the covariant derivative of α in the direction of v with a linear combination of covariant derivatives of β. This understood, then it follows from Lemma 2.1 that $|\alpha|$ at p'' is no greater than 10^{-4}. Given this upper bound for $|\alpha|$, use Lemma 2.1's upper bound for $|\nabla_A \alpha|$ to see that the contribution to the integral over $D_\diamond$ of $\frac{i}{2\pi} F_{\hat{A}}$ from the radius $c_0 \mathrm{r}^{-1/2}$ disk in $D_\diamond$ centered at p'' is greater than c_0^{-1}. This conclusion has the following consequence: There can be at most c_0 points in any subset of $D_\diamond$ such that the distance between any two distinct points is greater than $c_0 \mathrm{r}^{-1/2}$ and $1 - |\alpha|^2 > \frac{1}{32} \kappa_*^{-1}$ at each point.

Now, suppose that p is a point in $D_\diamond$ with $1 - |\alpha|^2 > \frac{1}{32} \kappa_*^{-1}$. It follows from what was just said and from Lemmas 2.1 and 2.3 that there exists an (A, ψ)–, μ– and r–independent constant $z_1 \geq 10^4$ and a subdisk $D_p \subset D_\diamond$ centered at p with radius $z_1 \mathrm{r}^{-1/2}$ with the following properties: First, $1 - |\alpha|^2 < \frac{1}{32} \kappa_*^{-1}$ on the annular neighborhood of the boundary of D_p with inner and outer radii equal to $\frac{1}{2} z_1 \mathrm{r}^{-1/2}$ and $z_1 \mathrm{r}^{-1/2}$. Second, the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over D_p is at least c_0^{-1}. This last property implies that the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over D_p is at least 1 since the connection $\hat{A}$ is flat with $\alpha/|\alpha|$ covariantly constant on the annular boundary neighborhood.

Since the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over the whole of $D_\diamond$ is equal to 1, the conclusions of the preceding paragraph have the following consequence: Any two versions of D_p are certain to overlap. It follows that $1 - |\alpha|^2$ is less than $\frac{1}{32} \kappa_*^{-1}$ at any point in $D_\diamond$ with distance $c_0 \mathrm{r}^{-1/2}$ or greater from the center point since α is zero at this point. This last observation verifies Proposition 2.4's fourth bullet for $M_\delta \cup \mathcal{H}_0 \cup \big(\bigcup_{\mathfrak{p} \in \Lambda} \mathcal{H}_{\mathfrak{p}, \mathrm{D}(n)}\big)$.

Step 7 Suppose that $t \in [0, t_0]$. Use $D_{\diamond t} \subset D_t$ to denote the image of $D_\diamond$. The definition of t_0 is such as to guarantee that $1 - |\alpha|^2$ is no greater than $\frac{1}{2} \kappa_*$ on the image in $D_{\diamond t}$ of the set of points in $D_\diamond$ with distance at most $c_0 \mathrm{r}^{-1/2}$ from the origin. This understood, the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over $D_{\diamond t}$ is equal to 1. Granted these last conclusions, an essentially verbatim repetition of what is said in Part 7 proves that the zero locus of α in the cylinder $\bigcup_{1 \leq t \leq t_0} D_{\diamond t}$ is transverse and consists of a properly embedded arc whose tangent vector has angle at most $c_0 \mathrm{r}^{-1/2}$ from v and whose points have distance at most $c_0 \mathrm{r}^{-1/2}$ from the integral curve of v through the center point in $D_\diamond$. Since $D_{\diamond t_0} \subset \mathcal{H}_{\mathfrak{p}, \mathrm{D}(n+1)}$, this arc smoothly extends the zero locus of α in $\mathcal{H}_{\mathfrak{p}, \mathrm{D}(n+1)}$. The preceding observations verify Proposition 2.4's first bullet for $M_\delta \cup \mathcal{H}_0 \cup \big(\bigcup_{\mathfrak{p} \in \Lambda} \mathcal{H}_{\mathfrak{p}, \mathrm{D}(n)}\big)$. □

3 The map Φ^{r} from $\mathcal{Z}^L_{\mathrm{ech},M}$ to $\mathcal{Z}_{\mathrm{SW,r}}$

Fix $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 and fix $L \geq 1$. The map $\hat{\Phi}_{\mathrm{r}}\colon \hat{\mathcal{Z}}^L_{\mathrm{ech},M} \to \hat{\mathcal{Z}}_{\mathrm{SW,r}}$ for Theorem 1.5 is a principal $\mathbb{Z}$–bundle covering map over a map from $\mathcal{Z}^L_{\mathrm{ech},M}$ into $\mathcal{Z}_{\mathrm{SW,r}}$ that is denoted in what follows by Φ^{r}.

The following proposition makes a formal assertion as to the existence of the desired map Φ^{r}. It then says something about the form of the solutions to the relevant version of (1-13) that lie in the $C^\infty(Y; S^1)$ orbits in $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ that form Φ^{r}'s image. The proposition uses the isomorphism in (1-19) to identify $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ with $\mathcal{Z}_{\mathrm{SW,r}} \times \mathbb{Z}$ and it uses the isomorphism described in the paragraph preceding Theorem 1.5 to identify $\hat{\mathcal{Z}}_{\mathrm{ech},M}$ with $\mathcal{Z}_{\mathrm{ech},M} \times \mathbb{Z}$ and thus $\hat{\mathcal{Z}}^L_{\mathrm{ech},M}$ with $\mathcal{Z}^L_{\mathrm{ech},M} \times \mathbb{Z}$. The proposition also uses the following notation: when γ denotes a closed integral curve of v, then ℓ_γ denotes its length.

Proposition 3.1 *There exists $\kappa > \pi$ such that given $\mathrm{E} > 1$ and $L > \kappa \mathrm{E}$, there exists $\kappa_L > \kappa$ with the following significance: Fix $\mathrm{r} \geq \kappa_L$ and an element $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Use the solutions to the (r, μ) version of (1-13) to define $\mathcal{Z}_{\mathrm{SW,r}}$. There exists a 1–1 map $\Phi^{\mathrm{r}}\colon \mathcal{Z}^L_{\mathrm{ech},M} \to \mathcal{Z}_{\mathrm{SW,r}}$ whose image contains the subset of $\mathcal{Z}_{\mathrm{SW,r}}$ with $\mathrm{M} < \mathrm{E}$. Moreover, suppose that $\Theta \in \mathcal{Z}^L_{\mathrm{ech},M}$ and that $\mathfrak{c} = (A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of (1-13) on the $C^\infty(Y; S^1)$ orbit defined by $\Phi^{\mathrm{r}}(\Theta)$. Then:*

- *$\mathfrak{c}$ is nondegenerate and holonomy nondegenerate.*
- $\mathrm{M}(\mathfrak{c}) < 2\pi \sum_{\gamma \in \Theta} \ell_\gamma + \kappa^{-1}$.
- *The zero locus of α is a disjoint union of embedded circles whose components are in 1–1 correspondence with the integral curves of v that constitute Θ. This correspondence is such that:*
  (1) *Any given component of $\alpha^{-1}(0)$ lies in the radius $\kappa \mathrm{r}^{-1/2}$ tubular neighborhood of its partner from Θ and it is isotopic in this neighborhood to its partner.*
  (2) $\big|1 - |\alpha|^2\big| \leq \kappa(e^{-\sqrt{\mathrm{r}}d/\kappa} + \mathrm{r}^{-1})$ *at points with distance d or more from $\bigcup_{\gamma \in \Theta} \gamma$.*
  (3) *Let $D \subset Y$ denote an oriented, embedded disk with all boundary points at distance greater than $\kappa \mathrm{r}^{-1/2}$ from $\bigcup_{\gamma \in \Theta} \gamma$ and with algebraic intersection number 1 with $\bigcup_{\gamma \in \Theta} \gamma$. Then $\frac{i}{2\pi} \int_D F_{\hat{A}} = 1$.*

- *View Φ^{r} as a map from $\mathcal{Z}^L_{\mathrm{ech},M} \times \mathbb{Z}$ to $\mathcal{Z}_{\mathrm{SW,r}} \times \mathbb{Z}$ that acts as the identity on the $\mathbb{Z}$ factor. As such, Φ^{r} defines a $\mathbb{Z}$–equivariant covering map $\hat{\Phi}^{\mathrm{r}}\colon \hat{\mathcal{Z}}^L_{\mathrm{ech},M} \to \hat{\mathcal{Z}}_{\mathrm{SW,r}}$ via the isomorphisms described above that reverses the sign of the relative $\mathbb{Z}$ or $\mathbb{Z}/p_M\mathbb{Z}$ degrees.*

The map Φ^{r} is constructed by copying in an almost verbatim fashion some of what is done in Section 3 of [20] to construct an analogous map in the context where $\hat{a}$ is a contact 1–form and $w = \frac{1}{2} d\hat{a}$. The latter version of Φ^{r} is the map that is described in Theorems 4.2 of [19] and Theorem 1.1 of [20]. This contact 1–form incarnation of Φ^{r} is constructed in Section 3 of [20] and its salient properties are stated as Theorem 1.1 in [21] and Theorem 1.1 in [22]. These theorems are proved respectively in Section 2 of [21] and Section 2 of [22]. As explained below, only the simplest case of what is done in Section 3 of [20], Section 2 of [21] and Section 2 of [22] are needed for what follows because of certain special features of the closed integral curves of v that arise from elements in $\mathcal{Z}_{\mathrm{ech},M}$.

By way of a look ahead, Section 3.1 summarizes some basic facts about a particular subset of solutions to the vortex equations that are used to construct Φ^{r}. The proof of Proposition 3.1 is given in Section 3.2. Section 3.3 has the proof of Lemma 3.2 from Section 3.1.

3.1 The vortex equations, II

The proof of Proposition 3.1 makes reference to (2-8)'s vortex equations. Of relevance here are the solutions which are such that $1 - |\alpha_0|^2$ is integrable on $\mathbb{C}$. The discussion of this subset of solutions to (2-8) has four parts. What is said in Parts 1–3 summarize various observations from Section 2 in [20].

Part 1 As all complex line bundles over $\mathbb{C}$ are isomorphic to the product line bundle $\mathbb{C} \times \mathbb{C}$, no generality is lost by the focus in what follows on solutions (A_0, α_0) with A_0 a connection on this product bundle and α_0 a complex function. Introduce θ_0 to denote the product connection on the product line bundle $\mathbb{C} \times \mathbb{C}$. Write any given connection on $\mathbb{C} \times \mathbb{C}$ as $\theta_0 + \hat{a}$ with $\hat{a}$ being an $i\mathbb{R}$–valued 1–form on $\mathbb{C}$. Doing so identifies the set of solutions to (2-8) with a subset of the space $C^\infty(\mathbb{C}; iT^*\mathbb{C}) \times C^\infty(\mathbb{C}; \mathbb{C})$. This identification endows the set of solutions with a topology. Meanwhile, there is a free action of $C^\infty(\mathbb{C}; S^1)$ on the space of solutions to (2-8) whereby a given map u sends a given solution (A_0, α_0) to $(A_0 - u^{-1}du, u\alpha_0)$. This action is continuous, and so the set of $C^\infty(\mathbb{C}; S^1)$ equivalence classes of solutions has the induced quotient topology. The resulting subspace of solutions with $1-|\alpha_0|^2$ being integrable is a disjoint union of components labeled by the nonnegative integers. The integer m component consists of the set of equivalence classes of solutions that obey

$$\frac{1}{2\pi}\int_{\mathbb{C}}(1-|\alpha_0|^2)=m. \tag{3-1}$$

The integer m component is denoted by $\mathfrak{C}_m$.

The space $\mathfrak{C}_m$ has the structure of a complex manifold that is holomorphically isomorphic to $\mathbb{C}^m$. The m complex functions $\{\sigma_q\}_{1\le q\le m}$ defined by

$$\sigma_q=\frac{1}{2\pi}\int_{\mathbb{C}}z^q(1-|\alpha|^2) \tag{3-2}$$

define such an isomorphism. (In (3-2) and in what follows, what is denoted by z is the complex coordinate for $\mathbb{C}$.) There are no convergence issues with regards to the integral in (3-2) by virtue of the fact that a solution to (2-8) and (3-1) obeys

(3-3)
- $|\alpha_0|\le 1$, with equality if and only if $|\alpha_0|=1$;
- $1-|\alpha_0|^2\le c_0e^{-\operatorname{dist}(\cdot,\alpha_0^{-1}(0))}$.

Here, c_0 depends only on the integer m. As it turns out, the zeros of α are the roots of the polynomial $z\mapsto \wp(z)=z^m+\sigma_1z^{m-1}+\sigma_2z^{m-2}+\cdots+\sigma_m$.

Part 2 Let $L\to\mathbb{C}$ denote a Hermitian line bundle. Suppose for the moment that (A_0,α_0) defines a pair of unitary connection on L and section of L. Define the operator ϑ on $C^\infty(\mathbb{C},L)$ by the rule

$$(x,\iota)\mapsto\big(\partial x+\tfrac{1}{\sqrt{2}}\bar\alpha_0\iota,\ \bar\partial_{A_0}\iota+\tfrac{1}{\sqrt{2}}\alpha_0x\big). \tag{3-4}$$

Here, ∂ is the holomorphic derivative on $C^\infty(\mathbb{C};\mathbb{C})$. The formal L^2–adjoint of ϑ is denoted by $\vartheta^\dagger$ and it is given by the rule

$$\vartheta^\dagger(c,\varsigma)=\big(-\bar\partial c+\tfrac{1}{\sqrt{2}}\bar\alpha_0\varsigma,\ -\partial_{A_0}\varsigma+\tfrac{1}{\sqrt{2}}\alpha_0c\big). \tag{3-5}$$

The corresponding Laplacian $\vartheta\vartheta^\dagger$ can be written as

$$\vartheta\vartheta^\dagger(c,\varsigma)=\big((-\partial\bar\partial+\tfrac12|\alpha_0|^2)c,\ (-\bar\partial_{A_0}\partial_{A_0}+\tfrac12|\alpha_0|^2)\varsigma\big)+\tfrac{1}{\sqrt{2}}(\partial_{A_0}\bar\alpha_0\varsigma,\ \bar\partial_{A_0}\alpha_0c). \tag{3-6}$$

What follows is an important observation to keep in mind: the right-most term in (3-6) is zero when (A_0,α_0) obeys the vortex equations.

Part 3 Suppose now that (A_0, α_0) is a solution to (2-8) and (3-1). The $(1,0)$–tangent space to the orbit of (A_0, α_0) in $\mathfrak{C}_m$ is canonically isomorphic to the vector space of square-integrable pairs (x, ι) of complex-valued functions that are annihilated by ϑ. This identification is used implicitly in what follows. The vector space of square-integrable elements annihilated by ϑ is called the *kernel* of ϑ.

Introduce the Hermitian inner product on the kernel of ϑ defined by the rule that sends an ordered pair $(\mathfrak{w} = (x, \iota), \mathfrak{w}' = (x', \iota'))$ in the kernel of ϑ to

$$\langle \mathfrak{w}, \mathfrak{w}' \rangle = \frac{1}{\pi} \int_{\mathbb{C}} (\bar{x} x' + \bar{\iota} \iota'). \tag{3-7}$$

This Hermitian inner product is compatible with the complex structure and it defines a complete Kähler metric on $\mathfrak{C}_m$. In the case $m = 1$, this is the standard metric on $\mathfrak{C}_m = \mathbb{C}$, but such is not the case if $m > 1$.

Given a real number ν and a complex number μ, define the function $\hbar$ on $\mathfrak{C}_m$ by the following rule: if $\mathfrak{c} = (A_0, \alpha_0)$, then

$$\hbar(\mathfrak{c}) = \frac{1}{4\pi} \int_{\mathbb{C}} (2\nu |z|^2 + \mu \bar{z}^2 + \bar{\mu} z^2)(1 - |\alpha_0|^2). \tag{3-8}$$

Suppose now that ν and μ depend on $t \in \mathbb{R}$, so that (3-6) defines a function on $\mathbb{R} \times \mathfrak{C}_m$. The Kähler metric on $\mathfrak{C}_m$ defines an associated symplectic form, and the latter with the $\mathbb{R}$–dependent function $\hbar$ define a corresponding 1–parameter family of Hamiltonian vector fields on $\mathfrak{C}_m$. An integral curve of this 1–parameter family of vector fields constitutes a map, $t \mapsto \mathfrak{c}(t) \in \mathfrak{C}_m$, from $\mathbb{R}$ to $\mathfrak{C}_m$ that obeys the equation

$$\tfrac{1}{2} i\, \mathfrak{c}' + \nabla^{(1,0)} \hbar|_{\mathfrak{c}} = 0, \tag{3-9}$$

where $\mathfrak{c}'$ is shorthand for the $(1,0)$ part of $\mathfrak{c}_* \frac{d}{dt}$, and where $\nabla^{(1,0)} \hbar$ denotes the $(1,0)$ part of the gradient of $\hbar$. Of interest in what follows are the solutions to (3-7) that obey the condition $\mathfrak{c}(t+T) = \mathfrak{c}(t)$ for some $T \geq 0$. Such a solution is said to be a *periodic* solution.

Let $\mathfrak{c}\colon S^1 \to \mathfrak{C}_m$ denote a given map. Associate to $\mathfrak{c}$ the bundle $\mathfrak{c}^* T_{1,0} \mathfrak{C}_m \to S^1$. The pullback of the Riemannian connection on $T\mathfrak{C}_m$ defines a unitary connection on S^1. The map $\mathfrak{c}$ is said to be *nondegenerate* when the operator

$$\zeta \mapsto \tfrac{i}{2} \nabla_t \zeta + (\nabla_{\zeta_{\mathbb{R}}} \nabla^{(1,0)} \hbar)|_{\mathfrak{c}} \tag{3-10}$$

on $C^\infty(S^1; \mathfrak{c}^* T_{1,0} \mathfrak{C}_m)$ has trivial kernel. The notation here is such that ∇_t denotes the covariant derivative of the aforementioned unitary connection. Also, $(\nabla_{\zeta_{\mathbb{R}}} \nabla^{(1,0)} \hbar)|_{\mathfrak{c}}$

denotes the covariant derivative at $\mathfrak{c}$ along the vector defined by ζ in $T\mathfrak{C}_m|_{\mathfrak{c}}$ of the vector field $\nabla^{1,0} \hbar \in C^\infty(\mathfrak{C}_m; T_{1,0}\mathfrak{C}_m)$.

Part 4 Let γ denote an integral curve of v. The proof of Proposition 3.1 refers to a certain pair of $\mathbb{R}$–valued and $\mathbb{C}$–valued functions on γ that are associated to a given unitary isomorphism between $K^{-1}|_\gamma$ and $\gamma \times \mathbb{C}$. To define these functions, fix a $\mathbb{C}$–linear, unitary isomorphism between $K^{-1}|_\gamma$ and $\gamma \times \mathbb{C}$. Let z denote the complex coordinate on $\mathbb{C}$ and let t denote an affine parameter for γ such that $\gamma_* \frac{\partial}{\partial t} = \frac{1}{2\pi}\ell_\gamma v$. Use the metric's exponential map along γ to give a tubular neighborhood of γ in Y with the coordinates (t, z) with it understood that these coordinates are only valid when z is in a small radius disk about the origin in $\mathbb{C}$. Use these coordinates with the first-order Taylor's expansion to write w as $w = \frac{i}{2}\, dz \wedge d\bar{z} - 2(\nu z + \mu \bar{z})\, d\bar{z} \wedge dt - 2(\nu \bar{z} + \bar{\mu} z)\, dz \wedge dt + \cdots$, where ν and μ are respectively $\mathbb{R}$– and $\mathbb{C}$–valued functions on S^1, and where the unwritten terms are bounded by $c_0|z|^2$. Note that ν must be $\mathbb{R}$–valued because w is closed.

The pair (ν, μ) is the desired pair. Use this pair to define the function $\hbar$. This done, fix a nonnegative integer m and use $\mathfrak{C}_{(\gamma,m)}$ to denote the set of periodic solutions to (3-9) on $\mathfrak{C}_m$.

Lemma 3.2 *Suppose that* $\Theta \in \mathcal{Z}_{\mathrm{ech},M}$ *and that* γ *is a closed integral curve of* v *from* Θ.

- *The space* $\mathfrak{C}_{(\gamma,1)}$ *consists of the constant map from* S^1 *to the* $\sigma_1 = 0$ *point in* $\mathfrak{C}_1$. *This is the equivalence of solutions to* (2-8) *and the* $m = 1$ *version of* (3-1) *with* $\alpha_0^{-1}(0) = 0$. *This solution is nondegenerate.*
- *Suppose that* $\mathfrak{p} \in \Lambda$ *and that* $\gamma \in \{\hat{\gamma}_{\mathfrak{p}}^+, \hat{\gamma}_{\mathfrak{p}}^-\}$. *Then* $\mathfrak{C}_{(\gamma,m)} = \varnothing$ *when* $m > 1$.

The proof of Lemma 3.2 constitutes Section 3.3.

3.2 Proof of Proposition 3.1

The proof differs only cosmetically from that of Theorem 4.2 in [19]. As with the proof of the latter, there are three parts: Part 1 constructs the map Φ^{r} and verifies what is said in the second and third bullets; Part 2 proves what is said in the first and fourth bullets; and Part 3 verifies that the image of Φ^{r} contains the $\mathrm{M} < \mathrm{E}$ subset of $\mathcal{Z}_{\mathrm{SW,r}}$.

Part 1 The map Φ^{r} is constructed by copying what is done for its namesake in Theorem 1.1 from Section 1d in [20]. The construction here constitutes what is perhaps the simplest of cases because only $\Theta \in \mathcal{Z}^L_{\mathrm{ech},M}$ and $\gamma \in \Theta$ versions of $\mathfrak{C}_{(\gamma,1)}$ are used. By way of a parenthetical remark, the first step in the construction of Φ^{r} uses the data from Θ to build a pair in $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ that comes very close to solving (1-14). This construction is described in the first subsection of the appendix.

The first bullet of Lemma 3.2 guarantees that each map from $\{\{\mathfrak{C}_{(\gamma,1)}\}_{\gamma\in\Theta}\}_{\Theta\in\mathcal{Z}_{\mathrm{ech},M}}$ is nondegenerate, and so this set can be used as input for Theorem 1.1 in [20]. The assertions made by the second and third bullets all follow directly from the construction and from Lemmas 2.1 and 2.3.

Part 2 To see about the first bullet of the proposition, suppose that $\Theta \in \mathcal{Z}^L_{\mathrm{ech},M}$ and that $\mathfrak{c} = (A,\alpha)$ is a solution to (1-13) that defines the equivalence class $\Phi^{\mathrm{r}}(\Theta)$. The assertion that $\mathfrak{c}$ is nondegenerate can be had by copying almost verbatim the arguments in Section 2a of [21] that prove the analogous assertion in Theorem 1.1 from [21]. There are no substantive changes to these arguments from [21]. The assertion that $\mathfrak{c}$ is holonomy nondegenerate follows from the third and fourth bullets of Proposition 2.4. To elaborate, these bullets imply that the connection $\hat{A}$ has a covariantly constant section on a neighborhood of the curve $\gamma^{(z_0)}$. Because of this, the number $\mathrm{X}(\hat{A} - A_E)$ is necessarily an integer because A_E was chosen to have holonomy 1 around $\gamma^{(z_0)}$.

To argue for the fourth bullet, fix $\Theta \in \mathcal{Z}^L_{\mathrm{ech},M}$ and suppose that (A,α) is a solution to (1-13) that defines the equivalence class $\Phi^{\mathrm{r}}(\Theta)$. Fix a 2–cycle $Z \in H_2(M,[\Theta]-[\Theta_0])$ that has algebraic intersection number zero with $\gamma^{(z_0)}$. The $\mathbb{Z}$–equivariant covering map $\hat{\Phi}^{\mathrm{r}}$ sends the equivalence class of a pair (Θ, Z) to the $\mathcal{G}_{M_\Lambda}$–orbit of a solution (A,ψ) of (1-13) with the property that $\mathrm{X}(\hat{A} - A_E) = 0$. With this point understood, the argument for the third bullet differ only cosmetically from those used [21] to prove an analogous assertion from Theorem 4.2 in [19]. The latter theorem follows directly from the relative degree assertion about the namesake Φ^{r} that appear in Theorem 1.1 in [21]. The proof of this part of [21, Theorem 1.1] constitute Sections 2b and 2c of [21]. Note in this regard that the assumption that is made in equation (2.56) in [21] is not needed by virtue of the fact that the map Φ^{r} is constructed using only elements from the set $\{\{\mathfrak{C}_{(\gamma,1)}\}_{\gamma\in\Theta}\}_{\Theta\in\mathcal{Z}_{\mathrm{ech},M}}$.

Part 3 But for one additional substantive remark, the arguments for Theorem 1.1 in [22] can copied with only notational changes to prove that if L is large, then Φ^{r}

maps $\mathcal{Z}^L_{\mathrm{ech},M}$ onto the $\mathrm{M} < \mathrm{E}$ subset of $\mathcal{Z}_{\mathrm{SW,r}}$ when r is large. The extra remark concerns the input to Theorem 1.1 of [22] of a set denoted by $\mathfrak{C}\mathcal{Z}^L$ and a subset $\mathfrak{C}\mathcal{Z}^{L*} \subset \mathfrak{C}\mathcal{Z}^L$. Theorem 1.1 in [22] requires $\mathfrak{C}\mathcal{Z}^{L*}$ to be the whole of $\mathfrak{C}\mathcal{Z}^L$. As explained below, $\mathfrak{C}\mathcal{Z}^L$ in this case is $\{\prod_{\gamma\in\Theta} \mathfrak{C}_{(\gamma,1)}\}_{\Theta\in\mathcal{Z}^L_{\mathrm{ech},M}}$ and $\mathfrak{C}\mathcal{Z}^{L*}$ in this case is indeed all of $\mathfrak{C}\mathcal{Z}^L$.

To define $\mathfrak{C}\mathcal{Z}^L$, introduce first $\mathcal{Z}$ to denote the set whose typical element consists of a finite collection of pairs whose first entry is a closed integral curve of v and whose second entry is a positive integer. Let Θ denote such a collection. This set is constrained in two ways: Distinct pairs from Θ contain distinct closed orbits of v. The second constraint requires $[\Theta] = \sum_{(\gamma,m)\in\Theta} m[\gamma] \in H_1(Y;\mathbb{Z})$ to be the class that is defined by the elements in $\mathcal{Z}_{\mathrm{ech},M}$. The set $\mathcal{Z}_{\mathrm{ech},M}$ sits in $\mathcal{Z}$, but $\mathcal{Z}$ is strictly larger than $\mathcal{Z}_{\mathrm{ech},M}$; this can be seen using the parametrization given next.

Invoke Proposition II.2.8 or Theorem I.2.1 to write $\mathcal{Z}$ as $\mathcal{Z}_{\mathrm{HF}} \times \big(\prod_{\mathfrak{p}\in\Lambda}(\mathbb{Z}\times\hat{\mathrm{O}})\big)$, where $\hat{\mathrm{O}}$ is the set $\{0,1,2,\dots\}\times\{0,1,2,\dots\}$. This parametrization is such that the factor $\mathcal{Z}_{\mathrm{HF}} \times \big(\prod_{\mathfrak{p}\in\Lambda}\mathbb{Z}\big)$ parametrizes pairs of the form $(\gamma,1)$ with $\gamma \subset M_\delta \cup \big(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}\big)$ crossing each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ once. To explain the factors of $\hat{\mathrm{O}}$, fix $\mathfrak{p}\in\Lambda$. The element $\{0,0\}$ in $\mathfrak{p}$'s version of $\hat{\mathrm{O}}$ signifies that neither $\hat{\gamma}_{\mathfrak{p}}^+$ nor $\hat{\gamma}_{\mathfrak{p}}^-$ appears in Θ. The element $(m_+,0)$ from $\hat{\mathrm{O}}$ with $m_+>0$ signifies that Θ contains $(\hat{\gamma}_{\mathfrak{p}}^+, m_+)$ and that Θ lacks a pair with $\hat{\gamma}_{\mathfrak{p}}^-$. By the same token, the element $(0,m_-)$ from $\hat{\mathrm{O}}$ with $m_->0$ signifies that Θ contains the pair $(\hat{\gamma}_{\mathfrak{p}}^-, m_-)$ and that Θ lacks a pair with $\hat{\gamma}_{\mathfrak{p}}^+$. The element (m_+,m_-) with both entries positive signifies the appearance in Θ of $(\hat{\gamma}_{\mathfrak{p}}^+, m_+)$ and $(\hat{\gamma}_{\mathfrak{p}}^-, m_-)$. Use $\mathcal{Z}^L$ to denote the subset of $\Theta = (\hat{v}, \mathfrak{k}_{\mathfrak{p}}, (m_{\mathfrak{p}+}, m_{\mathfrak{p}-})_{\mathfrak{p}\in\Lambda}) \in \mathcal{Z}$ with $\sum_{\mathfrak{p}\in\Lambda}(\mathfrak{k}_{\mathfrak{p}} + 2m_{\mathfrak{p}+} + 2m_{\mathfrak{p}-}) < L$.

The set $\mathfrak{C}\mathcal{Z}^L$ maps to $\mathcal{Z}^L$ with fiber over any given Θ being $\prod_{(\gamma,m)\in\Theta}\mathfrak{C}_{(\gamma,m)}$. The fiber over Θ of $\mathfrak{C}\mathcal{Z}^{L*}$ consists of the elements in $\prod_{(\gamma,m)\in\Theta}\mathfrak{C}_{(\gamma,m)}$ whose entries are nondegenerate.

Granted these definitions, invoke the second bullet in Lemma 3.2 to see that $\mathfrak{C}\mathcal{Z}^L$ is indeed $\{\prod_{\gamma\in\Theta}\mathfrak{C}_{(\gamma,1)}\}_{\Theta\in\mathcal{Z}^L_{\mathrm{ech},M}}$. This being the case, invoke the first bullet in Lemma 3.2 to see that $\mathfrak{C}\mathcal{Z}^{L*} = \mathfrak{C}\mathcal{Z}^L$. □

3.3 Proof of Lemma 3.2

To prove the first bullet, view $\mathfrak{C}_1$ as $\mathbb{C}$ using (3-2)'s coordinate σ_1. Viewed in this way, then (3-9) is an equation for a function $t \mapsto z(t)$ from $\mathbb{R}$ to $\mathbb{C}$, this being the equation

$$\frac{i}{2}\frac{d}{dt}z + \nu z + \mu\bar{z} = 0. \tag{3-11}$$

Let $z(\cdot)$ denote a solution, but viewed as a map from $\mathbb{R}$ to $\mathbb{R}^2$. Then $z(2\pi)$ can be written as $U_\gamma z(0)$, where $U_\gamma \in \mathrm{SL}(2;\mathbb{Z})$ is the linear return map that is described in Part 3 of Section II.1F. Proposition 2.7 of [9] asserts that all of the relevant integral curves are hyperbolic, and by definition, this means that U_γ has real eigenvalues with neither being 1 or -1. Thus (3-11) has a single solution, this being the constant map $t \mapsto 0$. The operator (3-10) in this case is the operator that appears on the left side of (3-11), and so its kernel is trivial.

The proof of the second bullet has ten parts. Parts 1–3 say more about the solutions to the vortex equation. The remaining parts contain the proof proper. The arguments in Parts 4–10 focus on the case when $\gamma = \hat{\gamma}_{\mathrm{p}}^{+}$. The arguments when $\gamma = \hat{\gamma}_{\mathrm{p}}^{-}$ are essentially identical.

Part 1 The lemma that follows supplies three facts that play a central role in the proof of Lemma 3.2.

Lemma 3.3 *Fix $m \geq 1$ and suppose that (A_0, α_0) is a solution to* (2-8) *that defines a point in $\mathfrak{C}_m$. Then:*

- $\frac{1}{\pi}\int_{\mathbb{C}}\bigl(\frac{1}{2}(1-|\alpha_0|^2)+|\partial_{A_0}\alpha_0|^2\bigr) = m$.
- $\frac{1}{2\pi}\int_{\mathbb{C}}\frac{1}{2}(1-|\alpha_0|^2) \geq \frac{2}{5}m$ *and* $\frac{1}{\pi}m\int_{\mathbb{C}}|\partial_{A_0}\alpha_0|^2 \leq \frac{3}{5}m$.
- $|\partial_{A_0}\alpha_0| \leq \frac{\sqrt{3}}{2}(1-|\alpha_0|^2)$.

Proof Use Δ in what follows to denote the Laplacian on $\mathbb{C}$. Meanwhile, let w denote the function $(1-|\alpha_0|^2)$ and use g to denote $\partial_{A_0}\alpha_0$. It follows from (2-8) that

$$-\tfrac{1}{4}\Delta\mathrm{w} + \tfrac{1}{2}|\alpha|^2\mathrm{w} = |\mathrm{g}|^2 \quad\text{and}\quad -\tfrac{1}{4}\nabla_A^\dagger\nabla_A\mathrm{g} + \tfrac{1}{2}|\alpha|^2\mathrm{g} = \tfrac{3}{4}\mathrm{wg}. \tag{3-12}$$

Write $|\alpha_0|^2 = -\mathrm{w}+1$ to write the left-most equation in (3-12) as

$$-\tfrac{1}{4}\Delta\mathrm{w} + \tfrac{1}{2}\mathrm{w} = \tfrac{1}{2}\mathrm{w}^2 + |\mathrm{g}|^2. \tag{3-13}$$

Integrate both sides of this equation and appeal to (3-1) to obtain the first bullet in the lemma.

The second bullet follows from the first and the third. To elaborate, use the third bullet of the lemma to see that the integral on the left-hand side of the first bullet is less than $\frac{5}{4\pi}$ times the integral of w^2. As a consequence, the contribution of the term $\frac{1}{2}\mathrm{w}^2$ to the integral on the left side of the first bullet is no less than $\frac{2}{5}m$ and so the contribution to this integral of $|\mathrm{g}|^2$ is no greater than $\frac{3}{5}m$.

To obtain the third bullet, use the right-hand identity in (3-12) to see that

$$-\tfrac{1}{4}\Delta|\mathrm{g}| + \tfrac{1}{2}|\alpha|^2|\mathrm{g}| \le \tfrac{3}{4}\mathrm{w}|\mathrm{g}|. \tag{3-14}$$

To exploit (3-14), set $\mathrm{x} = |\mathrm{g}| - \frac{\sqrt{3}}{2}\mathrm{w}$. The left-most equation in (3-12) and (3-14) require

$$-\tfrac{1}{4}\Delta\mathrm{x} + \tfrac{1}{2}|\alpha|^2\mathrm{x} \le |\mathrm{g}|\big(\tfrac{3}{4}\mathrm{w} - \tfrac{\sqrt{3}}{2}|\mathrm{g}|\big) = -\tfrac{\sqrt{3}}{2}|\mathrm{g}|\mathrm{x}. \tag{3-15}$$

Granted (3-15), use the maximum principle to see that x cannot have a positive local maximum. Given that x is square-integrable, this implies that $\mathrm{x} \le 0$, which is what is asserted by the second bullet of the lemma. □

Part 2 Various additional facts about any given $m \in \{1, 2, \dots\}$ version of $\mathfrak{C}_m$ are needed for the proof. The first of these facts concerns an isometric and holomorphic action of the semidirect product of S^1 and $\mathbb{C}$ on $\mathfrak{C}_m$. This action is induced by the group's action on $\mathbb{C}$, where S^1 acts as the group of rotations about the origin and $\mathbb{C}$ acts on itself by translation. The generator of the action of $\mathbb{C}$ on $\mathfrak{C}_m$ at the equivalence class of a solution (A_0, α_0) to (2-8) is the tangent vector that is defined by the element

$$\mathfrak{w}_1 = \big(\tfrac{1}{\sqrt{2}}(1 - |\alpha_0|^2), \partial_{A_0}\alpha_0\big) \tag{3-16}$$

in the kernel of ϑ. The action of S^1 on $\mathfrak{C}_m$ is such that $\eta \in S^1$ pulls back (3-2)'s functions $\{\sigma_q\}_{1\le q\le m}$ to $\{\eta^q\sigma_q\}_{1\le q\le m}$. The action has a unique fixed point in $\mathfrak{C}_m$, this given by the point where all σ_q are zero. The latter point is the equivalence class of the solutions to (2-8) with $\alpha^{-1}(0) = 0$. The fixed point of the S^1 action is called the *symmetric vortex*.

Part 3 Define (3-4)'s operator ϑ using the solution (A_0, α_0). The absence of the right-most term in (3-6) and the integrability of $1 - |\alpha_0|^2$ imply that $\vartheta\vartheta^\dagger$ has a bounded inverse that maps $L^2(\mathbb{C}; \mathbb{C}\times\mathbb{C})$ to the L^2–orthogonal complement of the kernel of ϑ.

The other Laplacian, $\vartheta^\dagger\vartheta$, can be written as $\vartheta^\dagger\vartheta = \vartheta\vartheta^\dagger + \mathfrak{e}$, where $\mathfrak{e}$ is a zeroth-order term that is bounded by $c_0(1 - |\alpha_0|^2)$. Given this last fact, the Bochner–Weitzenböck formula for $\bar\partial_A\partial_A$ can be used in conjunction with the left-most equation in (3-12) and the maximum principle to see that any given element in the kernel of ϑ with L^2–norm 1 is bounded pointwise by $c_0(1 - |\alpha_0|^2)$. The argument also invokes the third bullet of Lemma 3.3 and (3-3). Granted the latter as input, the argument differs little from the argument in Part 1 proving the third bullet in Lemma 3.3. This being the case, the details are omitted.

Part 4 Use (3-2)'s coordinates $\{\sigma_q\}_{1\le q\le m}$ for $\mathfrak{C}_m$ so as to view the equation in (3-9) as an equation for a map, $t \mapsto (\sigma_1(t), \sigma_2(t), \dots, \sigma_m(t))$, from $\mathbb{R}$ to $\mathbb{C}^m$. In particular, the map $t \mapsto \sigma_2(t)$ must obey the equation

$$\frac{i}{2}\frac{d}{dt}\sigma_2 + 2\nu\sigma_2 + \mu g^{2\bar{2}} = 0, \tag{3-17}$$

where $g^{2\bar{2}}$ is the norm of $d\sigma_2$ as defined by the Kähler metric on $\mathfrak{C}_m$. Note that $g^{2\bar{2}}$ is a strictly positive function on $\mathfrak{C}_m$. Parts 6–10 explain why

$$g^{2\bar{2}} > 2|\sigma_2|. \tag{3-18}$$

Meanwhile, Part 5 explains why the functions ν and μ in (3-8) and (3-17) can be assumed constant, with μ real and such that $\mu > |\nu| + c_0^{-1}$. With the preceding understood, write σ_2 as $\sigma_x + i\sigma_y$ with σ_x and σ_y real-valued functions. Then (3-17) and (3-18) require $-\frac{d}{dt}\sigma_y < 0$ and so there are no periodic solutions.

Part 5 The functions ν and μ that appear in (3-8) and (3-17) depend on a chosen unit-length basis vector for the bundle K^{-1} along the given closed integral curve, this being $\hat{\gamma}_{\mathrm{p}}^{+}$. Even so, the question of existence or not of solutions to the corresponding version of (3-9) does not depend on the trivialization. This fact is exploited in what follows to choose a convenient trivialization.

The metric on $\mathcal{H}_{\mathrm{p}}$ is invariant with respect to rotations of the coordinate ϕ and as v is a constant multiple of $-\frac{\partial}{\partial\phi}$ along $\hat{\gamma}_{\mathrm{p}}^{+}$ the basis vector for K^{-1} along $\hat{\gamma}_{\mathrm{p}}^{+}$ can be chosen so as to be covariantly constant along $\hat{\gamma}_{\mathrm{p}}^{+}$. Choosing such a basis vector gives a pair (ν, μ) with both being constants. As noted previously, ν must be real, and if μ is not real and nonnegative to begin with, a suitable constant rotation of $\mathbb{C}$ changes the coordinates so that the resulting version of μ is real and nonnegative.

The assertion $\mu > |\nu|$ follows from the fact that $\hat{\gamma}_{\mathrm{p}}^{+}$ is hyperbolic. By way of an explanation, the fact that μ and ν are constant can be used to solve (3-11) and thus write the matrix U_γ and see directly its eigenvalues. These are real and neither 1 nor -1 if and only if $\mu > |\nu|$.

By way of a parenthetical remark, Part 2 of Section III.5A introduces the coordinates $(s_+, \phi_+, \theta_+, u_+)$ for the product of $\mathbb{R}$ with a tubular neighborhood in Y of $\hat{\gamma}_{\mathrm{p}}^{+}$. These are such that the locus $\theta_+ = 0$, $u_+ = 0$ is the cylinder $\mathbb{R} \times \hat{\gamma}_{\mathrm{p}}^{+}$ with s_+ being the Euclidean coordinate for the $\mathbb{R}$ factor and ϕ_+ an $\mathbb{R}/(2\pi\mathbb{Z})$–valued coordinate for $\hat{\gamma}_{\mathrm{p}}^{+}$. This understood, the differentials $d\theta_+$ and du_+ together define a trivialization of the normal bundle of $\hat{\gamma}_{\mathrm{p}}^{+}$. Given this trivialization, the coefficients that appear in equation (III.5-1) determine ν and μ as functions of the constants x_0 and R that are used in Section 1.1 to define the geometry of Y. A direct calculation using these coordinates will also verify the claim that ν and μ can be assumed constant, with μ real and greater than $|\nu|$.

Part 6 Suppose that (A_0, α_0) is a solution to (2-8) that defines a point in $\mathfrak{C}_m$. Let $\mathfrak{w} = (x, \iota)$ denote an element in the kernel of the operator ϑ. The $(1,0)$ part of $d\sigma_2$ pairs with the tangent vector defined by $\mathfrak{w}$ to give

$$-\frac{1}{2\pi}\int_{\mathbb{C}} z^2\bar{\alpha}\iota. \tag{3-19}$$

Since $\vartheta\mathfrak{w} = 0$, the first entry of (3-4) is zero and so the integrand in (3-19) can be replaced by $-\sqrt{2}z^2\partial x$. Having done so, integration by parts writes (3-19) as

$$-\frac{\sqrt{2}}{\pi}\int_{\mathbb{C}} zx. \tag{3-20}$$

Note in this regard that such an integration by parts is possible here (and in a subsequent integration by parts) by virtue of what is said in Part 3 to the effect that $|\mathfrak{w}|$ is bounded by a multiple of $1-|\alpha_0|^2$, and thus is exponentially small where $|z|$ is large.

The integrand in (3-20) is the same as $z(1-|\alpha_0|^2)x + z\bar{\alpha}_0\alpha_0 x$. As $\vartheta\mathfrak{w} = 0$, the left-hand entry in (3-4) is zero, and so this is the same as $z(1-|\alpha_0|^2)x - \sqrt{2}z\bar{\alpha}_0\bar{\partial}_{A_0}\iota$. Use this fact with a second integration by parts to see that (3-19) is equal to

$$-\frac{2}{\pi}\int_{\mathbb{C}}\bigl(\tfrac{1}{\sqrt{2}}z(1-|\alpha|^2)x + z\overline{\bar{\partial}_A\alpha}\iota\bigr). \tag{3-21}$$

Introduce Π to denote the L^2–orthogonal projection from $L^2(\mathbb{C};\mathbb{C}\oplus\mathbb{C})$ to the kernel of ϑ. This last identity implies that $d\sigma_2$ acts on the kernel of ϑ as $\mathfrak{w} \mapsto 2\langle\Pi(\bar{z}\mathfrak{w}_1), \mathfrak{w}\rangle$ with $\mathfrak{w}_1$ as defined by (3-16). It follows as a consequence that

$$g^{2\bar{2}} = 4\langle\Pi(\bar{z}\mathfrak{w}_1), \Pi(\bar{z}\mathfrak{w}_1)\rangle. \tag{3-22}$$

Meanwhile, $\Pi(\bar{z}\mathfrak{w}_1)$ can be written as $\bar{z}\mathfrak{w}_1 + \vartheta^{\dagger}\mathfrak{z}$ and so

$$\langle\Pi(\bar{z}\mathfrak{w}_1), \Pi(\bar{z}\mathfrak{w}_1)\rangle = \frac{1}{\pi}\int_{\mathbb{C}}|z|^2\bigl(\tfrac{1}{2}(1-|\alpha_0|^2)^2 + |\partial_{A_0}\alpha_0|^2\bigr) - \langle\vartheta^{\dagger}\mathfrak{z}, \vartheta^{\dagger}\mathfrak{z}\rangle. \tag{3-23}$$

With (3-23) in hand, write (3-22) as

$$g^{2\bar{2}} = \frac{4}{\pi}\int_{\mathbb{C}}|z|^2\bigl(\tfrac{1}{2}(1-|\alpha_0|^2)^2 + |\partial_{A_0}\alpha_0|^2\bigr) - 4\langle\vartheta^{\dagger}\mathfrak{z}, \vartheta^{\dagger}\mathfrak{z}\rangle. \tag{3-24}$$

The comparison of $g^{2\bar{2}}$ with $2|\sigma_2|$ uses (3-24) with the rewriting of σ_2 as

$$\sigma_2 = \frac{1}{\pi}\int_{\mathbb{C}} z^2\big(\tfrac{1}{2}(1-|\alpha_0|^2)^2 + |\partial_{A_0}\alpha_0|^2\big). \tag{3-25}$$

To obtain this last identity, multiply both sides of (3-13) by $\frac{1}{2\pi}z^2$ and integrate the resulting equation over $\mathbb{C}$. The integral of $\frac{1}{2\pi}z^2\Delta \mathrm{w}$ is zero.

The inequality $g^{2\bar{2}} > 2|\sigma_2|$ follows directly from (3-24) and (3-25) if

$$\frac{1}{\pi}\int_{\mathbb{C}} |z|^2\big(\tfrac{1}{2}(1-|\alpha_0|^2)^2 + |\partial_{A_0}\alpha_0|^2\big) - 2\langle\vartheta^\dagger\mathfrak{z}, \vartheta^\dagger\mathfrak{z}\rangle \ge 0. \tag{3-26}$$

The remaining Parts 7–10 supply a proof of this inequality.

Part 7 This step supplies an upper bound for $\langle\vartheta^\dagger\mathfrak{z}, \vartheta^\dagger\mathfrak{z}\rangle$. To this end, use (3-6) to see that $\mathfrak{z} = (0, \varsigma)$ with ς being the L^2–solution on $\mathbb{C}$ to the equation

$$-\bar{\partial}_{A_0}\partial_{A_0}\varsigma + \tfrac{1}{2}|\alpha_0|^2\varsigma = -\partial_{A_0}\alpha_0. \tag{3-27}$$

It follows as a consequence that

$$\langle\vartheta^\dagger\mathfrak{z}, \vartheta^\dagger\mathfrak{z}\rangle = \frac{1}{\pi}\int_{\mathbb{C}}\big(|\partial_{A_0}\varsigma|^2 + \tfrac{1}{2}|\alpha_0|^2|\varsigma|^2\big). \tag{3-28}$$

Granted (3-28), it then follows from (3-27) that $\langle\vartheta^\dagger\mathfrak{z}, \vartheta^\dagger\mathfrak{z}\rangle \le \frac{1}{\pi}\|\varsigma\|_2\|\partial_{A_0}\alpha_0\|_2$ with $\|\cdot\|_2$ denoting here the L^2–norm. To see about the L^2–norm of ς, commute derivatives using the top bullet in (2-2) to write the left-hand side of (3-27) as $-\partial_{A_0}\bar{\partial}_{A_0}\varsigma + \frac{1}{2}\varsigma$. Take the L^2 inner product of both sides of the resulting equation with ς. This leads to an equality between integrals. An application of Hölder's inequality to the latter equality finds $\frac{1}{2}\|\varsigma\|_2^2 \le \|\varsigma\|_2\|\partial_{A_0}\alpha_0\|_2$ and so $\|\varsigma\|_2 \le 2\|\partial_{A_0}\alpha_0\|_2$. This being the case, then

$$\langle\vartheta^\dagger\mathfrak{z}, \vartheta^\dagger\mathfrak{z}\rangle \le \frac{2}{\pi}\int_{\mathbb{C}} |\partial_{A_0}\alpha_0|^2. \tag{3-29}$$

Part 8 This part of the proof exploits another identity coming from (3-13),

$$\frac{1}{\pi}\int_{\mathbb{C}} |z|^2\big(\tfrac{1}{2}(1-|\alpha_0|^2)^2 + |\partial_{A_0}\alpha_0|^2\big) = \frac{1}{2\pi}\int_{\mathbb{C}} |z|^2(1-|\alpha_0|^2) - 2m. \tag{3-30}$$

To derive (3-30), multiply both sides of (3-13) by $\frac{1}{\pi}|z|^2$ and then integrate the resulting equation over $\mathbb{C}$. An integration by parts identifies the integral over $\mathbb{C}$ of $-\frac{1}{4\pi}|z|^2\Delta \mathrm{w}$ with that of $-\frac{1}{\pi}\mathrm{w}$. According to (3-1), the latter integral is equal to $-2m$.

With (3-30) in mind, digress for a moment to consider a certain constrained minimization problem for a real-valued, measurable function on $\mathbb{C}$. The problem asks for an infimum of the functional

$$u \mapsto \mathfrak{s}(u) = \frac{1}{2\pi}\int_{\mathbb{C}} |z|^2 u - 2m - 4\Big(m - \frac{1}{2\pi}\int_{\mathbb{C}} u^2\Big) \tag{3-31}$$

subject to the two constraints $0 \le u \le 1$ and $\frac{1}{2\pi}\int_{\mathbb{C}} u = m$. By way of an explanation, the function $u = 1 - |\alpha_0|^2$ obeys the constraints, and it follows from (3-13) with the first bullet of Lemma 3.3 that the value $\mathfrak{s}$ in this case is no greater than what is written on the left-hand side of (3-26). As a consequence, (3-26) follows if the infimum of $\mathfrak{s}$ is positive.

As explained in a moment, the functional $\mathfrak{s}$ takes on its minimum with the function, u_*, given as follows: Set $\lambda = 2m + 4$. Then

$$\begin{cases} u_* = 1 & \text{if } |z|^2 \le \lambda - 8, \\ u_* = \frac{1}{8}(\lambda - |z|^2) & \text{if } \lambda - 8 \le |z|^2 \le \lambda, \\ u_* = 0 & \text{if } |z|^2 \ge \lambda. \end{cases} \tag{3-32}$$

The value of $\mathfrak{s}$ on u_* is $\frac{1}{4}\lambda^2 - \frac{16}{3} - 6m = m^2 - 2m - \frac{4}{3}$, and this is positive for all $m \ge 3$.

With regards to (3-32), note first that an averaging argument shows that any minimizer is a function of the radial coordinate on $\mathbb{C}$. Meanwhile, the variational equations for $\mathfrak{s}$ assert that a constrained minimizer, u_*, is such that $|z|^3 + 8|z|u_* = \lambda|z|$ where $0 \le u_* \le 1$. Here, λ is the Lagrange multiplier for the constraint that the integral of $\frac{1}{2\pi}u$ is equal to m. Thus, the minimizer u_* has the form that is depicted in (3-32) with λ chosen so that this integral constraint is obeyed. A calculation finds that $\lambda = 2m + 4$ and a second calculation finds that $\mathfrak{s}(u_*) = \frac{1}{4}\lambda^2 - \frac{16}{3} - 6m$.

Part 9 The verification of (3-26) when $m = 2$ requires more care with regards to the difference between $|\sigma_2|$ and the integral of $\frac{1}{\pi}|z|^2\big(\frac{1}{2}(1 - |\alpha_0|^2)^2 + |\partial_{A_0}\alpha_0|^2\big)$. As explained in Part 10, this difference is no less than $2q$, where q is the integral of this same function in the case when (A_0, α_0) is a symmetric solution to (2-8) from the space $\mathfrak{C}_1$. By way of a reminder, the space $\mathfrak{C}_1$ is diffeomorphic to $\mathbb{C}$ with the diffeomorphism given by the function σ_1. The symmetric solution is the $\sigma_1 = 0$ point, this the solution with $\alpha^{-1}(0) = 0$. This step proves that $q > \frac{2}{3}$. Granted the latter, then (3-24) and what is said in Part 8 find

$$g^{2\bar{2}} > 2|\sigma_2| + 4q - \tfrac{8}{3} > 2|\sigma_2|. \tag{3-33}$$

To derive the asserted lower bound for q, introduce w to denote $1 - |\alpha_0|^2$ and introduce g to denote $\partial_{A_0}\alpha_0$ for a $\mathfrak{C}_1$ version of (A_0, α_0) with $\alpha_0^{-1}(0) = 0$. Use $\rho = |z|$ to denote the radial coordinate on $\mathbb{C}$. Then $\partial_\rho \mathrm{w} \le 0$ since w is rotationally symmetric and has no local maxima. Meanwhile, $|\partial_\rho \mathrm{w}| = \sqrt{2}\,|\alpha_0||\mathrm{g}| < \sqrt{2}\,|\mathrm{g}|$. What with Lemma 3.3, this finds $|\partial_\rho \mathrm{w}| < \frac{\sqrt{3}}{\sqrt{2}}|\mathrm{w}|$. Keeping in mind that w is exponentially small at large ρ, integration by parts finds that

$$0 = \int_0^\infty \partial_\rho(\rho^2 \mathrm{w})\, d\rho = 2\int_0^\infty \mathrm{w}\rho\, d\rho + \int_0^\infty (\partial_\rho \mathrm{w})\rho^2\, d\rho. \tag{3-34}$$

Given what was said about $\partial_\rho \mathrm{w}$, this last equation implies that

$$\int_0^\infty \mathrm{w}\rho^2\, d\rho > \frac{2\sqrt{2}}{\sqrt{3}} \int_0^\infty \mathrm{w}\rho\, d\rho = \frac{2\sqrt{2}}{\sqrt{3}}. \tag{3-35}$$

By way of explanation, (3-1) asserts that the integral on the right-hand side is equal to 1. To continue, use Hölder's inequality with (3-1) to see that the left-hand side of (3-35) is no less than

$$\left(\int_0^\infty \mathrm{w}\rho^3\, d\rho\right)^{1/2} \left(\int_0^\infty \mathrm{w}\rho\, d\rho\right)^{1/2} = \left(\int_0^\infty \mathrm{w}\rho^3\, d\rho\right)^{1/2}. \tag{3-36}$$

Taken together, (3-35) and (3-36) assert that

$$\int_0^\infty \mathrm{w}\rho^3\, d\rho > \tfrac{8}{3}. \tag{3-37}$$

This last equation with (3-30) say that $q > \frac{2}{3}$.

Part 10 Suppose now that (A_0, α_0) is a solution to (2-8) that defines a point in $\mathfrak{C}_2$. This step explains why

$$\frac{1}{\pi}\int_{\mathbb{C}} |z|^2\big(\tfrac{1}{2}(1-|\alpha|^2)^2 + |\partial_{A_0}\alpha_0|^2\big) > |\sigma_2| + 2q. \tag{3-38}$$

To this end, note that (3-38) holds if the left-hand side is greater than $2q$ plus the real part of $\bar{u}\sigma_2$ for any $u \in S^1$. Therefore, no generality is lost in proving that the left-hand side of (3-38) is greater than $2q$ plus the real part of σ_2. Reintroduce σ_x to denote this real part. Let $\sigma\colon \mathfrak{C}_2 \to (0,\infty)$ denote the function given by the left side integral in (3-38).

Lemma 3.4 *The function $\sigma - \sigma_x$ does not take on its infimum at any point in $\mathfrak{C}_2$. Furthermore, sequences in $\mathfrak{C}_2$ on which $\sigma - \sigma_x$ converges to its infimum have the following properties: Fix $R \ge 1$; all but a finite number of elements in the sequence are $C^\infty(\mathbb{C}; S^1)$–orbits of pairs (A_0, α_0) with α_0 such that its two zeros have distance R*

or greater between them. Moreover, these zeros have distance $\frac{1}{R}$ or less from the real z–axis in $\mathbb{C}$.

Granted for the moment Lemma 3.4, write the coordinate z as $x+iy$ with x and y being real, and then use (3-29) to write

$$\sigma - \sigma_x = \frac{1}{\pi}\int_{\mathbb{C}} y^2(1-|\alpha|^2) - 2m. \tag{3-39}$$

It follows from (2-4) in [21] that if $R \gg 1$ and (A_0, α_0) is as described in Lemma 3.4, then what is written on the left-hand side of (3-39) differs by at most $c_0 R^{-1}$ from twice its value for the case where $m = 1$ and (A_0, α_0) is the $\sigma_1 = 0$ solution in $\mathfrak{C}_1$. Meanwhile, the $\sigma_1 = 0$ solution in $\mathfrak{C}_1$ is invariant with respect to the S^1 action on $\mathfrak{C}_1$ and so the $m = 1$ and $\sigma_1 = 0$ version of (3-38) is equal to q. □

Proof of Lemma 3.4 Fix an element in $\mathfrak{C}_2$ and write the zeros of any corresponding solution to (2-8) as an unordered pair $(z_1, z_2) \in \mathrm{Sym}^2(\mathbb{C})$. Fix pairs, (A_1, α_1) and (A_2, α_2), of $m = 1$ solutions to (2-8) with $\alpha_1^{-1}(0) = z_1$ and with $\alpha_2^{-1}(0) = z_2$. Part 4 in Section 2a of [20] writes the given $\mathfrak{C}_2$ element as the $C^\infty(\mathbb{C}; S^1)$–orbit of an $m = 2$ solution to (2-8) that can be written as (A, α) with

$$A = A_1 + A_2 + (\bar\partial u\, d\bar z - \partial u\, dz) \quad \text{and} \quad \alpha = e^{-u}\alpha_1\alpha_2 \tag{3-40}$$

such that u is a smooth, real-valued function on $\mathbb{C}$ that obeys $|u| \le c_0 e^{-\operatorname{dist}(\cdot,\alpha^{-1}(0))/c_0}$. The top line in (2-8) requires u to obey

$$\Delta u = (1 - e^{-2u}|\alpha_1|^2|\alpha_2|^2) - (1 - |\alpha_1|^2) - (1 - |\alpha_2|^2). \tag{3-41}$$

Were $u \le 0$, then the right-hand side of (3-41) would be less than $-(1-|\alpha_1|^2)(1-|\alpha_2|^2)$ and thus not positive. This being the case, the maximum principle demands that $u > 0$. With this in mind, multiply both sides of (3-41) by y^2 and then integrate the result over $\mathbb{C}$. An integration by parts writes the integral of $y^2\Delta u$ as twice the integral of u. In particular, the integral of u is positive, and so

$$\int_{\mathbb{C}} y^2(1-|\alpha|^2) \ge \int_{\mathbb{C}} y^2(1-|\alpha_1|^2) + \int_{\mathbb{C}} y^2(1-|\alpha_2|^2). \tag{3-42}$$

Meanwhile, the bound $u \le c_0 e^{-\operatorname{dist}(\cdot,\alpha^{-1}(0))/c_0}$ implies that

$$\int_{\mathbb{C}} y^2(1-|\alpha|^2) \le \int_{\mathbb{C}} y^2(1-|\alpha_1|^2) + \int_{\mathbb{C}} y^2(1-|\alpha_2|^2) + c_0 e^{-|z_1 - z_2|/c_0}. \tag{3-43}$$

The assertions of Lemma 3.4 follow directly from (3-42) and (3-43). □

4 Instantons

The purpose of this section is to provide various facts about the solutions to the r and $\mathfrak{g} = \mathfrak{e}_\mu$ versions of (1-20), this being the version reproduced below:

$$\begin{cases} \frac{\partial}{\partial s} A + B_A - \mathrm{r}(\psi^\dagger \tau \psi - i\hat{a}) + \frac{1}{2} B_{A_K} - i * d\mu = 0, \\ \frac{\partial}{\partial s} \psi + D_A \psi = 0. \end{cases} \tag{4-1}$$

These facts assert a priori bounds on various integrals on pointwise norms.

4.1 A priori integral bounds

The analysis of (4-1) concerns the versions with $\mathrm{r} > \pi$ and with $\mu \in \Omega$ a given element with $\mathcal{P}$–norm bounded by 1. Assume in what follows that μ is such that all solutions to (1-13) are nondegenerate.

To set some notation, suppose that $\mathfrak{d}\colon \mathbb{R} \to \mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ is a given instanton solution to (4-1). The $s \to -\infty$ limit of $\mathfrak{d}$ is denoted by $\mathfrak{c}_-$ and the $s \to \infty$ limit by $\mathfrak{c}_+$. The latter are solutions to (1-13). The respective $\mathrm{Conn}(E)$ and $C^\infty(Y; \mathbb{S})$ components of $\mathfrak{d}$ are written as (A, ψ), and ψ is often written in two-component form as (α, β). The lemmas that follow use $\mathrm{A}_{\mathfrak{d}}$ to denote $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+)$.

Many of the lemmas here and in the rest of Section 4 have analogs in Section 3 of [22]. Except for one item, the statement of a given lemma here is virtually identical to the statement of its partner in Section 3 of [22]. Various lemmas in Section 3 of [22] give the option of assuming the lower bound $\mathfrak{f}_s(\mathfrak{c}_+) - \mathfrak{f}_s(\mathfrak{c}_-) > -\mathrm{r}^2$ in lieu of an upper bound on $\mathrm{A}_{\mathfrak{d}}$. Their partners here do not give such an option. This difference is due solely to the term $2\pi \mathrm{r} \mathfrak{f}_s$ in (1-29)'s formula for $\mathfrak{a}^{\mathfrak{f}}$. The version of $\mathfrak{a}^{\mathfrak{f}}$ used in Section 3 of [22] has $\mathfrak{f}_s$ appearing only as $-2\pi^2 \mathfrak{f}_s$ while the version here has $2\pi(\mathrm{r} - \pi)\mathfrak{f}_s$. Of relevance here is the sign difference when $\mathrm{r} > \pi$.

Except for what was just said about $\mathfrak{f}_s(\mathfrak{c}_+) - \mathfrak{f}_s(\mathfrak{c}_-)$, the proof of almost every lemma here is virtually identical to that of its partner in Section 3 of [22]. When this is the case, the reader is referred to Section 3 of [22] for the proofs. The correspondence between lemmas here and lemmas in Section 3 of [22] are noted below. Be forewarned however that the lemmas in Section 3 of [22] do not appear in the same order as those here.

The first lemma below supplies an inequality that relates $\mathrm{A}_{\mathfrak{d}}$ to the change in $\mathfrak{f}_s$.

Lemma 4.1 *There exists a constant $\kappa \geq 1$ with the following significance: Suppose that $\mathrm{r} \geq \kappa$ and that $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Suppose that $\mathfrak{c}_+$ and $\mathfrak{c}_-$ are solutions to the (r, μ) version of (1-13). Then*

$$\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+) \leq 2\pi(\mathrm{r} - \pi)(\mathfrak{f}_s(\mathfrak{c}_+) - \mathfrak{f}_s(\mathfrak{c}_-)) + \kappa \mathrm{r}(\mathrm{M}(\mathfrak{c}_+) + 1).$$

Proof Write $\mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+)$ as $\mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_-) - \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}_+) + 2\pi(\mathrm{r} - \pi)(\mathfrak{f}_s(\mathfrak{c}_+) - \mathfrak{f}_s(\mathfrak{c}_-))$ and then appeal to the third bullet of Proposition 2.7. □

The next lemma refers to a certain $i\mathbb{R}$–valued 1–form that can be associated to a given $(A, \psi) \in \mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$. This 1–form is denoted by $\mathfrak{B}_{(A,\psi)}$:

$$\mathfrak{B}_{(A,\psi)} = B_A - \mathrm{r}(\psi^\dagger \tau \psi - i\hat{a}) - i * d\mu + \tfrac{1}{2} B_{A_K}. \tag{4-2}$$

The upcoming Lemma 4.2 gives an a priori bound for the L^2–norms of $\frac{\partial}{\partial s}\psi$, $\frac{\partial}{\partial s}A$, $\mathfrak{B}_{(A,\psi)}$ and $D_A\psi$. Lemma 4.2 is partnered with Lemma 3.4 in [22] and its proof is identical to the latter's but for notation.

Lemma 4.2 *There exists a constant $\kappa \geq 1$ with the following significance: Suppose that $\mathrm{r} \geq \kappa$, that $\mu \in \Omega$ has $\mathcal{P}$–norm less than 1 and that (A, ψ) is an instanton solution to the (r, μ) version of (4-1). Let $s' \geq s \in \mathbb{R}$. Then*

$$\frac{1}{2}\int_{[s,s']\times Y} \Big(\Big|\frac{\partial}{\partial s}A\Big|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}\Big(\Big|\frac{\partial}{\partial s}\psi\Big|^2 + |D_A\psi|^2\Big)\Big) = \mathfrak{a}(\mathfrak{d}|_s) - \mathfrak{a}(\mathfrak{d}|_{s'}).$$

Moreover,

$$\frac{1}{2}\int_{\mathbb{R}\times Y} \Big(\Big|\frac{\partial}{\partial s}A\Big|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}\Big(\Big|\frac{\partial}{\partial s}\psi\Big|^2 + |D_A\psi|^2\Big)\Big) = \mathfrak{a}(\mathfrak{c}_-) - \mathfrak{a}(\mathfrak{c}_+).$$

Lemmas 4.1 and 4.2 with Lemma 2.5 have the following as a corollary: there is a constant κ that is independent of $\mathfrak{d}$, r and μ and is such that $\mathfrak{f}_s(\mathfrak{c}_+) > \mathfrak{f}_s(\mathfrak{c}_-) - \kappa \ln \mathrm{r}$.

The final lemma in this section speaks to the L^2–norms of B_A and the covariant derivative of ψ along the constant s slices of $\mathbb{R} \times Y$. The latter is denoted by $\nabla_A^Y \psi$.

Lemma 4.3 *There exists $\kappa \geq \pi$ with the following significance: Suppose that $\mathrm{r} \geq \kappa$ and that $\mu \in \Omega$ has $\mathcal{P}$–norm less than 1. Let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to the (r, μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}} < \mathrm{r}^2$. Fix a point $s \in \mathbb{R}$. Then*

$$\int_{[s,s+1]\times Y} \Big(\Big|\frac{\partial}{\partial s}A\Big|^2 + |B_A|^2 + 2\mathrm{r}\Big|\frac{\partial}{\partial s}\psi\Big|^2 + 2\mathrm{r}|\nabla_A^Y \psi|^2\Big) \leq \kappa \mathrm{r}^2.$$

The proof of Lemma 4.3 is identical to its [22] analog, Lemma 3.3 in [22].

4.2 A priori bounds on α, β and B_A and $\frac{\partial}{\partial s}A$

The lemma that follows supplies the first of a series of a priori pointwise bounds on the size of the components of ψ, B_A and $\frac{\partial}{\partial s}A$. The bounds in this first lemma are the fundamental ones from which all else follows. This upcoming Lemma 4.4 is the analog of Lemma 3.1 in [22] and its proof essentially the same as that of the latter.

Lemma 4.4 *There exists $\kappa \geq \pi$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and fix $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1. Suppose that $\mathfrak{d} = (A, \psi)$ is an instanton solution to the corresponding (r, μ) version of (4-1). Then*

- $|\alpha| \leq 1 + \kappa \mathrm{r}^{-1}$,
- $|\beta|^2 \leq \kappa \mathrm{r}^{-1}(1 - |\alpha|^2) + \kappa^2 \mathrm{r}^{-2}$.

Proof The second line of (4-1) implies that $\big(-\frac{\partial}{\partial s} + D_A\big)\big(-\frac{\partial}{\partial s} + D_A\big)\psi = 0$. Taking the respective E and $E \otimes K^{-1}$ summands of this identity and commuting derivatives where appropriate leads to Laplacian-type equations for α and β,

(4-3)
- $\nabla_A^* \nabla_A \alpha + \mathrm{r}(|\alpha|^2 - 1 + |\beta|^2)\alpha + \mathfrak{c}_0 \alpha + \mathfrak{c}_1 \nabla_A \beta + \mathfrak{c}_2 \beta = 0,$
- $\nabla_A^* \nabla_A \beta + \mathrm{r}(|\alpha|^2 + 1 + |\beta|^2)\beta + \mathfrak{c}_3 \nabla_A \beta + \mathfrak{c}_4 \beta + \mathfrak{c}_5 \nabla_A \alpha + \mathfrak{c}_6 \alpha = 0,$

where $\mathfrak{c}_0, \dots, \mathfrak{c}_6$ are endomorphism-valued functions on Y that are independent of r, A and (α, β). They are determined solely by the geometric data for Y and the choice of μ and the connection A_K Taking the inner product of the top equation with α and the lower one with β leads to corresponding Laplacian-type differential inequalities for $|\alpha|^2$ (which is $1 - \mathrm{w}$) and $|\beta|^2$. The latter are then used in the manner of their [22] analogs (equation (3.1) in [22]) to establish the assertions of Lemma 4.4. □

The next set of bounds are for $|B_A|$ and $\big|\frac{\partial}{\partial s}A\big|$. Those stated by the next lemma are the analog of Lemma 3.2 in [22]. The proof of the next lemma is virtually identical to the proof of the latter with Lemma 4.3 serving as the substitute for Lemma 3.3 in [22].

Lemma 4.5 *There exists $\kappa \geq \pi$ with the following significance: Suppose that $\mathrm{r} \geq \kappa$ and that $\mu \in \Omega$ has $\mathcal{P}$–norm less than 1. Suppose in addition that $\mathfrak{d} = (A, \psi)$ is an instanton solution to the (r, μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}} < \mathrm{r}^2$. Then $|B_A| + \big|\frac{\partial}{\partial s}A\big| \leq \kappa \mathrm{r}$.*

The bound supplied by this lemma is used to prove the next one. This upcoming Lemma 4.6 is the analog of Lemma 3.6 in [22] and its proof is identical with Lemma 4.5 serving as a substitute for Lemma 3.2 in [22].

The notation used in Lemma 4.6 and subsequently has ∇_A denoting the covariant derivative on sections of the pullback of E over $\mathbb{R}\times Y$ that is defined by viewing the connection A as an $\mathbb{R}$–dependent connection on this pullback bundle. By way of an example, $\nabla_A\psi = \frac{\partial}{\partial s}\psi\, ds + \nabla_A^Y\psi$.

Lemma 4.6 *There exists $\kappa \ge \pi$ with the following significance: Suppose that $\mathrm{r}\ge\kappa$ and that $\mu\in\Omega$ has $\mathcal{P}$–norm less than 1. Suppose in addition that $\mathfrak{d}=(A,\psi)$ is an instanton solution to the (r,μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}}<\mathrm{r}^2$. Then*

- $|\nabla_A\alpha|^2 \le \kappa \mathrm{r}$,
- $|\nabla_A\beta|^2 \le \kappa$.

In addition, for each $q\ge 1$, there exists a constant κ_q which is independent of $\mathfrak{d}$, μ and r, and is such that when $\mathrm{r}\ge\kappa$ then

- $|\nabla_A^q\alpha| + \mathrm{r}^{1/2}|\nabla_A^q\beta| \le \kappa_q \mathrm{r}^{q/2}$.

The upcoming Lemma 4.7 is the analog of Lemma 3.7 in [22]. This lemma and subsequent lemmas refer to the function $\underline{\mathrm{M}}$ on $\mathbb{R}$ that is defined by the rule

$$s\mapsto \underline{\mathrm{M}}(s) = \mathrm{r}\int_{[s-1,s+1]\times Y}(1-|\alpha|^2). \tag{4-4}$$

The proof of the upcoming lemma differs little from that of Lemma 3.7 in [22] with Lemma 4.5 serving in lieu of Lemma 3.2 in [22].

Lemma 4.7 *There exists $\kappa\ge\pi$ with the following significance: Fix $\mathrm{r}\ge\kappa$ and $\mu\in\Omega$ with $\mathcal{P}$–norm less than 1. Let $\mathfrak{d}=(A,\psi)$ denote an instanton solution to the (r,μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}}<\mathrm{r}^2$. Assume in addition that $s_0\in\mathbb{R}$ and that $\mathcal{K}\ge 1$ are such that $\sup_{s\in[s_0-2,s_0+2]}\underline{\mathrm{M}}(s)\le\mathcal{K}$. Then*

$$\left|\frac{\partial}{\partial s}A - B_A\right| \le \mathrm{r}(1+\kappa\mathcal{K}^{1/2}\mathrm{r}^{-1/2})(1-|\alpha|^2)+\kappa$$

at all points in $[s_0-1,s_0+1]\times Y$.

The upcoming Lemma 4.8 is a refinement of Lemma 3.8 in [22] in that it makes no reference to $\underline{\mathrm{M}}$. The proof given below works just as well in the context of Lemma 3.8 in [22] and so the assertion of the latter lemma holds also with no reference to $\underline{\mathrm{M}}$.

Lemma 4.8 *There exists $\kappa \geq \pi$ with the following significance: Fix $\mathrm{r} > \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to the (r, μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}} < \mathrm{r}^2$. Fix $s_0 \in \mathbb{R}$ and let $X_* \subset [s_0 - 2, s_0 + 2] \times Y$ denote the set of points where $1 - |\alpha| \leq \kappa^{-1}$. The bounds stated below hold on X_*:*

- $|\nabla_A \alpha|^2 + \mathrm{r}|\nabla_A \beta|^2 \leq \kappa \mathrm{r}(1 - |\alpha|^2) + \kappa^2$.
- $|\nabla_A \alpha|^2 + \mathrm{r}|\nabla_A \beta|^2 \leq \kappa(\mathrm{r}^{-1} + \mathrm{r}e^{-\sqrt{\mathrm{r}}\,\mathrm{dist}(\cdot, X_*)/\kappa})$.
- $|\beta|^2 \leq \kappa(\mathrm{r}^{-2} + \mathrm{r}^{-1} e^{-\sqrt{\mathrm{r}}\,\mathrm{dist}(\cdot, X_*)/\kappa})$.
- $\mathrm{r}(1 - |\alpha|^2) \leq \kappa(1 + \mathrm{r}e^{-\sqrt{\mathrm{r}}\,\mathrm{dist}(\cdot, X_*)/\kappa})$.

Remark Lemma 3.8 in [22] misstates the bound on $\mathrm{r}(1 - |\alpha|^2)$ in its second bullet; the correct bound is of the form given by the fourth bullet in Lemma 4.8 here. The proof of the second bullet of Lemma 3.8 in [22] has a corresponding misstep. See [24] for a corrected version of Lemma 3.8 in [22] and its proof.

Proof The proof of the top bullet is the same as the proof of the analog in [14, Proposition 2.8]. It uses only the bounds from Lemma 4.6 on $|B_A|$ and $\left|\frac{\partial}{\partial s} A\right|$. The bounds in the second and third bullets of Lemma 4.8 are derived in the three steps that follow. See also [24] for a different proof.

Step 1 Mimic what is done in Step 2 of the proof of Proposition 4.4 in [14] to find positive (A, ψ)– and r–independent constants $c > 1$ and z_1 and z_2 such that the function $y_1 = (|\nabla_A \alpha|^2 + z_1 \mathrm{r}|\nabla_A \beta|^2 + z_2 \mathrm{r}^2 |\beta|^2)$ obeys a differential inequality of the form

$$d^* d y_1 + c^{-2} \mathrm{r} y_1 \leq c_0 \mathrm{r}(1 - |\alpha|^2) y_1 + c_0. \tag{4-5}$$

With regards to the derivation, differentiating the equations in (4-3) and commuting covariant derivatives leads to second-order, Laplacian-type equations for $\nabla_A \alpha$ and $\nabla_A \beta$. Note that the 2–form $F_A = \frac{\partial}{\partial s} A \wedge ds + B_A$ on $\mathbb{R} \times Y$ will appear in these equations because of the covariant derivative commutators. A $c_0 \mathrm{r}(1 - |\alpha|^2) + c_0$ bound on the norm of F_A should be kept in mind; it follows from (4-1) and from the $\mathcal{K} = c_0 \mathrm{r}$ version of Lemma 4.7, which is always available because $\underline{\mathrm{M}}$ is in no event greater than that version of $\mathcal{K}$. The covariant derivative commutators will also give a term proportional to either $d^* F_A \alpha$ or $d^* F_A \beta$ as the case may be. Keep in mind in this regard that $d^* F_A$ can evaluated using just the top equation of (4-1) because $dF_A = 0$. Taking the inner product of the respective second-order, Laplacian-type equations for $\nabla_A \alpha$ and $\nabla_A \beta$ with $\nabla_A \alpha$ and $\nabla_A \beta$ leads to second-order, elliptic differential inequalities for $|\nabla_A \alpha|^2$

and $|\nabla_A \beta|^2$. Equation (4-5) is obtained from the latter plus the equation for $|\beta|^2$ that results from taking the inner product of the second equation in (4-3) with β.

Step 2 Fix $x_0 \in X_*$ and let d_0 denote the distance from x_0 to the boundary of X_*. There exists $c_0 > 1$ such that the function $x \mapsto h_0(x) = e^{-\sqrt{\mathrm{r}}(\mathrm{dist}(x,x_0)-d_0)/2c}$ obeys the differential inequality $d^*dh_0 + \frac{1}{2}c^{-2}\mathrm{r}h_0 \le 0$ when $\mathrm{dist}(x,x_0) \le c_0^{-1}$. Lemma 4.4 and the top bullet in Lemma 4.8 bound y_1 by $c_0\mathrm{r}$ in any event, and so $y_2 = y_1 - c_0(\mathrm{r}h_0 + \mathrm{r}^{-1})$ is nonpositive where $\mathrm{dist}(x,x_0) \ge d_0$. Meanwhile, (4-5) implies that $d^*dy_2 \le 0$ if X_* is defined to be where $1-|\alpha|^2 \le \frac{1}{2}c_0^{-1}c^{-2}$. Granted this definition, then the maximum principle asserts that $y_2 \le 0$ on X_*. In particular, this is the case at x_0 and so $y_1 \le c_0(\mathrm{r}e^{-\sqrt{\mathrm{r}}d_0/2c} + \mathrm{r}^{-1})$. The latter implies the second and third bullets in Lemma 4.8.

Step 3 Take the inner product of both sides of the top bullet of (4-3) to obtain a differential inequality for $\mathrm{w} = 1-|\alpha|^2$ that has the form

$$d^*d\mathrm{w} + 2\mathrm{r}\mathrm{w} \le 2\mathrm{r}\mathrm{w}^2 + c_0(y_1+1). \tag{4-6}$$

Granted (4-6), and granted the bounds from the second and third bullets of Lemma 4.8, then Step 2's maximum principle argument using h_0 can be repeated with only cosmetic changes to prove the lemma's fourth bullet. By way of a parenthetical remark with regards to w, keep in mind that Lemma 4.4 bounds w from below by $-c_0\mathrm{r}^{-1}$. □

Lemmas 4.9 and 4.10 are the respective analogs of Lemmas 3.9 and 3.10 of [22]. To set the stage for these lemmas, suppose that $x \in \mathbb{R}\times Y$ and $\rho \in (\mathrm{r}^{-1/2}, c_0^{-1})$ have been specified. The lemmas use $\mathrm{M}_{(x,\rho)}$ to denote the integral of $\mathrm{r}(1-|\alpha|^2)$ over the radius ρ ball in $\mathbb{R}\times Y$ centered at x.

Lemma 4.9 *There exists $\kappa \ge \pi$, and, given data consisting of an open set $U \subset \mathbb{R}\times Y$, an open subset $V \subset U$ with compact closure and $\mathcal{K} \ge 1$, there exists $\kappa_{\mathcal{K},U,V} \ge 1$ with the following significance: Fix $\mathrm{r} \ge \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Let $\mathfrak{d} = (A,\psi)$ denote an instanton solution to the (r,μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}} < \mathrm{r}^2$. Assume that $\mathfrak{d}$ is such that $\sup_{x\in U}\mathrm{M}_{(x,1/\kappa)} \le \mathcal{K}$. Then*

$$\left|\frac{\partial}{\partial s}A + B_A\right| \le \mathrm{r}(1-|\alpha|^2) + \kappa_{\mathcal{K},U,V},$$
$$\left|\frac{\partial}{\partial s}A - B_A\right| \le \mathrm{r}(1-|\alpha|^2) + \kappa_{\mathcal{K},U,V}$$

at all points in V.

The proof of Lemma 4.9 is very similar to that of its analog in [22], the latter being very similar to the proof of Proposition 3.4 in [14].

The final lemma in this subsection is a monotonicity result and of a different flavor from the pointwise bounds given above. It plays a role in the proof of Lemma 4.9.

Lemma 4.10 *There exists $\kappa \geq \pi$, and, given $z \geq 1$, there exists $\kappa_z \geq 1$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to the (r, μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}} < \mathrm{r}^2$ and $\sup_{s\in\mathbb{R}} \underline{\mathrm{M}}(s) \leq \mathrm{r}^{1-1/z}$. Given $x \in \mathbb{R}\times Y$ and $\rho \in (\mathrm{r}^{-1/2}, \kappa_z^{-1})$, use $\mathrm{M}_{(x,\rho)}$ to denote the integral of $\mathrm{r}(1-|\alpha|^2)$ over the radius ρ ball in $\mathbb{R}\times Y$ centered at x. Then:*

- *If $\rho_1 > \rho_0$ are in $(\mathrm{r}^{-1/2}, \kappa_z^{-1})$, then $\mathrm{M}_{(x,\rho_1)} > \kappa_z^{-1}\rho_1^2/\rho_0^2\,\mathrm{M}_{(x,\rho_0)}$.*
- *Suppose that $|\alpha| \leq \frac{3}{4}$ at x. If $\rho \in (\mathrm{r}^{-1/2}, \kappa_z^{-1})$, then $\mathrm{M}_{(x,\rho)} \geq \kappa^{-1}\rho^2$.*
- *Suppose that $\mathcal{K} \in (1, \mathrm{r}^{1-1/z})$ and suppose that $d \in (\mathrm{r}^{-1/2}, \kappa^{-1})$ and $x \in \mathbb{R}\times Y$ are such that $\mathrm{M}_{(x,d)} \leq \mathcal{K}d^2$. If $\rho \in (\mathrm{r}^{-1/2}, d)$, then $\mathrm{M}_{(x,\rho)} \leq \kappa_z\mathcal{K}\rho^2$.*

As with the proof of Lemma 3.10 in [22], the proof of Lemma 4.10 differs little from the proof of Proposition 3.1 in [14]. This lemma also plays a role in the subsequent sections.

4.3 Instantons and holomorphic data on $\mathbb{C}^2$

The three parts of this section first introduce holomorphic notions on $\mathbb{C}^2$, and then explain how they model an instanton solution to (4-1) in a radius $\mathcal{O}(\mathrm{r}^{-1/2})$ ball.

Part 1 This part introduces the relevant holomorphic data on $\mathbb{C}^2$. To this end, introduce complex coordinates (x_0, x_1) for $\mathbb{C}^2 = \mathbb{R}^4$. Give $\mathbb{C}^2$ the standard metric with Kähler form $\omega_0 = \frac{i}{2}(dx_0 \wedge d\bar{x}_0 + dx_1 \wedge d\bar{x}_1)$. Use $P^+\colon \bigwedge^2 T^*\mathbb{C}^2 \to \bigwedge^2 T^*\mathbb{C}^2$ to denote the projection to the self dual subspace and P^- the projection to the anti-self dual subspace.

Of interest here are pairs (A_0, α_0) on $\mathbb{C}^2$ where A_0 is a unitary connection on the trivial bundle and α_0 is a section of this bundle, and where these are such that

$$\begin{aligned} &\bar{\partial}_{A_0}\alpha_0 = 0, \qquad P^+F_{A_0} = -\tfrac{1}{2}i(1-|\alpha_0|^2)\omega_0, \\ &|\alpha_0| \leq 1, \quad |P^-F_{A_0}| \leq |P^+F_{A_0}| \leq 2^{-1/2}(1-|\alpha_0|^2). \end{aligned} \tag{4-7}$$

Proposition 4.1 in [14] and Proposition 4.2 in [22] describe the pairs (A_0, α_0) that satisfy these conditions. Except for the second bullet, the following proposition restates

Proposition 4.2 in [22]. The proof of the second bullet is the same as that of the second bullet of this same Proposition 4.2 in [22].

Proposition 4.11 *Suppose that* (A_0, α_0) *obeys* (4-7).

- *If* $|\alpha_0| < 1$ *somewhere, then* $|\alpha_0|$ *is strictly less than* 1, *it has no positive local minimum and* $\inf_{\mathbb{C}^2} |\alpha_0| = 0$. *If* $\alpha_0^{-1}(0) \neq \varnothing$, *then* $\alpha_0^{-1}(0)$ *is either all of* $\mathbb{C}^2$ *or a complex analytic subvariety of complex dimension* 1.
- *There exists* $\kappa_0 > 1$ *that is independent of* (A_0, α_0) *and has the following significance: Let* $X_* \subset \mathbb{C}^2$ *denote the set of points where* $1 - |\alpha_0| \geq \frac{3}{4}$. *Then*
$$1 - |\alpha_0| + |\nabla_{A_0}\alpha_0| \leq \kappa_0 e^{-\operatorname{dist}(\cdot, X_*)/\kappa_0}.$$
- *If* $|\alpha_0| < 1$ *somewhere, and if there exists* $m \geq 1$ *such that the integral of* $1 - |\alpha_0|^2$ *over the ball of any given radius* $\mathrm{R} \geq 1$ *centered at the origin is less than* $m\mathrm{R}^2$, *then*:
  (a) *The locus* $\alpha^{-1}(0)$ *is a nonempty, complex algebraic subvariety with complex dimension* 1. *As such, this locus near any given point is the zero locus of a holomorphic polynomial.*
  (b) *The order of the latter polynomial has a purely* m*–dependent upper bound.*
  If, in addition, the integral over $\mathbb{C}^2$ *of* $|P^+ F_{A_0}|^2 - |P^- F_{A_0}|^2$ *is finite, then*:
  (c) *This integral is a nonnegative integer multiple of* $4\pi^2$.
  (d) *If the latter integral is zero, then* (A_0, α_0) *is the pullback via a projection* $\mathbb{C}^2 \to \mathbb{C}$ *of a solution on* $\mathbb{C}$ *to the vortex equations in* (1-4) *and* $\alpha_0^{-1}(0)$ *is a union of planes.*
- *The set of gauge equivalence classes of pairs* (A_0, α_0) *that obey* (4-1) *is sequentially compact with respect to convergence on compact subsets of* $\mathbb{C}^2$ *in the* C^∞ *topology.*

The solutions to (4-7) constitute the desired holomorphic data on $\mathbb{C}^2$.

Part 2 Fix a point $p \in \mathbb{R} \times Y$. A complex Gaussian coordinate system centered at p is a coordinate chart map from a ball about the origin in $\mathbb{C}^2$ to a neighborhood of p that takes the origin to x and defines a Gaussian coordinate chart when written in terms of the real coordinates for $\mathbb{C}^2$. In addition, the almost complex structure J at the point p must appear in these coordinates as the standard complex structure. The complex coordinates on $\mathbb{C}^2$ are written again as (x_0, x_1). No generality is lost by assuming that any given such Gaussian coordinate chart is defined where $|x_0|^2 + |x_1|^2 \le c_0^{-1}$ with c_0 being independent of p.

Introduce a new coordinate chart by composing the original one with the map from $\mathbb{C}^2$ to itself that sends (x_0, x_1) to $(\mathrm{r}^{-1/2}x_0, \mathrm{r}^{-1/2}x_1)$. The new coordinate chart is defined on the ball of radius $c_0^{-1}\mathrm{r}^{1/2}$ centered at the origin in $\mathbb{C}^2$. Use φ_r in what follows to denote this coordinate chart map from the ball of radius $c_0^{-1}\mathrm{r}^{1/2}$ in $\mathbb{C}^2$ to $\mathbb{R} \times Y$.

The φ_r–pullback of the metric from $\mathbb{R} \times Y$ differs from the standard Euclidean metric by no more than $c_0\mathrm{r}^{-1}$ on the radius 2^4 ball. The pullback of the Riemannian curvature is also bounded in absolute value on this ball by $c_0\mathrm{r}^{-1}$, and the latter's derivatives to a given order $k \ge 1$ on this ball have norm bounded by $c_k\mathrm{r}^{-1-k/2}$ with c_k depending on k only. Meanwhile, the φ_r–pullback of the almost complex structure on this ball differs from the standard one by at most $c_0\mathrm{r}^{-1/2}$ and its derivatives to order k have norm bounded by $c_k\mathrm{r}^{-(1+k)/2}$.

Part 3 Let $\mathfrak{d} = (A, \psi = (\alpha, \beta))$ denote an instanton solution to (4-1) with $\mathrm{A}_\mathfrak{d} \le \mathrm{r}^2$ and such that there exists $z > 1$ such that $\sup_{s\in\mathbb{R}} \underline{\mathrm{M}}(s) \le \mathrm{r}^{1-1/z}$. Introduce $(A_\mathrm{r}, \alpha_\mathrm{r})$ to denote the φ_r–pullback of the (A, α). Use F_{A_r} to denote the curvature 2–form of the connection A_r. Lemmas 4.4, 4.6 and 4.7 have implications with regards to $(A_\mathrm{r}, \alpha_\mathrm{r})$ that are described in what follows. To say more, fix $R \ge 1$. Given Part 2's remarks about the φ_r–pullbacks of the metric and almost complex structure, there exists $c_R > 1$ that is independent of p and such that if $\mathrm{r} \ge c_R$, then the $\mathfrak{d}$ version of $(A_\mathrm{r}, \alpha_\mathrm{r})$ is nearly a solution to (4-7) on the ball of radius R in $\mathbb{C}^2$ centered at the origin in the sense that

$$\begin{aligned} |\bar\partial_{A_\mathrm{r}}\alpha_\mathrm{r}| &\le c_R\mathrm{r}^{-1/2}, \\ \bigl|P^+F_{A_\mathrm{r}} + \tfrac{i}{2}(1-|\alpha_\mathrm{r}|^2)\omega_0\bigr| &\le c_R\mathrm{r}^{-1}, \\ |\alpha_\mathrm{r}| &\le 1 + c_R\mathrm{r}^{-1}, \\ |P^-F_{A_\mathrm{r}}| \le |P^+F_{A_\mathrm{r}}| + c_R\mathrm{r}^{-1/2z} &\le 2^{-1/2}(1-|\alpha_\mathrm{r}|^2) + c_R\mathrm{r}^{-1/2z}. \end{aligned} \tag{4-8}$$

Moreover, with the φ_r–pullback of β, the pair $(A_\mathrm{r}, \alpha_\mathrm{r})$ plus $\varphi_\mathrm{r}^*\beta$ obey an equation on the radius R ball in $\mathbb{C}^2$ that gives bounds on the covariant derivatives of α_r and F_{A_r} to any given order that are independent of p, $\mathfrak{d}$ and R. These bounds with (4-8) lead to the following lemma:

Lemma 4.12 *Given $q \ge 1$, $\mathrm{R} > 1$, $\varepsilon > 0$, $k \in \{1, 2, \dots\}$ and $m > 1$, there exists $\kappa > 10\mathrm{R}$ with the following significance: Fix $\mathrm{r} \ge \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 and suppose that (A, ψ) denotes an instanton solution to* (4-1) *with* $\mathrm{A}_\mathfrak{d} \le \mathrm{r}^2$

and $\sup_{s\in\mathbb{R}} \underline{\mathrm{M}}(s) \le \mathrm{r}^{1-1/q}$. *Given* $p \in \mathbb{R}\times Y$, *there exists a solution to* (4-7) *on* $\mathbb{C}^2$, *this denoted by* (A_0, α_0), *such that* $(A_\mathrm{r}, \alpha_\mathrm{r}) = (A_0 + \hat{a}, \alpha_0 + \eta)$ *with* $(\hat{a}, \eta)$ *having* C^k*–norm less than* ε *on the ball of radius* R *in* $\mathbb{C}^2$ *centered at the origin. Moreover, suppose that the integral of* $\mathrm{r}(1-|\alpha|^2)$ *on each radius* $\rho \in (\mathrm{r}^{-1/2}, \kappa\mathrm{r}^{-1/2})$ *ball centered on* p *is less than* $m\rho^2$. *Then* (A_0, α_0) *can be chosen so as to obey items* (a) *and* (b) *of the third bullet in Proposition 4.11.*

Lemma 4.12 is the analog here of Lemma 4.3 in [22]. As with the latter, the proof differs little from that of Proposition 4.2 in [14].

5 A priori bounds for the function $\underline{\mathrm{M}}$: the complement of $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$

Write $\upsilon_\diamond$ as $q_\diamond\hat{a} + \mathfrak{b}$ where the 1–form $\mathfrak{b}$ annihilates v. By way of a reminder, the function $q_\diamond$ differs from 1 only in $\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_\mathfrak{p}$, it vanishes only on $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$, and it is such that $q_\diamond \ge c_0^{-1}|\upsilon_\diamond|^2$. Fix $\mathrm{r} > c_0$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 so as to define (4-1). Suppose that $\mathfrak{d} = (A, \psi = (\alpha, \beta))$ is an instanton solution to this (r, μ) version of (4-1). This section supplies a $\mathfrak{d}$– and r–independent bound for the function on $\mathbb{R}$ given by the rule

$$s \mapsto \underline{\mathrm{M}}_\diamond(s) = \mathrm{r}\int_{[s-1,s+1]\times Y} q_\diamond^6(1-|\alpha|^2). \tag{5-1}$$

The proposition that follows makes a formal statement that such a bound exists:

Proposition 5.1 *There exists* $\kappa \ge \pi$ *and, given* $c \ge 1$, *there exists* $\kappa_c > 1$ *with the following significance: Suppose that* $\mathrm{r} \ge \kappa$ *and that* $\mu \in \Omega$ *has* $\mathcal{P}$*–norm less than* 1. *Suppose in addition that* $\mathfrak{d} = (A, \psi)$ *is an instanton solution to the* (r, μ) *version of* (4-1) *with* $\mathrm{A}_\mathfrak{d} < c\mathrm{r}$. *Then the corresponding function* $\underline{\mathrm{M}}_\diamond$ *obeys* $-\kappa < \underline{\mathrm{M}}_\diamond < \kappa_c$.

The lower bound follows directly from Lemma 4.4, so it holds without the bound for $\mathrm{A}_\mathfrak{d}$. The proof of the upper bound occupies the rest of this section. By way of a parenthetical remark, the proof looks much like the proof of Lemma 5.8 in [11].

5.1 Preliminary bounds for $\underline{\mathrm{M}}_\diamond$ and $\underline{\mathrm{M}}$

The lemma that follows supplies a preliminary and easy-to-come-by bound for $\underline{\mathrm{M}}$ that is used in the later subsections to invoke Lemma 4.7.

Lemma 5.2 *There exists* $\kappa \ge \pi$ *with the following significance: Fix* $\mathrm{r} \ge \kappa$ *and* $\mu \in \Omega$ *with* $\mathcal{P}$*–norm less than* 1. *Let* $\mathfrak{d} = (A, \psi)$ *denote an instanton solution to the* (r, μ)

version of (4-1). *The corresponding version of* $\underline{\text{M}}_\diamond$ *obeys* $-\kappa \le \underline{\text{M}}_\diamond \le \kappa(\text{A}_\mathfrak{d} + 1)^{1/2}$ *and the corresponding version of* $\underline{\text{M}}$ *obeys* $-\kappa < \underline{\text{M}} < \kappa \text{r}^{2/3}(1 + \text{A}_\mathfrak{d})^{1/6}$.

Proof The lower bounds follow from Lemma 4.4. The first step of what follows establishes the upper bound for $\underline{\text{M}}_\diamond$ and the second step establishes the upper bound for $\underline{\text{M}}$. The notation in these steps is that used earlier in the proof of the second bullet of Lemma 2.5.

Step 1 To prove the upper bound for $\underline{\text{M}}_\diamond$, take the inner product on Y between $\upsilon_\diamond$ and the 1–form on the right-hand side of the top line in (4-1). Integrate the result over $[s-1, s+1] \times Y$. This integral is, of course, equal to zero. Thus,

$$\text{(5-2)} \quad \int_{[s-1,s+1]\times Y} \Big(\upsilon_\diamond \wedge * \frac{\partial}{\partial s} A\Big) + \int_{[s-1,s+1]\times Y} (\upsilon_\diamond \wedge * B_A) = \text{r} \int_{[s-1,s+1]\times Y} (\upsilon_\diamond \wedge *(\psi^\dagger \tau \psi - i\hat{a})) + \mathfrak{e},$$

where $|\mathfrak{e}| \le c_0$. Write $\upsilon_\diamond$ as $q_\diamond \hat{a} + \mathfrak{b}$ with $\mathfrak{b}$ annihilating the vector field v and use this rewriting for the integrand of the integral on the right-hand side of (5-2). Then, use the bounds $|\mathfrak{b}| \le |\upsilon_\diamond|$ and $|\upsilon_\diamond| \le c_0 q_\diamond^{1/2}$ with Lemma 4.4's bounds for $|\beta|$ to see that this integrand is greater than $\frac{1}{2}|\upsilon_\diamond|^2(1 - |\alpha|^2) - c_0 \text{r}^{-1}$. This the case, a bound on the integral on the left-hand side of (5-2) supplies one for the integral on the right-hand side of (5-1).

To obtain an upper bound for the left-hand side of (5-2), use Lemma 4.2 to see that the integral of $\upsilon_\diamond \wedge * \frac{\partial}{\partial s} A$ that appears on the left-hand side of (5-2) is no greater than $c_0(1 + \text{A}_\mathfrak{d})^{1/2}$. Meanwhile, the integral of $\upsilon_\diamond \wedge * B_A$ is independent of A and r because it computes a pairing with the first Chern class of the bundle E. These last facts imply that the left-hand side of (5-2) is no greater than $c_0(\text{A}_\mathfrak{d} + 1)^{1/2}$.

Step 2 Fix $\rho > 0$ and let Y^ρ denote for the moment the set of points in Y with distance ρ or more from the curves in the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$. The integral in (5-1) is no less than the contribution from Y^ρ and this is no less than $c_0^{-1}\rho^4(\underline{\text{M}} - c_0 \text{r}\rho^2) - c_0$. It follows as a consequence that $\underline{\text{M}} \le c_0(\rho^{-4}(\text{A}_\mathfrak{d} + 1)^{1/2} + \text{r}\rho^2)$. This understood, take $\rho^2 = \text{r}^{-1/3}(\text{A}_\mathfrak{d} + 1)^{1/6}$ to obtain what is asserted by Lemma 5.2. □

5.2 A vortex-like inequality

This subsection shows how Proposition 5.1 follows from Lemma 5.3. This lemma asserts an inequality that is reminiscent of the equality asserted by the first bullet of Lemma 3.3.

Lemma 5.3 refers to a certain function, $\mathrm{Q}_\diamond$, on Y which is specified in the next subsection. For the purposes of the lemma, it is enough to know that $\mathrm{Q}_\diamond \geq c_0^{-1} q_\diamond^6$ and that $|d\mathrm{Q}_\diamond| \leq c_0$. Given $s \in \mathbb{R}$, this lemma uses χ_s to denote the function $\chi(2|s-(\cdot)|-1)$ on $\mathbb{R}$. This function is 1 on $\left[s-\frac{1}{2}, s+\frac{1}{2}\right]$ and it is equal to 0 on the complement of $[s-1, s+1]$.

Lemma 5.3 *There exists $\kappa > \pi$ and, given $c \geq 1$, there exists $\kappa_c \geq 1$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to the (r, μ) version of (4-1) with $\mathrm{A}_\mathfrak{d} < c\mathrm{r}$. Then*

$$\sup_{s\in\mathbb{R}} \underline{\mathrm{M}}_\diamond \leq \kappa_c \sup_{s\in\mathbb{R}} \int_{\mathbb{R}\times Y} \chi_s \mathrm{Q}_\diamond(\mathrm{r}|\alpha|^2(1-|\alpha|^2) - |\nabla_A\alpha|^2) + \kappa_c^2.$$

This lemma is proved in a moment.

Proof of Proposition 5.1 The proof assumes that Lemma 5.3 is true so as to deduce a suitable upper bound for $\underline{\mathrm{M}}_\diamond$. To deduce such a bound from Lemma 5.3, introduce by way of notation $\mathcal{D}_A$ to denote $\frac{\partial}{\partial s} + D_A$, this being an operator on the space of sections of $\mathbb{S}$ over $\mathbb{R} \times Y$. Use $\mathcal{D}_A^\dagger$ to denote its formal L^2–adjoint. Given that $\mathcal{D}_A\psi = 0$, $\mathcal{D}_A^\dagger\mathcal{D}_A\psi$ is also zero. Projecting the equation $\mathcal{D}_A^\dagger\mathcal{D}_A\psi = 0$ to the E summand of $\mathbb{S}$ gives an equation of the form

$$\nabla_A^*\nabla_A\alpha + \mathrm{r}\alpha(|\alpha|^2 - 1) + \mathfrak{r} = 0, \tag{5-3}$$

where $|\mathfrak{r}| \leq c_0(|\alpha| + |\beta| + |\nabla_A\beta|)$. Take the inner product of this equation with α to find an equation of the form

$$\tfrac{1}{2}d^*d(1-|\alpha|^2) + \mathrm{r}|\alpha|^2(1-|\alpha|^2) - |\nabla_A\alpha|^2 + \mathfrak{e} = 0, \tag{5-4}$$

where $|\mathfrak{e}| \leq c_0(|\alpha|^2 + |\beta|^2 + |\nabla_A\beta|^2)$. Multiply both sides of this last equation by $\chi_s\mathrm{Q}_\diamond$ and integrate the result over $\mathbb{R} \times Y$. Integrate by parts and appeal to Lemma 4.4 and the bound on $|\nabla_A\beta|^2$ from Lemma 4.6 to see that the integral that appears on the right-hand side of Lemma 5.3 has an (A, ψ)– and r–independent upper bound. □

5.3 Proof of Lemma 5.3

The four steps that follow derive Lemma 5.3 from the upcoming Lemma 5.4. The rest of the subsection supplies a proof of Lemma 5.4.

Step 1 This step specifies the function $\mathrm{Q}_\diamond$. To do this, first introduce the function f_* that is defined on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ by the rule $f_*(u, \theta) = \mathrm{g}(u)(1 - 3\cos^2\theta)$

with g as defined in the third line in (1-2). By way of a reminder from (1-5), this function is such that $\upsilon_\diamond = df_*$ on $\mathcal{H}_{\mathfrak{p}}$. Choose a smooth, nondecreasing function on $[0,\infty)$ with the properties listed next. This function is denoted by Q. It is such that $\mathrm{Q}(t) = t^5$ for $t \in \left(0, \frac{1}{2}\right]$ and $\mathrm{Q}(t) = 1$ for $t \geq 1$. With Q in hand, fix for the moment $\varepsilon \in (0,1)$ and use Q_ε to denote the function $\mathrm{Q}(\varepsilon^{-2} q_\diamond)$. Let $v(\mathrm{Q}_\varepsilon)$ denote the pairing between v and $d\mathrm{Q}_\varepsilon$. The function $\mathrm{Q}_\diamond$ is the function $\mathrm{Q}_\varepsilon q_\diamond + f_* v(\mathrm{Q}_\varepsilon)$ for a choice for ε that guarantees it to be greater than $c_0^{-1} q_\diamond^6$ and to have derivative norm bounded by c_0. This choice is such that $\varepsilon > c_0^{-1}$.

Step 2 The upcoming equation (5-5) supplies an integral form of the Bochner–Weitzenböck identity for the operator $D_A(Q_\diamond D_A)$. The formula reintroduces from (1-11) the Clifford multiplication endomorphism $\mathrm{cl}(\cdot)$. This formula is derived using integration by parts. Suppose for the moment that (A,ψ) is any given pair in $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$. What follows is the promised identity:

$$\begin{aligned}(5\text{-}5)\quad \int_Y \mathrm{Q}_\diamond(|B_A|^2 + \mathrm{r}^2|\psi^\dagger\tau\psi - i\hat{a}|^2 + 2\mathrm{r}|\nabla_A^Y\psi|^2)
&= 2i\mathrm{r}\int_Y \mathrm{Q}_\diamond \hat{a}\wedge *B_A - \mathrm{r}\int_Y (\psi^\dagger\,\mathrm{cl}(d\mathrm{Q}_\diamond)D_A\psi - (D_A)^\dagger\,\mathrm{cl}(d\mathrm{Q}_\diamond)\psi)\\
&\quad + \int_Y \mathrm{Q}_\diamond(|\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}|D_A\psi|^2) + \mathfrak{e},\end{aligned}$$

where $\mathfrak{e}$ obeys $|\mathfrak{e}| \leq c_0(1+\mathrm{r})$. The proof of Lemma 5.3 uses the a priori bounds given by the next lemma on the first two integrals that appear on the right-hand side of (5-5). Lemma 4.2 is used to bound the third, right-most integral on the right-hand side of (5-5).

Lemma 5.4 *There exists $\kappa > \pi$ and, given $c \geq 1$, there exists $\kappa_c > 1$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Let $\mathfrak{d} = (A,\psi)$ denote an instanton solution to the (r,μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}} < c\mathrm{r}$ and $\sup_{s\in\mathbb{R}} \underline{\mathrm{M}}_\diamond \geq 1$. Then*

- $$\sup_{s\in\mathbb{R}} 2\mathrm{r}\int_{\mathbb{R}\times Y} \chi_s\left(i\int_Y \mathrm{Q}_\diamond\hat{a}\wedge *B_A\right) \leq \tfrac{1}{1000}\mathrm{r}\sup_{s\in\mathbb{R}}\underline{\mathrm{M}}_\diamond + \kappa_c\mathrm{r},$$
- $$\sup_{s\in\mathbb{R}} \mathrm{r}\left|\int_{\mathbb{R}\times Y} \chi_s\big(\psi^\dagger\,\mathrm{cl}(d\mathrm{Q}_\diamond)D_A\psi - (D_A)^\dagger\,\mathrm{cl}(d\mathrm{Q}_\diamond)\psi\big)\right| \leq \tfrac{1}{1000}\mathrm{r}\sup_{s\in\mathbb{R}}\underline{\mathrm{M}}_\diamond + \kappa_c\mathrm{r}.$$

This lemma is proved in a moment. The remaining steps use Lemma 5.4 to complete the argument for Lemma 5.3.

Step 3 Take the $\mathrm{Conn}(E)\times C^\infty(Y;\mathbb{S})$ pair (A,ψ) in (5-5) to be the pair given in the statement of Lemma 5.4 at any given slice of $\mathbb{R}\times Y$ with constant $\mathbb{R}$ factor. Add the integral over this slice of $\mathrm{Q}_\diamond\bigl|\frac{\partial}{\partial s}\psi\bigr|^2$ to both sides of (5-5). View the result as an equality between functions on $\mathbb{R}$. Multiply this equality by χ_s and integrate over $\mathbb{R}$. Then use Lemmas 4.2 and 5.4 with (5-5) to see that

$$\int_{\mathbb{R}\times Y}\chi_s\mathrm{Q}_\diamond\bigl(|B_A|^2+\mathrm{r}^2(1-|\alpha|^2)^2+2\mathrm{r}|\nabla_A\alpha|^2\bigr)\le\tfrac{1}{100}\mathrm{r}\sup_{s\in\mathbb{R}}\underline{\mathrm{M}}_\diamond+c_{c*}\mathrm{r},\tag{5-6}$$

where c_{c*} denotes the version of κ_c that is given by Lemma 5.4. To make something of this, mimic what is done in Section 5.4 of [11] by writing

$$\begin{aligned}\frac{\partial}{\partial s}A&=-i(1-\sigma)(\mathrm{r}(1-|\alpha|^2)+\mathfrak{z}_A)\hat{a}+\mathfrak{r}+\mathfrak{X},\\ B_A&=-i\sigma(\mathrm{r}(1-|\alpha|^2)+\mathfrak{z}_B)\hat{a}+\mathfrak{r}-\mathfrak{X},\end{aligned}\tag{5-7}$$

where the notation uses σ to denote a function on $\mathbb{R}\times Y$. The notation has $\mathfrak{z}_A$ and $\mathfrak{z}_B$ denoting functions on $\mathbb{R}\times Y$ with norms bounded by 1, and it has both $\mathfrak{r}$ and $\mathfrak{X}$ annihilating υ. Lemma 4.4 finds $|\mathfrak{r}|\le c_0\bigl(\mathrm{r}^{1/2}\bigl|1-|\alpha|^2\bigr|^{1/2}+1\bigr)$. To say something more about $\mathfrak{X}$, use the top bullet in (4-1) and Lemma 4.7 with Lemma 5.2 to see that

$$4|\mathfrak{X}|^2+(1-2\sigma)^2\mathrm{r}^2(1-|\alpha|^2)^2\le\mathrm{r}^2(1+c_c\mathrm{r}^{-1/12})^2(1-|\alpha|^2)^2+c_c,\tag{5-8}$$

where c_c here and in what follows denotes a purely c–dependent constant with value greater than 1. The notation is such that c_c increases between subsequent appearances. This last inequality implies that

$$|\mathfrak{X}|^2\le\mathrm{r}^2\sigma(1-\sigma)(1-|\alpha|^2)^2+c_c\mathrm{r}^{23/12}(1-|\alpha|^2)^2+c_c.\tag{5-9}$$

Use (5-7) to write

$$\begin{aligned}\int_{\mathbb{R}\times Y}\chi_s\mathrm{Q}_\diamond\Bigl|\frac{\partial}{\partial s}A\Bigr|^2&=\int_{\mathbb{R}\times Y}\chi_s\mathrm{Q}_\diamond\bigl((1-\sigma)^2\mathrm{r}^2(1-|\alpha|^2)^2+|\mathfrak{X}|^2\bigr)+\mathfrak{e}_A,\\ \int_{\mathbb{R}\times Y}\chi_s\mathrm{Q}_\diamond|B_A|^2&=\int_{\mathbb{R}\times Y}\chi_s\mathrm{Q}_\diamond\bigl(\sigma^2\mathrm{r}^2(1-|\alpha|^2)^2+|\mathfrak{X}|^2\bigr)+\mathfrak{e}_B,\end{aligned}\tag{5-10}$$

where $\mathfrak{e}_A$ and $\mathfrak{e}_B$ are such that $|\mathfrak{e}_A|+|\mathfrak{e}_B|\le\frac{1}{1000}\mathrm{r}\sup_{s\in\mathbb{R}}\underline{\mathrm{M}}_\diamond+c_c\mathrm{r}$.

Step 4 Let $s\in\mathbb{R}$ denote a point where the function of $\underline{\mathrm{M}}_\diamond$ is greater than $\frac{3}{4}$ times its supremum. Following along the lines of what is done in Section 5.4 of [11], consider two cases: that when

$$\int_{\mathbb{R}\times Y}\chi_s\mathrm{Q}_\diamond\Bigl|\frac{\partial}{\partial s}A\Bigr|^2\ge\frac{1}{100}\int_{\mathbb{R}\times Y}\chi_s\mathrm{Q}_\diamond\mathrm{r}^2(1-|\alpha|^2)^2\tag{5-11}$$

and otherwise. If (5-11) holds, add 100 times the right-hand integral in (5-11) to both sides of (5-6) and invoke Lemma 4.2 to bound the resulting contribution to the right-hand side. Doing so leads to the inequality

$$\int_{\mathbb{R}\times Y} \chi_s \mathrm{Q}_\diamond\big(2\mathrm{r}^2(1-|\alpha|^2)^2 + 2\mathrm{r}|\nabla_A\alpha|^2\big) \le \tfrac{11}{100}\mathrm{r}\sup_{s\in\mathbb{R}} \underline{\mathrm{M}}_\diamond + c_c\mathrm{r}. \tag{5-12}$$

Write the left-hand side of this inequality as

$$2\mathrm{r}^2\int_{\mathbb{R}\times Y} \chi_s \mathrm{Q}_\diamond(1-|\alpha|^2) + \mathrm{r}^2\int_{\mathbb{R}\times Y} \chi_s \mathrm{Q}_\diamond\big(-2|\alpha|^2(1-|\alpha|^2)^2 + 2\mathrm{r}|\nabla_A\alpha|^2\big). \tag{5-13}$$

The left-most integral in (5-13) with the factor of 2 is no less than $\frac{3}{2}\sup_{s\in\mathbb{R}} \underline{\mathrm{M}}_\diamond$ and so (5-11) and (5-13) imply what is asserted by Lemma 5.3.

Now suppose that (5-11) is not true. If this is so, then (5-8) and (5-10) imply that

$$\int_{\mathbb{R}\times Y} \chi_s \mathrm{Q}_\diamond |B_A|^2 \ge \big(1-\tfrac{1}{50}\big)\int_{\mathbb{R}\times Y} \chi_s \mathrm{Q}_\diamond \mathrm{r}^2(1-|\alpha|^2)^2. \tag{5-14}$$

Use this last inequality in (5-6) with the top bullet in Lemma 3.3 to see that (5-12) still holds. This being the case, then what is said at the end of the last paragraph can be repeated so as to complete the proof of Lemma 5.3. □

Proof of Lemma 5.4 The first seven steps in the proof verify the top line and the eighth step verifies the lower inequality.

Step 1 As noted previously, $\upsilon_\diamond = df_*$ on any given $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_\mathfrak{p}$. This understood, a homologous closed 1–form, denoted by υ_ε, is defined to be $\upsilon_\diamond$ on $M_\delta \cup \mathcal{H}_0$ and to equal $d(\mathrm{Q}_\varepsilon f_*)$ on each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_\mathfrak{p}$. This form can be written as $\upsilon_\varepsilon = Q_\diamond\hat{a} + \mathfrak{b}_\varepsilon$ with $\mathfrak{b}_\varepsilon$ annihilating the vector field v. Writing $\upsilon_\diamond = q_\diamond\hat{a} + \mathfrak{b}$ allows $\mathfrak{b}_\varepsilon$ to be written as $\mathrm{Q}_\varepsilon\mathfrak{b} + f_* d^\perp\mathrm{Q}_\varepsilon$ with $d^\perp$ denoting here the orthogonal projection of the exterior derivative to the annihilator of v in T^*Y. Use the fact that $|\mathfrak{b}| \le |\upsilon_\diamond|$ and $|\upsilon_\diamond| \le c_0 q_\diamond^{1/2}$ and that $|f_*| \le c_0 q_\diamond$ to see that $|\mathfrak{b}_\varepsilon| \le c_0 q_\diamond^{11/2}$.

Step 2 Write $\mathrm{Q}_\diamond\hat{a}$ as $\upsilon_\varepsilon - \mathfrak{b}_\varepsilon$. The integral of $\frac{i}{2\pi}\upsilon_\varepsilon\wedge *B_A$ computes the cup product pairing between the cohomology class defined by $\upsilon_\diamond$ and the first Chern class of the line bundle E. This being the case, the lemma's top bullet follows given a suitable bound for the absolute value of the integral of $\mathfrak{b}_\varepsilon\wedge *B_A$. To obtain such a bound, use (5-9) to write this form as $\mathfrak{b}\wedge *(\mathfrak{r}-\mathfrak{X})$. As Lemma 4.4 finds $|\mathfrak{r}| \le c_0$, an appropriate bound for the integral of $\mathrm{r}|\mathfrak{b}\wedge *\mathfrak{X}|$ will suffice.

To obtain the desired bound on $|\mathfrak{b}_\varepsilon\wedge *\mathfrak{X}|$, first use (5-7) to see that

$$|\mathfrak{b}_\varepsilon\wedge *\mathfrak{X}| \le c_0\mathrm{r}q_\diamond^{11/2}(|1-\sigma|^{1/2} + c_c\mathrm{r}^{-1/12})\big|1-|\alpha|^2\big| + c_c. \tag{5-15}$$

Introduce the set X_* from Lemma 4.8. It follows from this lemma (and Lemma 4.4) that $\bigl|1-|\alpha|^2\bigr| \le c_0 \mathrm{r}^{-1}$ where the distance to X_* is greater than $c_0\mathrm{r}^{-1/2}\ln \mathrm{r}$. The right-hand side of (5-15) is therefore less than c_c where the distance to X_* is greater than $c_0\mathrm{r}^{-1/2}\ln\mathrm{r}$. Thus, this part of $\mathbb{R}\times Y$ contributes at most $c_c\mathrm{r}$ to the absolute value of any $s\in\mathbb{R}$ version of the integral over $\mathbb{R}\times Y$ of $2\mathrm{r}\chi_s|\mathfrak{b}_\varepsilon\wedge *\mathfrak{X}|$. With the preceding understood, the remainder of the proof of Lemma 5.4 restricts attention (implicitly for the most part) to the contribution to the integral of the function $2\mathrm{r}\chi_s|\mathfrak{b}_\varepsilon\wedge *\mathfrak{X}|$ from the part of $\mathbb{R}\times Y$ where $\mathrm{dist}(\cdot\,, X_*)\le c_0\mathrm{r}^{-1/2}\ln\mathrm{r}$. To set the notation, let m denote the particular value of this last incarnation of the number c_0, and use X_{**} to denote the part of $\mathbb{R}\times Y$ where $\mathrm{dist}(\cdot\,, X_*)\le m\mathrm{r}^{-1/2}\ln\mathrm{r}$.

To continue exploiting (5-15), fix $z\ge 1$ for the moment and use the inequality

$$q_\diamond^{11/2} c_c \mathrm{r}^{-1/12}\bigl|1-|\alpha|^2\bigr| \le z^{-1} q_\diamond^6 \bigl|1-|\alpha|^2\bigr|^{12/11} + z c_c \mathrm{r}^{-1} \tag{5-16}$$

to see that the term with factor $q_\diamond^{11/2}\mathrm{r}^{-1/12}$ in (5-15) contributes at most $z^{-1}\mathrm{r}\underline{\mathrm{M}}_\diamond + z c_c\mathrm{r}$ to the integral of $\mathrm{r}\chi_s|\mathfrak{b}_\varepsilon\wedge *\mathfrak{X}|$.

Meanwhile, the inequality

$$q_\diamond^{11/2}|1-\sigma|^{1/2}\bigl|1-|\alpha|^2\bigr| \le z^{-1} q_\diamond^{22/3}\bigl|1-|\alpha|^2\bigr|^{2/3} + z c_0 (1-\sigma)^2(1-|\alpha|^2)^2 \tag{5-17}$$

implies that the term in (5-15) with the factor $q_\diamond^{11/2}|1-\sigma|^{1/2}$ contributes at most

$$c_0 z^{-1}\mathrm{r}^2 \int_{(\mathbb{R}\times Y)\cap X_{**}} \chi_s q_\diamond^{22/3}\bigl|1-|\alpha|^2\bigr|^{2/3} + c_0 z \mathrm{r}^2 \int_{(\mathbb{R}\times Y)\cap X_{**}} \chi_s (1-\sigma)^2 \bigl|1-|\alpha|^2\bigr|^2 \tag{5-18}$$

to the integral of $\mathrm{r}\chi_s|\mathfrak{b}_\varepsilon\wedge *\mathfrak{X}|$. Use (5-7) to see that the right-most integral in (5-18) is no greater than $c_0 z\int_{\mathbb{R}\times Y}\bigl|\frac{\partial}{\partial s}A\bigr|^2$ and therefore no greater than $z c_c\mathrm{r}$ on account of Lemma 4.2.

Step 3 The conclusions of Step 2 supply the bound

$$\mathrm{r}\biggl|\int_{\mathbb{R}\times Y} \chi_s \mathfrak{b}_\varepsilon\wedge * B_A\biggr| \le c_0 z^{-1}\mathrm{r}^2 \int_{(\mathbb{R}\times Y)\cap X_{**}} \chi_s q_\diamond^{22/3}\bigl|1-|\alpha|^2\bigr|^{2/3} + z^{-1}\mathrm{r}\underline{\mathrm{M}}_\diamond + z c_c \mathrm{r}. \tag{5-19}$$

This step and the next supply an appropriate upper bound for

$$\mathrm{r}^2 \int_{(\mathbb{R}\times Y)\cap X_{**}} \chi_s q_\diamond^{22/3}\bigl|1-|\alpha|^2\bigr|^{2/3}. \tag{5-20}$$

To start this task, introduce κ_* to denote the version of κ given by Lemma 4.8. Separate the domain of integration in (5-20) into two parts: The first part is the set of points in X_* (where $1-|\alpha|^2 \ge \kappa_*^{-1}$) and the second part is the part in $X_{**} - X_*$ (which is the part of X_{**} where $1-|\alpha|^2 \le \kappa_*^{-1}$). Noting that $q_\diamond^{22/3} = q_\diamond^6 q_\diamond^{4/3}$, the contribution to the integral in (5-20) from the $1-|\alpha|^2 \ge \kappa_*^{-1}$ part of the domain is no greater than $c_0 \mathrm{r}\underline{\mathrm{M}}_\diamond$. Because of this, it is enough to bound the integral of (5-20) with the domain restricted to the subset in $[s-2, s+2] \times Y$ where $1-|\alpha|^2 < \kappa_*^{-1}$ and which is in X_{**} (which is where the distance to X_* is no greater than $m\mathrm{r}^{-1/2}\ln \mathrm{r}$). The strategy will be to show that the contribution to (5-20) from the part in $X_{**} - X_*$ is no greater than c_0 times the contribution from the X_* part. The upcoming Step 5 finds a lower bound for the contribution from X_*, and then Step 6 considers the contribution from $X_{**} - X_*$. Step 4 supplies some preliminary observations.

Step 4 It follows from Lemma 4.12 and the second bullet of Proposition 4.11 that there is a point where $|\alpha| < \frac{1}{2}$ with distance $c_0 \mathrm{r}^{-1/2}$ or less from each point in X_*. With the preceding in mind, let p denote a point where $|\alpha| < \frac{1}{2}$.

The function $q_\diamond$ in the radius $2m\mathrm{r}^{-1/2}\ln \mathrm{r}$ ball centered at p is no less than $\frac{1}{2} q_\diamond(p)$ and no greater than $2q_\diamond(p)$ unless p has distance less than $c_0 m\mathrm{r}^{-1/2}\ln \mathrm{r}$ from the zero locus of $q_\diamond$, this being $\bigcup_{\mathfrak{p}\in\Lambda}(\hat\gamma_\mathfrak{p}^+ \cup \hat\gamma_\mathfrak{p}^-)$. The contribution to (5-20) and to $\mathrm{r}\underline{\mathrm{M}}(s)$ from the set of such points is no greater than $c_0 \mathrm{r}^{-5}$ because $q_\diamond$ near any of these closed integral curves of v is bounded by c_0 times the square of the distance to the integral curve. If p does indeed have distance greater than $c_0 m\mathrm{r}^{-1/2}\ln \mathrm{r}$ from where $q_\diamond$ is zero, then the function $q_\diamond$ in the radius $m\mathrm{r}^{-1/2}\ln \mathrm{r}$ ball centered at p is bounded above and below (uniformly) by constant multiples of its value at p. Thus, if B is a radius $\rho = m\mathrm{r}^{-1/2}\ln \mathrm{r}$ ball centered at p, then

$$\frac{1}{c_0} q_\diamond(p)^{22/3} \int_{(\mathbb{R}\times Y)\cap B} \chi_s \big|1-|\alpha|^2\big|^{2/3} \le \int_{(\mathbb{R}\times Y)\cap B} \chi_s q_\diamond^{22/3} \big|1-|\alpha|^2\big|^{2/3} \le c_0 q_\diamond(p)^{22/3} \int_{(\mathbb{R}\times Y)\cap B} \chi_s \big|1-|\alpha|^2\big|^{2/3}. \tag{5-21}$$

This says, in effect, that the point-to-point variation of $q_\diamond$ on B is of no concern with regards to the derivation of upper or lower bounds for the middle integral in (5-21).

Step 5 Fix $n \in \{1, 2, \dots\}$ which is less than $c_0 m \ln \mathrm{r}$. Let p denote a point in X_* where $|\alpha| < \frac{1}{2}$ and where the distance to $\bigcup_{\mathfrak{p}\in\Lambda}(\hat\gamma_\mathfrak{p}^+ \cup \hat\gamma_\mathfrak{p}^-)$ is greater than $c_0 m \ln \mathrm{r}$.

It follows from what was is said in Step 4 and from Lemmas 4.10 and 5.2 that the contribution to $\mathrm{r}\underline{\mathrm{M}}_{\diamond}(s-1)+\mathrm{r}\underline{\mathrm{M}}_{\diamond}(s+1)$ from the ball of radius $n\mathrm{r}^{-1/2}$ centered at p is *no less* than $c_c^{-1}n^2 q_{\diamond}(p)^6$. Note that this is a lower bound for the contribution. With the preceding in mind, fix a maximal set $\mathfrak{U}_n \subset X_*$ obeying the following:

(5-22)
- The function $|\alpha|$ is less than $\frac{1}{2}$ at all points in $\mathfrak{U}_n$.
- The union of the balls of radius $2^8 n\mathrm{r}^{-1/2}$ centered at the points in $\mathfrak{U}_n$ cover the subset in X_{**} with distance $4n\mathrm{r}^{-1/2}$ or less from the subset where $|\alpha| < \frac{1}{2}$.
- The respective balls of radius $n\mathrm{r}^{-1/2}$ centered at distinct points in $\mathfrak{U}_n$ are disjoint.

The conditions in the second and third bullets of (5-22) imply that any given point in X_{**} with distance $2^8 n\mathrm{r}^{-1/2}$ or less from where $|\alpha| < \frac{1}{2}$ is in at most c_0 balls of radius $2^8 n\mathrm{r}^{-1/2}$ centered at the points in $\mathfrak{U}_n$.

It follows from what said in this step's opening paragraph and from the condition in the third bullet of (5-22) that

$$\mathrm{r}\underline{\mathrm{M}}_{\diamond}(s-1)+\mathrm{r}\underline{\mathrm{M}}_{\diamond}(s+1) \ge c_c^{-1}n^2 \sum_{p\in\mathfrak{U}_n} q_{\diamond}(p)^6. \tag{5-23}$$

Note that this is also asserting a lower bound.

Step 6 Supposing that $n \in \{1,2,\dots\}$ but less than $c_0 m\mathrm{r}^{-1/2}\ln\mathrm{r}$, let X_n for $n \in \{1,2,\dots\}$ denote the subset of $X_{**}\cap\big(\big[s-\frac{3}{2}, s+\frac{3}{2}\big]\times Y\big)$ with distance between $n\mathrm{r}^{-1/2}$ and $(n-1)\mathrm{r}^{-1/2}$ from X_* and with $1-|\alpha|^2 < \kappa_*^{-1}$. Lemma 4.8 with Proposition 4.11 and Lemma 4.12 have the following corollary: given that $\mathrm{r} \ge c_0$, there is a point in X_* where $|\alpha| < \frac{1}{2}$ and with distance less than $(n+c_0)\mathrm{r}^{-1/2}$ from each point in X_n.

With the preceding understood, let p denote a point in X_* where $|\alpha| < \frac{1}{2}$ and where the distance to $\bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_{\mathrm{p}}^{+} \cup \hat{\gamma}_{\mathrm{p}}^{-})$ is greater than $c_0 m\ln\mathrm{r}$. Use Lemma 4.8 to see that the contribution to (5-20) from the part of X_n that is in the ball of radius $(n+c_0)\mathrm{r}^{-1/2}$ centered at p is no greater than $c_0 e^{-n/c_0} n^4 q_{\diamond}^6$. This bound and the second bullet in (5-22) and the lower bound in (5-23) imply the following: the X_n contribution to the integral in (5-20) is less than $c_c e^{-n/c_0}\mathrm{r}\sup_{s\in\mathbb{R}}\underline{\mathrm{M}}_{\diamond}$. (Note that a straightforward inequality is used here: if $x > 0$, then $n^4 e^{-nx} = (n^2 e^{-nx/2})(n^2 e^{-nx/2})$, which is less than $c_0 x^{-2} n^2 e^{-nx/2}$.)

Sum these various $n \in \{1,2,\dots\}$ contributions to (5-20) to see that the contribution to (5-20) from X_{**} is at most $c_c\mathrm{r}\sup_{s\in\mathbb{R}}\underline{\mathrm{M}}_{\diamond}$.

Step 7 Being that the integral in (5-20) is no greater than $c_c \mathrm{r} \sup_{s\in\mathbb{R}} \underline{\mathrm{M}}_\diamond$, the right-hand side of (5-19) is no greater than $c_c(z^{-1}\mathrm{r}\sup_{s\in\mathbb{R}} \underline{\mathrm{M}}_\diamond + z\mathrm{r})$. The $z \ge c_c^2$ version of this last bound with what is said at the outset of Step 2 give the top inequality of Lemma 5.4.

Step 8 This step proves the second inequality of Lemma 5.4. To this end, use Lemma 4.8 to see that $|D_A\psi| \le c_0 \mathrm{r}^{1/2}((1-|\alpha|^2)+c_0\mathrm{r}^{-1})^{1/2}$. Meanwhile, $|d(Q_\varepsilon q_\diamond)| \le c_0 q_\diamond^{11/2}$ because $q_\diamond^{1/2} \ge c_0^{-1}|dq_\diamond|$. These observations have the following consequence: the supremum in the second bullet is no greater than $c_0(z^{-1}\mathrm{r}\sup_{s\in\mathbb{R}} \underline{\mathrm{M}}_\diamond + z\mathrm{r})$ for any $z \ge 1$. Any $z \ge 1000c_0$ version of this last fact gives the assertion of Lemma 5.4's second bullet. □

6 A priori bounds for $\underline{\mathrm{M}}$

Fix $\mathrm{r} > c_0$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Suppose that $\mathfrak{d}$ is an instanton solution to the corresponding (r, μ) version of (4-1). This section uses Proposition 5.1's bound on $\underline{\mathrm{M}}_\diamond$ to derive a $\mathfrak{d}$– and r–independent bound for the function $\underline{\mathrm{M}}$. The proposition that follows makes the formal statement that such a bound exists:

Proposition 6.1 *There exists $\kappa \ge \pi$ and, given $c \ge 1$, there exists $\kappa_c > 1$ with the following significance: Fix $\mathrm{r} \ge \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Suppose that $\mathfrak{d} = (A, \psi)$ is an instanton solution to the (r, μ) version of (4-1) with $\mathrm{A}_\mathfrak{d} < c\mathrm{r}$ and $\lim_{s\to\infty} \mathrm{M}(\mathfrak{d}|_s) < c$. Then the corresponding function $\underline{\mathrm{M}}$ obeys $-\kappa < \underline{\mathrm{M}} < \kappa_c$.*

But for one extra lemma, the proof of Proposition 6.1 is in Section 6.2. The extra lemma is proved in Section 6.4. Section 6.1 makes observations that are used in the proof of Proposition 6.1. Sections 6.1 and 6.2 borrow much from the proof of Lemma 5.2 in [22]. Section 6.3 supply some facts that are used in Section 6.4 and again in Section 7's proof of Theorem 1.5. The assertions in Section 6.3 all have analogs in Section 4 of [22].

6.1 Functions E and $\underline{\mathrm{E}}$

Let $\mathfrak{d} = (A, \psi)\colon \mathbb{R} \to \mathrm{Conn}(E) \times C^\infty(M; \mathbb{S})$ denote an instanton solution to the (r, μ) version of (4-1) with $\mathrm{A}_\mathfrak{d} < c\mathrm{r}$ and with $\lim_{s\to\infty} \mathrm{M}(\mathfrak{d}|_s) < c$. Introduce the function $\mathrm{E}(\cdot)$ on $\mathbb{R}$ given by

$$s \mapsto \mathrm{E}(s) = i\int_{\{s\}\times Y} \hat{a} \wedge *B_A, \tag{6-1}$$

and introduce the function $\underline{\mathrm{E}}$ on $\mathbb{R}$ given by the rule

$$s \mapsto \underline{\mathrm{E}}(s) = \int_{[s-1,s+1]\times Y} \mathrm{E}(\cdot). \tag{6-2}$$

This section explains how bounds on $\underline{\mathrm{E}}$ give bounds on $\underline{\mathrm{M}}$.

To do this, differentiate E and use the top bullet of (4-1) and an integration by parts to see that

$$\frac{d}{ds}\mathrm{E} = i \int_{\{s\}\times Y} d\hat{a} \wedge (-B_A + \mathrm{r}(\psi^\dagger \tau \psi - i\hat{a})) + \mathfrak{e} \tag{6-3}$$

with $|\mathfrak{e}| \le c_0$. The 2–form $d\hat{a}$ is zero on $M_\delta \cup \mathcal{H}_0$ and it is equal to w where $|u| < R + c_0 \ln \delta$ on each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$. In particular, $d\hat{a} = w$ where $q_\diamond \le c_0^{-1}$. This understood, use Lemma 4.4 with Proposition 5.1 to write (6-3) as

$$\frac{d}{ds}\mathrm{E} = -\mathrm{E} + \mathrm{M} + \mathfrak{r}_\mathrm{E}, \tag{6-4}$$

where $|\mathfrak{r}_\mathrm{E}| \le c_c$, with c_c denoting a purely c–dependent constant. By way of notation, c_c will henceforth denote a purely c–dependent constant that is greater than 1. Its value can be assumed to increase between successive appearances.

Integrate (6-4) to see that

$$\mathrm{E}(s) = e^{-s} \int_{-\infty}^{s} e^x (\mathrm{M}(x) + \mathfrak{r}_\mathrm{E})\, dx. \tag{6-5}$$

It follows from Lemma 5.2 that $\mathrm{M}(\cdot) \ge -c_0$, and thus (6-4) leads to the bound

$$-c_0 \le \mathrm{E}(s) \le e^t (\mathrm{E}(s+t) + c_c) \quad \text{for any } t \ge 0. \tag{6-6}$$

It then follows from (6-5) and (6-6) that

$$\mathrm{E}(s) \le c_c + c_c^2 \underline{\mathrm{E}}(s+2) \quad \text{and} \quad \underline{\mathrm{M}}(s) \le c_c(\underline{\mathrm{E}}(s+4) + 1). \tag{6-7}$$

Thus, a bound on $\underline{\mathrm{E}}$ gives a bound on $\underline{\mathrm{M}}$. By way of a converse to (6-7), note that

$$\mathrm{E}(s) \le (1 + c_c \mathrm{r}^{-1/12})\mathrm{M}(s) + c_c \quad \text{and} \quad \underline{\mathrm{E}}(s) \le (1 + c_c \mathrm{r}^{-1/12})\underline{\mathrm{M}}(s) + c_c, \tag{6-8}$$

this being a consequence of (5-7) and (5-9).

As explained next, the function E is closely related to the function $s \mapsto \mathrm{w}(A|_s)$ with w as defined in (1-27). The discussion that follows uses $\mathrm{w}_A(s)$ to denote $\mathrm{w}(A|_s)$. To say more about E and w_A, use (4-1) to see that

$$\frac{d}{ds}\mathrm{w}_A = -\mathrm{E} + \mathrm{M} + \mathfrak{r}_\mathrm{w} \tag{6-9}$$

where $|\mathfrak{r}_{\mathrm{w}}| \le c_0$. In particular, a comparison with (6-4) finds that

$$\left|\frac{d}{ds}(\mathrm{E} - \mathrm{w}_A)\right| \le c_c. \tag{6-10}$$

Most of $|\mathrm{E} - \mathrm{w}_A|$ is accounted for by the restriction of A to $M_\delta \cup \mathcal{H}_0$ in the sense that the $\hat{a} = \hat{a}_A|_s$ version of (2-7) with (5-7), (5-9) and Proposition 5.1 can be used to write

$$\mathrm{w}_A = \mathrm{E} + i \sum_{z \in ¥} \mathrm{C}_{\mathbb{S},z} \int_{\{s\}\times\gamma^{(z)}} \hat{a}_A + \mathfrak{r}_*, \tag{6-11}$$

where $\{\mathrm{C}_{\mathbb{S},z}\}_{z\in ¥}$ are integers and where $\mathfrak{r}_*$ is a function on $\mathbb{R}$ with $|\mathfrak{r}_*| \le c_c$. A given $z \in ¥$ version of the integral that appears in (6-11) is the value at $A|_s$ of (2-4)'s function $\mathrm{X}^{(z)}$. The lemma that follows says more about the ¥–indexed sum in (6-11). This lemma writes the $s \to \infty$ limit of $\mathfrak{d}$ as (A_+, ψ_+) and writes A_+ as $A_E + \hat{a}_{A_+}$.

Lemma 6.2 *There exists $\kappa \ge \pi$ and, given $c \ge 1$, there exists $\kappa_c > 1$ with the following significance: Fix $\mathrm{r} \ge \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to the (r, μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}} < c\mathrm{r}$ and $\lim_{s\to\infty} \mathrm{M}(\mathfrak{d}|_s) < c$. Then*

$$\left|\sum_{z\in ¥} \mathrm{C}_{\mathbb{S},z}\left(\int_{\{s\}\times\gamma^{(z)}} \hat{a}_A\right) - \sum_{z\in ¥} \mathrm{C}_{\mathbb{S},z}\left(\int_{\gamma^{(z)}} \hat{a}_{A_+}\right)\right| < \kappa_c$$

for all $s \in \mathbb{R}$.

This lemma is proved in Section 6.4. Accept it as true in the meantime. By way of a look ahead, Proposition 5.1 plays a key role in the proof of this lemma; it plays no role otherwise.

6.2 An algebraic inequality for $\underline{\mathrm{E}}$

The equation that follows asserts the desired algebraic inequality for $\underline{\mathrm{E}}$:

$$\underline{\mathrm{E}}(s) \le c_c\big(1 + \mathrm{r}^{-1/3} \sup_{x \ge s+1} |\underline{\mathrm{E}}(x)|^{4/3}\big). \tag{6-12}$$

The derivation of this formula is given in a moment. What follows directly assumes (6-12) to prove Proposition 6.1.

Proof of Proposition 6.1 The $s \to \infty$ limit of $\underline{\mathrm{E}}$ is by assumption bounded by c. Fix for the moment $T > c$ and let s_T denote the largest value of s with $\underline{\mathrm{E}}(s) \ge T$. Let c_* denote the version of c_c that appears in (6-12). The $s = s_T$ version of (6-12) reads $T < c_*(1 + \mathrm{r}^{-1/3} T^{4/3})$. This has no solutions for $T \in (2c_*, c_*^{-1}\mathrm{r})$ if $\mathrm{r} > 2^8 c_*^8$. The bound $\underline{\mathrm{E}} < 2c_*$ leads via (6-7) to a purely c–dependent bound for $\underline{\mathrm{M}}$. □

The four parts that follow derive the inequality in (6-12).

Part 1 The derivation starts with an inequality that involves the functions $\mathfrak{a}(\mathfrak{d}|_s)$, E and M and a function, O, on $\mathbb{R}$ that is defined at any given s by

$$\mathrm{O}(s) = \int_{\{s\}\times Y} (|\mathfrak{B}_{(A,\psi)}|^2 + \mathrm{r}|D_A\psi|^2), \tag{6-13}$$

with $\mathfrak{B}_{(A,\psi)}$ as defined in (4-2). The derivation of this first inequality mimics the derivation of an analogous inequality in Section 5b of [22].

To start, use (1-28) to see that

$$\mathfrak{a}(\mathfrak{d}|_s) \le \mathfrak{cs}(A|_s) - \mathrm{r}\mathrm{w}_A(s) + c_0\mathrm{r}^{1/2}\mathrm{O}(s)^{1/2}. \tag{6-14}$$

The next part of the derivation talks about the function $\mathfrak{cs}$.

Part 2 The formula for $\mathfrak{cs}$ is given in (1-26) as a sum of two integrals. To say more about the right-most integral in (1-26), keep in mind that the $i\mathbb{R}$–valued 2–form $2F_{A_E} + F_{A_K}$ that appears in this formula is cohomologous to $-2\pi i w$. This being the case, their difference is the exterior derivative of a fixed, smooth 1–form, this denoted by $\mathfrak{y}$. As a consequence, integration by parts equates the right-most integral in (1-26) with

$$-2\int_{\{s\}\times Y} \hat{a}_A \wedge \left(F_{A_E} + \tfrac{1}{2}F_{A_K}\right) = 2\pi\mathrm{w}_A + i\int_{\{s\}\times Y} *B_A \wedge \mathfrak{y}. \tag{6-15}$$

Use Lemma 5.2's preliminary bound for $\underline{\mathrm{M}}$ in Lemma 4.7 to bound the absolute value of the right-most integral in (6-15) by $c_c(\mathrm{M}+1)$.

The remaining term in (1-26) is the integral of $\hat{a}_A \wedge d\hat{a}_A$. This term can be bounded by writing $\hat{a}_A = \hat{a}_A^{\perp} + \mathfrak{q}$ with $\hat{a}_A^{\perp}$ a coclosed 1–form that is orthogonal to the space of harmonic 1–forms on Y. Meanwhile, $\mathfrak{q}$ is a closed 1–form on Y. (This is just the Hodge decomposition of the 1–form $\hat{a}_A$.) The integral of $\hat{a}_A \wedge d\hat{a}_A$ is the same as that of $\hat{a}_A^{\perp} \wedge d\hat{a}_A^{\perp}$. Meanwhile, the norm of $\hat{a}_A^{\perp} \wedge d\hat{a}_A^{\perp}$ obeys

$$\left|\int_{\{s\}\times Y} \hat{a}_A \wedge d\hat{a}_A\right| \le c_c(\mathrm{r}^{2/3}\mathrm{M}^{4/3} + 1). \tag{6-16}$$

The derivation of the latter bound has two steps.

Step 1 This step bounds $|\hat{a}_A|$ pointwise by $c_c(\mathrm{r}^{2/3}\mathrm{M}^{1/3} + 1)$. This is done with the help of Lemmas 4.7 and 5.2 and the Green's function for the Laplacian $*d*d - d*d*$ acting on 1–forms that are L^2–orthogonal to the space of harmonic 1–forms. (The strategy here mimics what is done in the proof of Lemma 2.4 of [17].) To elaborate:

Because $d\hat{a}_A^{\perp} = *(B_A - B_{A_E})$ and $d *\hat{a}_A^{\perp} = 0$, and because $\hat{a}_A^{\perp}$ is orthogonal to the harmonic forms, it can be written as $*d\mathcal{G}(B_A - B_{A_E})$, where $\mathcal{G}$ is the aforementioned Green's function. With $\mathcal{G}$ viewed as a homomorphism-valued function on $M \times M$, it obeys $|d\mathcal{G}|_{(p,q)} \le c_0 \operatorname{dist}(p,q)^{-2}$. Given the latter bound, then Lemma 4.7 (with Lemma 5.2's preliminary bound for $\underline{\mathrm{M}}$) can be used to see that $|d\mathcal{G}(B_A - B_{A_E})| \le c_c(\rho^{-2}\mathrm{M} + \rho\mathrm{r} + 1)$, where ρ can be any given number in $(0, c_0^{-1})$. This bound is obtained by bounding $|d\mathcal{G}|_{(p,q)}$ by $c_0\rho^{-2}$ where $\operatorname{dist}(p,q) > \rho$ and bounding $|B_A|$ by $c_c\mathrm{r}$ where $\operatorname{dist}(p,q) \le \rho$. (This last bound comes from Lemma 4.7 with Lemma 5.2's preliminary bound for $\underline{\mathrm{M}}$.) Taking $\rho = \mathrm{r}^{-1/3}\mathrm{M}^{1/3}$ gives the asserted bound for $|\hat{a}_A|$.

Step 2 Use the bound $|\hat{a}_A| \le c_c\mathrm{r}^{2/3}\mathrm{M}^{1/3}$ to obtain a bound $c_c\mathrm{r}^{2/3}\mathrm{M}^{1/3}|B_A - B_{A_E}|$ for the integrand in (6-16). Then use Lemma 4.7 again with Lemma 5.2's bound for $\underline{\mathrm{M}}$ to bound the latter integral by $c_c(\mathrm{r}^{2/3}\mathrm{M}^{4/3} + 1)$.

Use the bound in (6-16) and the bound by $c_c(\mathrm{M}+1)$ for the right-most integral in (6-15) to see that

$$\mathfrak{cs}(A|_s) \le 2\pi\mathrm{w}_A + c_c(1 + \mathrm{M} + \mathrm{r}^{2/3}\mathrm{M}^{4/3}). \tag{6-17}$$

The next step exploits this inequality for $\mathfrak{cs}$.

Part 3 Replacing $\mathfrak{cs}$ in (6-14) with the right-hand side of (6-17) leads to the inequality

$$\mathfrak{a}(\mathfrak{d}|_s) \le -(\mathrm{r}-\pi)\mathrm{w}_A + c_c(\mathrm{M} + \mathrm{O} + \mathrm{r} + \mathrm{r}^{2/3}\mathrm{M}^{4/3}). \tag{6-18}$$

Replace the function w_A in (6-18) by E using Lemma 6.2. Having done so, rearrange terms to obtain the following inequality for E:

$$(\mathrm{r}-\pi)\mathrm{E} \le -\mathfrak{a}(\mathfrak{d}|_s) - (\mathrm{r}-\pi)\sum_{z\in\yen} \mathrm{C}_{\mathbb{S},z}\left(\int_{\gamma^{(z)}} \hat{a}_{A_+}\right) + c_c(\mathrm{O} + \mathrm{r} + \mathrm{r}^{2/3}\mathrm{M}^{4/3}). \tag{6-19}$$

As the function $s \mapsto \mathfrak{a}(\mathfrak{d}|_s)$ is nonincreasing, the right-hand side of (6-19) is no less than

$$-\mathfrak{a}(\mathfrak{c}_+) - (\mathrm{r}-\pi)\sum_{z\in\yen} \mathrm{C}_{\mathbb{S},z}\left(\int_{\gamma^{(z)}} \hat{a}_{A_+}\right) + c_c(\mathrm{O} + \mathrm{r} + \mathrm{r}^{2/3}\mathrm{M}^{4/3}). \tag{6-20}$$

Use the A_+ versions of (6-11), (6-15) and (6-16) to bound the combined two left-most terms in (6-20) by $c_0(\mathrm{r}\mathrm{M}(\mathfrak{c}_+) + \mathrm{r}^{2/3}\mathrm{M}(\mathfrak{c}_+)^{4/3})$. Using this bound leads to the inequality

$$\mathrm{r}\mathrm{E} \le c_c(\mathrm{r}\mathrm{M}(\mathfrak{c}_+) + \mathrm{r}^{2/3}\mathrm{M}(\mathfrak{c}_+)^{4/3} + \mathrm{O} + \mathrm{r} + \mathrm{r}^{2/3}\mathrm{M}^{4/3}) \tag{6-21}$$

when $\mathrm{r} > 2\pi$. The assumed $\mathrm{M}(\mathfrak{c}_+) \le c$ bound and (6-21) imply that

$$\mathrm{r}\mathrm{E} \le c_c(\mathrm{O} + \mathrm{r} + \mathrm{r}^{2/3}\mathrm{M}^{4/3}).$$

Part 4 Let F for the moment denote any given nonnegative function on $[-1,1]$ and let $\underline{\mathrm{F}}$ denote its integral over this interval. The measure of the set of points where F is less than $8\underline{\mathrm{F}}$ must be greater than $\frac{8}{15}$. This being the case, suppose that F' is a second nonnegative function. Then there are points in $[s-1,s+1]$ where both F and F' are less than $8\underline{\mathrm{F}}$ and $8\underline{\mathrm{F}}'$, respectively.

With the preceding in mind, introduce $\underline{\mathrm{O}}(s)$ to denote the integral of $\mathrm{O}(\cdot)$ over the interval $[s-1,s+1]$. Fix $s' \in [s-1,s+1]$ where $\mathrm{O}(s') \le 8\underline{\mathrm{O}}(s)$ and $\mathrm{M}(s') \le 8\underline{\mathrm{M}}(s)$. The s' version of the inequality $\mathrm{rE} \le c_c(\mathrm{O} + \mathrm{r} + \mathrm{r}^{2/3}\mathrm{M}^{4/3})$ implies that

$$\mathrm{rE}(s') \le c_c(\underline{\mathrm{O}}(s) + \mathrm{r} + \mathrm{r}^{2/3}\underline{\mathrm{M}}(s)^{4/3}). \tag{6-22}$$

As explained next, the inequality in (6-12) follows from (6-20) with three additional replacements. The first replacement invokes Lemma 4.2 to substitute $2\,\mathrm{A}_{\mathfrak{d}}$ for $\underline{\mathrm{O}}(s)$. The second replacement invokes (6-7) to replace $\underline{\mathrm{M}}(s)$ with $\sup_{s \ge \mathbb{R}} \underline{\mathrm{E}}(x)$.

To explain the final replacement, fix for the moment $s'' \in [s-3,s-1]$ and invoke (6-6) with $t = s' - s''$. With the first and second replacements made, (6-22) and (6-6) imply

$$\mathrm{rE}(s'') \le c_c\Big(\mathrm{r} + \mathrm{A}_{\mathfrak{d}} + \mathrm{r}^{2/3} \sup_{x \ge s} \underline{\mathrm{E}}(x)^{4/3}\Big). \tag{6-23}$$

View both sides of (6-23) as functions on $[s-3,s-1]$, with the right-hand side being the constant function. Integrate both sides over this interval. The integral of the left-hand side is $\mathrm{r}\underline{\mathrm{E}}(s-2)$ and that of the right is twice what is written in (6-23). The resulting inequality with the assumed $\mathrm{A}_{\mathfrak{d}} \le c\mathrm{r}$ bound leads directly to (6-12) when evaluated at $s+2$ rather than s.

6.3 Local convergence to pseudoholomorphic subvarieties

The upcoming Proposition 6.3 describes how certain instanton solutions to (4-1) can be used to determine pseudoholomorphic subvarieties in bounded subsets of $\mathbb{R} \times Y$. Proposition 6.3 and the subsequent two lemmas about pseudoholomorphic subvarieties are used to prove Lemma 6.2 and they are invoked again in Section 7 to prove Theorem 1.5. Proposition 6.3 is the analog of Proposition 4.1 in [22] and subsequent two lemmas are the respective analogs of Lemma 4.6 and Corollary 4.7 in [22].

Proposition 6.3 and the two lemmas use Y_* to denote either $M_\delta \cup \mathcal{H}_0$ or Y. Their assertions with regards to $Y_* = M_\delta \cup \mathcal{H}_0$ are used in the upcoming proof of Lemma 6.2 and those that concern $Y_* = Y$ are used in Section 7.

Proposition 6.3 introduces the notion of a pseudoholomorphic subvariety in an open set of $\mathbb{R}\times Y$. To define this term, let $U \subset \mathbb{R}\times Y$ denote the open set. A subset $C \subset U$ is said to be a pseudoholomorphic subvariety in U when the conditions below are met:

(6-24)
- C is the intersection between U and a closed subset, C', of a neighborhood of U.
- The complement in C' of a finite set of points is a smoothly embedded submanifold of this neighborhood with J–invariant tangent space.
- C' has no totally disconnected components.
- The integral of w over C' is finite.
- There exists an $s\in\mathbb{R}$ independent upper bound for the integral of $ds\wedge\hat{a}$ over the intersection between C' and $[s-1,s+1]\times Y$.

The subvariety C is said to be irreducible when the complement in C of any finite set of points is connected.

Proposition 6.3 *Fix $c \ge 1$ and, in the case $Y_* = Y$, also $\mathcal{K} > 1$. There exists $\kappa_c > 1$ and, given $m > \max(\kappa_c, 100)$, there exists $\kappa_{c,m} > \pi$, these having the following significance: Fix $r \ge \kappa_{c,m}$ and fix $\mu\in\Omega$ with $\mathcal{P}$–norm less than 1. Suppose that $\mathfrak{d} = (A, \psi = (\alpha,\beta))$ is an instanton solution to the (r,μ) version of (4-1) with $A_{\mathfrak{d}} < cr$. If $Y_* = Y$, assume in addition that $\sup_{s\in\mathbb{R}} \underline{M}(s) < \mathcal{K}$. Let $I\subset\mathbb{R}$ denote an interval of length $2m$. Then each point in $I\times Y_*$ where $|\alpha| < 1-\kappa_{c,m}^{-1}$ has distance $\kappa_{c,m} r^{-1/2}$ or less from $\alpha^{-1}(0)$. Also, there exists a finite set, ϑ, of at most κ_c elements with each element having the form (C,m) with C an irreducible, pseudoholomorphic subvariety in $I\times Y_*$ and m a positive integer no greater than κ_c. Distinct pairs from ϑ have distinct subvariety components. Furthermore:*

- $$\sup_{z\in\bigcup_{(C,m)\in\vartheta}(C\cap(I\times Y_*))} \operatorname{dist}(z,\alpha^{-1}(0)) + \sup_{z\in(\alpha^{-1}(0)\cap(I\times Y_*))} \operatorname{dist}\Big(\bigcup_{(C,m)\in\vartheta} C, z\Big) < m^{-1}.$$
- *Let υ denote the restriction to $I\times Y_*$ of a smooth 2–form on $\mathbb{R}\times Y$ with $\|\upsilon\|_\infty = 1$ and $\|\nabla\upsilon\|_\infty \le m$. Then*
$$\left|\frac{i}{2\pi}\int_{I_*\times Y} \upsilon\wedge F_{\hat{A}} - \sum_{(C,m)\in\vartheta} m\int_{C\cap(I\times Y_*)} \upsilon\right| \le m^{-1}.$$

Proof But for one extra remark in the $Y_* = M_\delta\cup\mathcal{H}_0$ case, the arguments for the first bullet and items (a) and (b) of the second bullet of Proposition 4.1 in [22] can be used with only notational changes to prove Proposition 6.3. The extra remark concerns the assumption made by Proposition 6.1 for a bound on $\underline{\text{M}}(s)$. The arguments for Proposition 4.1 in [22] require only a bound on Lemma 4.10's function $\underline{\text{M}}_{(x,\rho)}$ for $\rho = c_0^{-1}$ and for all $x \in \mathbb{R} \times Y_*$. Such a bound comes from Proposition 5.1's bound on $\underline{\text{M}}_\diamond$ when $Y = M_\delta \cup \mathcal{H}_0$. □

The next two lemmas are used to say more about the subvarieties that can arise in the context of Proposition 6.3. The lemma that follows directly is an analog of Lemma 4.6 in [22]. The notation is such that if $s \in \mathbb{R}$ and C is a pseudoholomorphic subvariety that is defined in a neighborhood of $\{s\} \times Y_*$, then $C|_s$ denotes $C \cap (\{s\} \times Y_*)$.

Lemma 6.4 *Given $m > 1$ and $\varepsilon > 0$, there exists $\kappa_{m_\varepsilon} > 1$ with the following significance: Suppose that C is a closed, irreducible, pseudoholomorphic subvariety in a neighborhood of $\mathcal{J} := [-4, 4] \times Y_*$ with $\int_{C \cap \mathcal{J}} w < \kappa_{m_\varepsilon}^{-1}$ and $\int_{C \cap \mathcal{J}} ds \wedge \hat{a} \le m$. Then each point of $C|_s$ for $|s| \le 1$ has distance along Y_* no greater than ε from a closed integral curve, γ, of length less than $m + \varepsilon$. Moreover, there is a positive integer q which is bounded by an m and ε independent multiple of m, and is such that if υ is a smooth 2–form on $[-1, 1] \times Y_*$ with $\|\upsilon\|_\infty = 1$ and $\|\nabla\upsilon\|_\infty \le \varepsilon^{-1}$, then $\left|\int_{C \cap ([-1,1] \times Y_*)} \upsilon - q \int_{[-1,1] \times \gamma} \upsilon\right| \le \varepsilon$.*

Proof The proof is the same but for notation of Lemma 4.6 in [22]. □

The next lemma is an analog of [22, Corollary 4.7]. Note in this regard that [22, Corollary 4.7] makes an assumption that is not guaranteed here, this being that all integral curves of v with an a priori length bound are nondegenerate. The upcoming Lemma 6.5 is a version of [22, Corollary 4.7] that suffices for the present purposes.

Lemma 6.5 *Given $m > 1$ and $\varepsilon > 0$, there exists $\kappa_{m_\varepsilon} > 1$ with the following significance: Let $\mathbb{I} \subset \mathbb{R}$ denote an interval of length at least 4, and suppose that C is an irreducible, pseudoholomorphic subvariety in a neighborhood of $\mathbb{I} \times Y_*$ with $\int_{C \cap (I' \times Y_*)} w < \kappa_{m_\varepsilon}^{-1}$ and $\int_{C \cap (I' \times Y_*)} ds \wedge \hat{a} < m$ for all intervals $I' \subset \mathbb{I}$ of length 1. Assume in addition that C has intersection number zero with all submanifolds in $\mathbb{R} \times Y$ of the form $\{s\} \times S$ with S being a cross-sectional sphere in $\mathcal{H}_0$. Let $I \subset \mathbb{I}$ denote the subset with distance at least 3 from any boundary point of $\mathbb{I}$. There exists a finite set Θ consisting of pairs (γ, q) with γ a closed, integral curve of v and q a positive integer. The set Θ is such that no two pairs share the same closed integral curve. Moreover:*

- *The intersection of γ with M_δ is a collection of arcs that begin on the boundary of radius δ coordinate balls about the index 1 critical points of f in M and end* *on the boundary of radius δ coordinate balls about the index 2 critical points of f in M.*

- $\sum_{(\gamma,q)\in\Theta} q\ell_\gamma < m+\varepsilon$.
- *Each point of $C|_s$ for $s \in I$ has distance along Y less than ε from $\bigcup_{(\gamma,q)\in\Theta} \gamma$. Conversely, each point in $\bigcup_{(\gamma,q)\in\Theta} \gamma$ has distance no greater than ε from $C|_s$.*
- *If υ is a smooth 2–form on $I \times Y$ with $\|\upsilon\|_\infty = 1$ and $\|\nabla\upsilon\|_\infty \le \varepsilon^{-1}$, then*

$$\left|\int_{C\cap(I\times Y)} \upsilon - \sum_{(\gamma,q)\in\Theta} q\int_{I\times Y} \upsilon\right| < \varepsilon.$$

Proof But for one additional remark, the argument in [22] that explains how [22, Corollary 4.7] follows from Lemma 4.6 in [22] explains why Lemma 6.5 follows from Lemma 6.4. The additional remark concerns both the first bullet of the lemma and the assumption for Corollary 4.7 in [22] of nondegenerate Reeb orbits. The assumption that C has intersection number zero with submanifolds of the form $\{s\}\times S$ with $S \subset \mathcal{H}_0$ being a cross-sectional sphere replaces the nondegeneracy assumption in Lemma 4.6 of [22] and it leads directly to the first bullet of Lemma 6.5. To see how this comes about, note that if $S \subset \mathcal{H}_0$ is a constant u sphere, then $\{s\}\times S$ is pseudoholomorphic, so if $C|_s$ is close to a closed integral curve of v that crosses $\mathcal{H}_0$, then C will have positive intersection number with $\{s\}\times S$. This is ruled out by assumption. Meanwhile, Section II.2 finds that the only closed integral curves of v that don't intersect $\mathcal{H}_0$ are hyperbolic and so nondegenerate. Moreover, those that intersect M_δ are described by the first bullet of Lemma 6.5. □

6.4 Proof of Lemma 6.2

The proof has four parts. These parts use c_c to denote a number greater than 1 that depends only on c. Its value can be assumed to increase between successive appearances.

Part 1 The curvature of the version (1-15) that defines $\hat{A}$ can be written as

$$F_{\hat{A}} = (1-\wp)\Big(ds \wedge \frac{\partial}{\partial s}A + *B_A\Big) + \wp'\nabla_A\alpha \wedge \nabla_A\bar{\alpha} \tag{6-25}$$

with it understood that the ds component of $\nabla_A\alpha$ is $\frac{\partial}{\partial s}\alpha$. The notation here uses $\wp$ and $\wp'$ to denote the respective functions on $I_* \times Y_*$ given by $\wp|_{t=|\alpha|^2}$ and

$\left(\frac{d}{dt}\wp\right)\big|_{t=|\alpha|^2}$. Use (6-25) to see that $w \wedge F_{\hat{A}}$ can be written as $-i\mathrm{F}\,ds \wedge \hat{a} \wedge w$ with F being

$$i(1-\wp)\Big(\frac{\partial}{\partial s}A\Big)_v - i\wp'\Big(\Big(\frac{\partial}{\partial s}\alpha\Big)(\nabla_A\bar{\alpha})_v - (\nabla_A\alpha)_v\Big(\frac{\partial}{\partial s}\bar{\alpha}\Big)\Big), \tag{6-26}$$

where $\left(\frac{\partial}{\partial s}A\right)_v$ and $(\nabla_A\alpha)_v$ denote the pairing of these 1–forms with the vector field v.

Part 2 Let I denote an interval of length 2 and introduce V to denote the subset of $I \times (M \times \mathcal{H}_0)$ where $|\alpha|^2 < \frac{5}{8}$. The support of $F_{\hat{A}}$ in $I \times (M \times \mathcal{H}_0)$ is in V. Use (6-25) and (6-26) to see that

$$c_0 \int_V \Big|\frac{\partial}{\partial s}\alpha\Big|^2 \geq \Big|\frac{i}{2\pi}\int_{I\times(M\times\mathcal{H}_0)} w \wedge F_{\hat{A}}\Big| - c_0\,\mathrm{vol}(V), \tag{6-27}$$

where $\mathrm{vol}(V)$ denotes the volume of the set V. Proposition 5.1 bounds the integral of $\mathrm{r}(1-|\alpha|^2)$ over V by c_c and this implies that $\mathrm{vol}(V) \leq \mathrm{r}^{-1}c_c$. Therefore, (6-27) implies that

$$c_0\mathrm{r}\int_{I\times Y}\Big|\frac{\partial}{\partial s}\psi\Big|^2 \geq \mathrm{r}\Big|\frac{i}{2\pi}\int_{I\times(M\times\mathcal{H}_0)} w \wedge F_{\hat{A}}\Big| - c_c. \tag{6-28}$$

This last bound leads directly to the following conclusion: if $\varepsilon \in (\mathrm{r}^{-1}c_c, 1)$, there are at most $\varepsilon^{-1}c_c$ disjoint intervals in I of the form $[s-1, s+1]$ with

$$\Big|\frac{i}{2\pi}\int_{[s-1,s+1]\times(M\times\mathcal{H}_0)} w \wedge F_{\hat{A}}\Big| > \varepsilon.$$

Part 3 Apply the $Y_* = M_\delta \cup \mathcal{H}_0$ version of Proposition 6.3 to intervals of length 200 in $\mathbb{R}$. Use the first bullet of the latter, Lemma 6.5 and the final conclusion in Part 2 to see that there exists a set of at most c_c points in $\mathbb{R}$ with the following property: if $s \in \mathbb{R}$ has distance 1 or more from all points in this set, then $F_{\hat{A}} = 0$ and $\alpha/|\alpha|$ is $\hat{A}$–covariantly constant at points with distance c_0^{-1} or less from any $z \in$ ¥ version of $\{s\} \times \gamma^{(z)}$. Let $\mathcal{Q}$ denote this finite set in $\mathbb{R}$.

Suppose that $s \in \mathbb{R}$ has distance less than 2 from some point in $\mathcal{Q}$. The fact that $\mathcal{Q}$ has at most c_c elements implies that there are points in $[s-c_c, s+c_c]$ with distance at least 2 from each point in $\mathcal{Q}$. Let s' denote such a point. Use the s and s' versions of (6-11) with the derivative bound in (6-10) to conclude that Lemma 6.2 is true for s if and only if it is true for s'.

Introduce $\mathcal{Q}_* \subset \mathbb{R}$ to denote the set of points with distance less than 2 from some point in $\mathcal{Q}$. Let $(s, s') \subset \mathbb{R}$ denote a connected component of $\mathcal{Q}_*$. Then $|s'-s| < c_c$ because

$\mathcal{Q}$ has at most c_c elements. This understood, use the s and s' versions of (6-11) with (6-10) again to see that

$$\left|\sum_{z\in ¥} \mathrm{c}_{\mathbb{S},z}\left(\int_{\{s\}\times\gamma^{(z)}} \hat{a}_A\right) - \sum_{z\in ¥} \mathrm{c}_{\mathbb{S},z}\left(\int_{\{s'\}\times\gamma^{(z)}} \hat{a}_A\right)\right| < \kappa_c. \tag{6-29}$$

Given the conclusions of the preceding two paragraphs, the fact that $\mathcal{Q}$ has at most c_c elements implies that Lemma 6.2 holds if (6-29) is also true when s and s' are any two elements in the same component of $\mathbb{R} - \mathcal{Q}_*$.

Part 4 To see about (6-29) when s and s' are in the same component of $\mathbb{R} - \mathcal{Q}_*$, fix for the moment a point $z \in ¥$. Write $\hat{A}$ as $\hat{A} = A_E + \hat{a}_{\hat{A}}$ and use the $\mathbb{R}\times Y$ version of (1-15) with Lemma 4.8 to see that

$$\left|\int_{\{s\}\times\gamma^{(z)}} \hat{a}_A - \int_{\{s\}\times\gamma^{(z)}} \hat{a}_{\hat{A}}\right| \le c_0 \tag{6-30}$$

when s has distance 2 or more from every point in $\mathcal{Q}$. Note that this inequality also holds with A replaced by A_+ and with $\hat{A}$ replaced by $\hat{A}_+$, this being the $s \to \infty$ limit of $\hat{A}$.

With (6-30) in mind, suppose that $s' > s$ are in the same component of $\mathbb{R} - \mathcal{Q}_*$. Use Stokes' theorem to see that

$$\int_{\{s'\}\times\gamma^{(z)}} \hat{a}_{\hat{A}} - \int_{\{s\}\times\gamma^{(z)}} \hat{a}_{\hat{A}} = \int_{[s,s']\times\gamma^{(z)}} F_{\hat{A}}. \tag{6-31}$$

The right-hand side of (6-31) is zero, so it follows using the s and s' versions of (6-30) that the integral of $\hat{a}_A$ over $\{s\}\times\gamma^{(z)}$ differs by at most c_0 from its integral over $\{s'\}\times\gamma^{(z)}$. Thus, (6-29) does indeed hold for any pair $s' > s$ in the same component of $\mathbb{R} - \mathcal{Q}_*$. □

7 Propositions 1.1–1.4 and Theorem 1.5

This last section supplies the proofs for Section 1's propositions and theorem.

7.1 Proofs of Propositions 1.1–1.3

Leave out for the moment the second and third bullets of Proposition 1.1 and the assertions of Propositions 1.2 and 1.3 that refer to $\hat{\mathcal{Z}}^{\ge}_{\mathrm{SW,r}}$. The remaining assertions of these propositions, those that refer only to $\hat{\mathcal{Z}}^{\ge}_{\mathrm{SW,r}}$, are all special cases of theorems from [7]. To elaborate, the essential concern is a compactness theorem for the space of instanton solutions (1-20). See in particular the discussion at the beginning of Chapter 29.2 in [7]. The desired compactness theorem is Proposition 29.2.1 in [7]. This is because the $\mathrm{r} > \pi$ versions of (1-14) and (1-20) are defined by what is said in [7] to be a *monotone* perturbation.

The second bullet of Proposition 1.1 follows directly from Proposition 2.4. To elaborate: The bullets in Proposition 2.4 imply that $|\alpha|$ is nearly 1 along $\gamma^{(z_0)}$. This being the case, then (by construction) the section $\alpha/|\alpha|$ is $\hat{A}$–covariantly constant along $\gamma^{(z_0)}$. Therefore, the holonomy of $\hat{A}$ is 1 along $\gamma^{(z_0)}$, which implies that $\mathrm{x}(\hat{A} - A_E)$ is an integer (because A_E also has holonomy 1 along $\gamma^{(z_0)}$).

The third bullet of Proposition 1.1 and the assertions about $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ in Propositions 1.2 and 1.3 follow from a proof that the value of the function x in (1-16) on the $s \to \infty$ limit of any relevant instanton is no less than its value on the $s \to -\infty$ limit if the instanton contributes to the differential on the chain complex, or to one of the other homomorphisms. This property of x follows from the upcoming Proposition 7.1 together with Lemmas 2.5, 4.1 and 4.2. Note in this regard that Proposition 7.1 proves this assertion about x for instanton solutions to (4-1), this being the version of (1-20) that uses $\mathfrak{g} = \mathfrak{e}_\mu$ with $\mu \in \Omega$ having $\mathcal{P}$–norm less than 1. Even so, the fact that $\lim_{s\to\infty} \mathrm{x}(\mathfrak{d}|_s) \geq \lim_{s\to-\infty} \mathrm{x}(\mathfrak{d}|_s)$ for the instanton solutions to (4-1) implies this inequality is also true for any instanton solutions to a $\mathfrak{g} = \mathfrak{e}_\mu + \mathfrak{p}$ version of (1-20) that contributes to the differential or the other relevant homomorphisms if $\mathfrak{p}$ comes from a certain residual set in $\mathcal{P}_\mu$ and has small $\mathcal{P}$–norm. More is said about why this is after the statement of Proposition 7.1.

To set the stage for Proposition 7.1, let γ denote a closed, integral curve of v. Define the function x_γ on $\mathrm{Conn}(E)$ by the rule that assigns to a connection A on E the integral over the curve γ of the 1–form $\frac{i}{2\pi}(\hat{A} - A_E)$. The $\gamma = \gamma^{(z_0)}$ version of x_γ is the function x in (1-16).

The proposition assumes that $\gamma \subset M_\delta \cup \mathcal{H}_0$ and that γ has a tubular neighborhood of the sort described directly. Let ℓ denote the length of γ and let $t \in \mathbb{R}/(\ell\mathbb{Z})$ denote an affine parameter for γ. Use z to denote the complex coordinate for $\mathbb{C}$. The operative assumption is that γ has a tubular neighborhood with coordinates (t, z) that are defined for $|z|$ less than a positive constant and are such that:

(7-1) • The curve γ is the $z = 0$ locus.

- The vector field v, the 2–form w and the 1–form $\hat{a}$ appear as
$$v = \frac{\partial}{\partial t} + \cdots, \quad w = \tfrac{1}{2} i\, dz \wedge d\bar{z} + \cdots \quad \text{and} \quad \hat{a} = dt,$$
where the unwritten terms are bounded by a constant multiple of $|z|$.
- The vector field v annihilates $|z|^2$.

It follows from the constructions in Sections II.1C and II.1D that each $z \in ¥$ version of $\gamma^{(z)}$ has a tubular neighborhood with coordinates of the sort described by (7-1), and, in particular, the curve $\gamma^{(z_0)}$ has such a tubular neighborhood.

Proposition 7.1 *There exists $\kappa \geq \pi$ and, given $c \geq 1$, there exists $\kappa_c > 1$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 and suppose that $\mathfrak{d} = (A, \psi)$ is an instanton solution to the (r, μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}} < c\mathrm{r} \ln \mathrm{r}$. Let γ denote a closed, integral curve of v that lives entirely in $M_\delta \cup \mathcal{H}_0$ and has a tubular neighborhood with coordinates of the sort described by (7-1). Then $\lim_{s\to\infty} \mathrm{x}_\gamma(A|_s) \geq \lim_{s\to-\infty} \mathrm{x}_\gamma(A|_s)$.*

Given what is said by the third bullet of Proposition 2.7 and Lemma 4.1, the assumption $\mathrm{A}_{\mathfrak{d}} < c\mathrm{r}\ln \mathrm{r}$ is satisfied if the difference between the value of $\mathfrak{f}_s$ on the $s \to \infty$ limit of $\mathfrak{d}$ and the value of $\mathfrak{f}_s$ on the $s \to -\infty$ limit of $\mathfrak{d}$ is no greater than $c \ln \mathrm{r}$. The bound $\mathrm{A}_{\mathfrak{d}} < c\mathrm{r}\ln \mathrm{r}$ in Proposition 7.1 is used to invoke Lemma 5.2 so as to bound $\mathfrak{d}$'s version of the function $\underline{\mathrm{M}}$ by $c_c \mathrm{r}^{6/7}$. This bound on $\underline{\mathrm{M}}$ is then used to invoke Lemma 4.7. Lemma 4.7 in turn is used to write $\frac{\partial}{\partial s} A$ and B_A as in (5-7). A crucial point in subsequent arguments is that the function σ that appears in (5-7) is constrained to obey

$$-c_c \mathrm{r}^{-1/q} < \sigma < 1 + c_c \mathrm{r}^{-1/q} \quad \text{where } \big|1 - |\alpha|^2\big| > \mathrm{r}^{-1+1/2q}, \tag{7-2}$$

with q being any number greater than 12. Indeed, this follows from (5-9) because its left-hand side is nonnegative. (In general, $\mathrm{r}\,|\alpha|\,\big|1 - \frac{1}{2}|\alpha|^2\big|$ and $\mathrm{r}\,\big|1 - |\alpha|\big|\,\big|1 - \frac{1}{2}|\alpha|^2\big|$ are bounded by c_c, which is a consequence of Lemma 4.9.)

The proof of Proposition 7.1 is given in Section 7.2. What follows directly explains how Proposition 7.1 is used to prove the assertions in Propositions 1.1–1.3 that refer to $\hat{\mathcal{Z}}^{\geq}_{\mathrm{SW,r}}$. To this end, suppose that the conclusions of Proposition 7.1 hold for instanton solutions to a given $\mathfrak{g} = \mathfrak{e}_\mu + \mathfrak{p}$ version of (1-20) if (r, μ) obey its assumptions and if the perturbation $\mathfrak{p}$ is in $\mathcal{P}_\mu$. If $\mathfrak{d}$ is an instanton solution to this version of (1-20) and if it contributes to either the differential or one of the other relevant homomorphisms of the Seiberg–Witten chain complex, then the $s \to \infty$ limit of $\mathfrak{f}_s(\mathfrak{d}|_s)$ is either 1 or 2 more than the $s \to -\infty$ limit of $\mathfrak{f}_s(\mathfrak{d}|_s)$. Use this observation with Proposition 2.7 and Lemma 4.1 to see that $\mathfrak{d}$ obeys Proposition 7.1's bound $\mathrm{A}_{\mathfrak{d}} \le c\mathrm{r}\ln \mathrm{r}$.

Given what was said in the preceding paragraph, the assertions in Propositions 1.1–1.3 hold if the conclusions of Proposition 7.1 hold for instanton solutions to any $\mathfrak{g} = \mathfrak{e}_\mu + \mathfrak{p}$ version of (1-20) if (r, μ) obey its assumptions and if $\mathfrak{p} \in \mathcal{P}_\mu$ has small $\mathcal{P}$–norm. To see why this is so, assume it to be false so as to derive nonsense. Under this contrary assumption, there is a sequence $\{\mathfrak{p}_n\}_{n=1,2,\ldots}$ with the following two properties: the $\mathcal{P}$–norm of each $n \in \{1, 2, \ldots\}$ version of $\mathfrak{p}_n$ is less than n^{-1} and the conclusions of Proposition 7.1 fail for some instanton solution to the $\mathfrak{g} = \mathfrak{e}_\mu + \mathfrak{p}_n$ version of (1-20) with $\mathrm{A}_{\mathfrak{d}} < c\mathrm{r}\ln \mathrm{r}$. Let $\{\mathfrak{d}_{\mathfrak{p}_n}\}_{n=1,2,\ldots}$ denote a corresponding sequence of recalcitrant instantons. This sequence can be chosen so that all its constituent members have the same $s \to \infty$ limit, and all have the same $s \to -\infty$ limit. The latter are denoted respectively by $\mathfrak{c}_+$ and $\mathfrak{c}_-$. Since the function X takes integer values on the solutions to (1-13), the operative assumption in what follows is that $\mathrm{X}(\mathfrak{c}_+) \le \mathrm{X}(\mathfrak{c}_-) - 1$.

The function $s \mapsto \mathfrak{a}(\mathfrak{d}_{\mathfrak{p}_n}|_s) + \mathfrak{p}_n(\mathfrak{d}_{\mathfrak{p}_n}|_s)$ is a nonincreasing function on $\mathbb{R}$ and as the sequence $\{\mathfrak{p}_n\}_{n=1,2,\ldots}$ is bounded and converges to zero, the fact that the set of $\mathfrak{d} = \mathfrak{d}_{\mathfrak{p}_n}$ versions of $\mathrm{A}_{\mathfrak{d}}$ is bounded implies that the sequence $\{\mathfrak{d}_{\mathfrak{p}_n}\}_{n=1,2,\ldots}$ has a subsequence that converges in the sense described in Chapter 16 of [7] to what is said in Definition 16.1.2 of [7] to be a broken trajectory. In the situation here, such a trajectory consists of a nonempty, finite, ordered set $\{\mathfrak{d}_k\}_{k=1,2,\ldots,N}$ of instanton solutions to (4-1) with the following property: the $s \to \infty$ limit of $\mathfrak{d}_k$ is the $s \to -\infty$ limit of $\mathfrak{d}_{k+1}$ for $k < N$. In addition, $\mathfrak{c}_-$ is the $s \to -\infty$ limit of $\mathfrak{d}_1$ and $\mathfrak{c}_+$ is the $s \to \infty$ limit of $\mathfrak{d}_N$. This being the case, $\mathrm{X}(\mathfrak{c}_+) - \mathrm{X}(\mathfrak{c}_-)$ can be written as $\sum_{k=1,2,\ldots,N} \big(\lim_{s\to\infty} \mathrm{X}(\mathfrak{d}_k|_s) - \lim_{s\to\infty} \mathrm{X}(\mathfrak{d}_k|_s)\big)$. This sum is nonnegative if each $\mathfrak{d}_k$ obeys $\mathrm{A}_{\mathfrak{d}_k} \le c\mathrm{r}\ln \mathrm{r}$ so as to invoke Proposition 7.1.

To see that this last bound is obeyed, use Lemma 4.2 to see that each $\mathfrak{d} = \mathfrak{d}_k$ version of $\mathrm{A}_{\mathfrak{d}}$ is positive, so the desired bound holds if it holds for $\sum_{1\le k\le N} \mathrm{A}_{\mathfrak{d}_k}$. Meanwhile, the fact that the function on $\mathbb{R}$ given by the rule $s \mapsto \mathfrak{a}(\mathfrak{d}_{\mathfrak{p}_n}|_s) + \mathfrak{p}_n(\mathfrak{d}_{\mathfrak{p}_n}|_s)$ is nonincreasing and the fact that $\{\mathfrak{p}_n\}_{n=1,2,\ldots}$ converges to zero implies that the $n \to \infty$ limit of the set of $\mathfrak{d} = \mathfrak{d}_{\mathfrak{p}_n}$ versions of $\mathrm{A}_{\mathfrak{d}}$ exists. Moreover, the manner of convergence of $\{\mathfrak{d}_{\mathfrak{p}_n}\}_{n=1,2,\ldots}$ to $\{\mathfrak{d}_k\}_{k=1,2,\ldots}$ as described in Chapter 16 of [7] guarantees that the limit of the corresponding set of $\mathfrak{d} = \mathfrak{d}_{\mathfrak{p}_n}$ versions of $\mathrm{A}_{\mathfrak{d}}$ is no less than $\sum_{1\le k\le N} \mathrm{A}_{\mathfrak{d}_k}$. In fact, the limit equals the sum if all solutions to the (r, μ) version of (1-13) are nondegenerate.

The conclusion that $\mathrm{x}(\mathfrak{c}_+) - \mathrm{x}(\mathfrak{c}_-) \geq 0$ violates the assumptions and so constitutes the desired nonsense.

7.2 Proof of Proposition 7.1

The proof has six parts. By way of notation, c_c is used in what follows to denote a constant which is greater than 1 that is determined solely by c, γ and the geometry of Y. In particular, c_c does not depend on $\mathfrak{d}$, nor does it depend on the chosen values of r or μ. The value of c_c can be assumed to increase between successive appearances.

Part 1 Let $\mathfrak{c}_+ = (A_+, \psi_+)$ denote the $s \to \infty$ limit of $\mathfrak{d}$ and let $\mathfrak{c}_- = (A_-, \psi_-)$ denote the corresponding $s \to -\infty$ limit of $\mathfrak{d}$. Both $\hat{A}_+$ and $\hat{A}_-$ are flat with trivial holonomy on a fixed, but small radius tubular neighborhood of γ if r is greater than a purely γ–dependent constant. This fact implies that $\mathrm{x}_\gamma(A_+) - \mathrm{x}_\gamma(A_-) \in \mathbb{Z}$.

With the preceding in mind, fix for the moment a smooth, closed 2–form on Y with compact support in this tubular neighborhood whose de Rham cohomology class is the image of the Poincaré dual of the class in $H_1(Y;\mathbb{Z})$ that is defined by viewing the oriented loop γ as a 1–cycle. Use υ_γ to denote the chosen 2–form.

Reintroduce $\hat{A}$ to denote the connection that is defined by A using the formula in (1-15) with it understood that $\nabla_A \alpha$ has ds component equal to $\frac{\partial}{\partial s}\alpha$. The curvature of this connection is depicted in (6-25). Stokes' theorem writes $\mathrm{x}_\gamma(A_+) - \mathrm{x}_\gamma(A_-)$ as the integral over $\mathbb{R} \times Y$ of the curvature 2–form $\frac{i}{2\pi} F_{\hat{A}} \wedge \upsilon_\gamma$.

By way of a parenthetical remark, if $\mathrm{A}_{\mathfrak{d}} \leq c\mathrm{r}$, then Propositions 5.1 and 6.3 can be invoked if r is greater than a purely c–dependent constant. Assume this to be the case. It follows from Proposition 6.3 and what is said in Part 4 of the proof of Lemma 6.2 that the integral of $\frac{i}{2\pi} F_{\hat{A}} \wedge \upsilon_\gamma$ is a weighted, algebraic count with positive weights of the intersections between the submanifold $\mathbb{R} \times \gamma$ and a pseudoholomorphic subvariety that is defined in some neighborhood of $\mathbb{R} \times \gamma$. Thus $\mathrm{x}_\gamma(A_+) - \mathrm{x}_\gamma(A_-) \geq 0$.

The equality between $\mathrm{x}_\gamma(A_+) - \mathrm{x}_\gamma(A_-)$ and the integral of $\frac{i}{2\pi} F_{\hat{A}} \wedge \upsilon_\gamma$ does not depend on the chosen version of υ_γ as long as its support is in a radius c_0^{-1} tubular neighborhood of γ. This being the case, the remainder of this Part 1 defines a useful choice. To this end, let T denote a radius c_0^{-1} tubular neighborhood of γ with coordinates of the sort that are described in (7-1). Assume that T appears in these coordinates as $\mathbb{R}/(\ell\mathbb{Z}) \times D_0$, where D_0 is a radius c_0^{-1} disk centered at the origin in $\mathbb{C}$.

The desired version of υ_γ is constructed with the help of a nonnegative, nonincreasing function on $[0,\infty)$ with support in $[0,2]$. The latter is denoted in what follows by q and it has the following properties:

(7-3)
- $q(x)=1$ where $x<1$ and $q(x)=0$ where $x>2$.
- $q(x)=e^{-1/(2-x)}/(1-e^{-1/(1-x)}+e^{-1/(2-x)})$ where $x\in[1,2]$.

Use ζ to denote the integral of the function $x\mapsto 2\pi x q(x)$. With ζ in hand, fix D so as to be greater than 100 times the inverse of the radius of D_0. The value of D can be taken smaller than a constant that is determined ultimately by c and γ. With D fixed, let q_D denote the function on D_0 that is defined by the rule $z\mapsto\zeta^{-1}\mathrm{D}^2q(\mathrm{D}z)$. Use the coordinates for T in (7-1) to view q_D as a function on Y with compact support in T.

The desired version of υ_γ is defined to be zero on the complement of T and defined to equal $q_\mathrm{D}w$ on T. So defined, the condition in the third bullet of (7-1) guarantees that υ_γ is closed. This understood, it follows from the definition that its de Rham cohomology class is the image in de Rham cohomology of the Poincaré dual of γ's class in $H_1(Y;\mathbb{Z})$.

Part 2 The upcoming Lemma 7.2 refers to certain notions that are defined directly. The first of these is ρ_r, this used to denote $(\ln\mathrm{r})^{-4}$. To define the rest, fix $s_0\in\mathbb{R}$, $t_0\in\mathbb{R}/(\ell\mathbb{Z})$ and a point $z_0\in D_0$ with $|z_0|$ at most half the diameter of D_0. The lemma uses $Q^{(s_0,t_0,z_0)}$ to denote the set in $\mathbb{R}\times\mathbb{R}/(\ell\mathbb{Z})\times D_0$ whose (s,t,z) coordinates obey the two conditions $|s-s_0|\le 2\rho_\mathrm{r}$ and $|t-t_0|+|z-z_0|<4\rho_\mathrm{r}$. The integral of $iF_{\hat{A}}\wedge(ds\wedge\hat{a}+w)$ over $Q^{(s_0,t_0,z_0)}$ is denoted by $\Delta_{(s_0,t_0,z_0)}$. The lemma uses $Q_{(s_0,z_0)}$ to denote $\bigcup_{t\in\mathbb{R}/(\ell\mathbb{Z})}Q^{(s_0,t,z_0)}$; this is the set whose (s,t,z) coordinates obey the two constraints $|s-s_0|<2\rho_\mathrm{r}$ and $|z-z_0|<4\rho_\mathrm{r}$ with no constraint on t.

Lemma 7.2 *Given $c>1$ there exists $\kappa_c>1$ with the following significance: Fix $\mathrm{r}\ge\kappa_c$ and $\mu\in\Omega$ with $\mathcal{P}$–norm less than 1 and suppose that $\mathfrak{d}=(A,\psi)$ is an instanton solution to the (r,μ) version of (4-1) with $\mathrm{A}_\mathfrak{d}<c\mathrm{r}\ln\mathrm{r}$. Fix $s_0\in\mathbb{R}$ and $z_0\in D_0$ with $|z_0|$ less than half the radius of D_0. If $\sup_{t\in\mathbb{R}/(\ell\mathbb{Z})}\Delta_{(s_0,t,z_0)}>\kappa_c\rho_\mathrm{r}^2$, then*

$$\int_{Q_{(s_0,z_0)}} iF_{\hat{A}}\wedge w>\kappa_c^{-1}\rho_\mathrm{r}\sup_{t\in\mathbb{R}/(\ell\mathbb{Z})}\Delta_{(s_0,t,z_0)}.$$

Step 1 This step states five observations that play central roles in the subsequent steps. The first observation constitutes the next equation. This writes $\frac{\partial}{\partial s}A$ and B_A as in (5-7).

These equations with (6-25), (4-1) and Lemma 4.6 lead to

$$(7\text{-}4)\qquad \begin{aligned} *(iF_{\hat{A}} \wedge w) &= (1-\wp)(1-\sigma)\mathrm{r}(1-|\alpha|^2) + 2\wp'|(\nabla^{(1,0)}\alpha)_0|^2 + \mathfrak{e}_A, \\ *(iF_{\hat{A}} \wedge ds \wedge \hat{a}) &= (1-\wp)\sigma\mathrm{r}(1-|\alpha|^2) + 2\wp'|(\nabla^{(1,0)}\alpha)_1|^2 + \mathfrak{e}_B, \end{aligned}$$

with the notation as follows. What is denoted by σ is the function that appears in (5-7). To define $(\nabla^{(1,0)}\alpha)_0$ and $(\nabla^{(1,0)}\alpha)_1$, first introduce $\nabla^{(1,0)}\alpha$ to denote the $T^{1,0}(\mathbb{R}\times Y)$ part of the covariant derivative of α. View the latter as a homomorphism from the $(1,0)$ summand in $T(\mathbb{R}\times Y)\otimes\mathbb{C}$ to E. What is denoted by $(\nabla^{(1,0)}\alpha)_0$ is the restriction of this homomorphism to the span of $\frac{\partial}{\partial s} - iv$, and what is denoted by $(\nabla^{(1,0)}\alpha)_1$ is the restriction of $\nabla^{(1,0)}\alpha$ to the $+i$ eigenspace of J in the $K^{-1}\otimes\mathbb{C}$ summand in $T(\mathbb{R}\times Y)\otimes\mathbb{C}$. Meanwhile, $\mathfrak{e}_A$ and $\mathfrak{e}_B$ are such that their absolute values are bounded by $c_0((1-\wp)+\wp')$. The $1-\wp$ contribution to the latter bounds follows from the bounds on $\mathfrak{z}_A$ and $\mathfrak{z}_B$ in (5-7), and the $\wp'$ contribution follows by using (4-1) to write the $T^{0,1}$ part of $\nabla_A\alpha$ as a linear combination of covariant derivatives of β and then invoking Lemma 4.6.

Adding the two equalities in (7-4) leads to the second observation:

$$(7\text{-}5)\qquad *(iF_{\hat{A}} \wedge (ds\wedge\hat{a} + w)) \geq \tfrac{1}{4}(1-\wp)\mathrm{r} + 2\wp'|\nabla^{(1,0)}\alpha|^2 + \mathfrak{e},$$

with $\mathfrak{e}=0$ where $\wp=1$ and $|\mathfrak{e}|\leq c_0\mathrm{r}^{-3/2}$ in any event. By way of an explanation, this inequality follows from the fact that $|\alpha|^2 < \frac{9}{16}$ on the support of $1-\wp$ and from the fact that $\wp'\leq c_0(1-\wp)^{3/4}$. To elaborate, remember that $\wp<1$ only where $|\alpha|^2\leq\frac{9}{16}$. Therefore, summing the two inequalities in (7-4) yields

$$(7\text{-}6)\qquad *(iF_{\hat{A}} \wedge (ds\wedge\hat{a} + w)) \geq \tfrac{7}{16}(1-\wp)\mathrm{r} + 2\wp'|\nabla^{(1,0)}\alpha|^2 + \mathfrak{e}_A + \mathfrak{e}_B.$$

Now, $\mathfrak{e}_A$ and $\mathfrak{e}_B$ obey

$$(7\text{-}7)\qquad |\mathfrak{e}_A| + |\mathfrak{e}_B| \leq c_0((1-\wp)+\wp'),$$

and $\wp'$ is nonnegative and it obeys $\wp' < c_0(1-\wp)^{3/4}$. The $c_0(1-\wp)$ term in (7-7) is accounted for by replacing the $\frac{7}{16}(1-\wp)\mathrm{r}$ term in (7-6) with $\frac{3}{8}(1-\wp)\mathrm{r}$; that is, $*(iF_{\hat{A}}\wedge(ds\wedge\hat{a}+w)) \geq \frac{3}{8}(1-\wp)\mathrm{r} + 2\wp'|\nabla^{(1,0)}\alpha|^2 + \mathfrak{e}_0$, where $|\mathfrak{e}_0|\leq c_0(1-\wp)^{3/4}$. Writing $(1-\wp)^{3/4}$ as $\mathrm{r}^{3/8}(1-\wp)^{3/4}\mathrm{r}^{-3/8}$ and using a standard algebraic inequality shows that $(1-\wp)^{3/4}\leq c_0(\mathrm{r}^{1/2}(1-\wp)+\mathrm{r}^{-3/2})$. For r greater than c_0, this implies that $|\mathfrak{e}_0|\leq\frac{1}{8}\mathrm{r}(1-\wp)+c_0\mathrm{r}^{-3/2}$.

Fix $q\in[12,c_0]$ to invoke (7-2). The third observation is

$$(7\text{-}8)\qquad \int_{Q_{(s_0,z_0)}} iF_{\hat{A}}\wedge w \geq -c_c(\mathrm{r}^{-1/(q+1)}\Delta_* + \mathrm{r}^{-3/2}).$$

This is a consequence of (7-5), the first bullet of (7-4) and (7-2). (Remember that (7-2) says that σ is at most $1+c_c\mathrm{r}^{-1/q}$ on the support of $(1-\wp)$.) The point is that the expression on the right side of the first bullet in (7-4) is negative only where σ is greater than 1 or the $\mathfrak{e}_A$ term is negative and dominates the others. What with (7-2), this leads to a $-c_c((1-\wp)\mathrm{r}^{1-1/q}+(1-\wp)^{3/4})$ lower bound for the right-hand side of the top bullet in (7-4). (Remember that $\wp' \le c_0(1-\wp)^{3/4}$.) Because of the inequality $(1-\wp)^{3/4} \le c_0\mathrm{r}^{1/2}(1-\wp)+c_0\mathrm{r}^{-2/3}$ from the preceding paragraph, the right-hand side of the top bullet in (7-4) is in no account less than $-c_c((1-\wp)\mathrm{r}^{1-1/q}+\mathrm{r}^{-3/2})$. Now, by virtue of (7-5) and the definition of Δ_*, the integral of $\mathrm{r}(1-\wp)$ over $Q_{s,t}$ is at most $c_c\ell((\ln \mathrm{r})^4\Delta_*+\mathrm{r}^{-3/2})$. (Keep in mind here that the length of γ in units of ρ_{r} is at most $\ell(\ln\mathrm{r})^4$.) Therefore, the integral of $-c_c(1-\wp)\mathrm{r}^{1-1/q}-c_c\mathrm{r}^{-3/2}$ is at most $\mathrm{r}^{-1/q}$ times $c_c\ell((\ln\mathrm{r})^4\Delta_*+\mathrm{r}^{-3/2})$, which is at most $-c_c(\mathrm{r}^{-1/(1+q)}\Delta_*+\mathrm{r}^{-3/2})$.

The fourth observation is a direct corollary to (7-8):

(7-9) Fix $t\in\mathbb{R}/(\ell\mathbb{Z})$. If $m>0$ and if the integral of $iF_{\hat{A}}\wedge w$ over $Q^{(s_0,t,z_0)}$ is greater than $m+c_c\mathrm{r}^{-1/(q+1)}(\Delta_*+1)$, then $\int_{Q_{(s_0,z_0)}} iF_{\hat{A}}\wedge w > m$.

The fifth observation is a tautology that comes by writing $iF_{\hat{A}}\wedge(ds\wedge\hat{a}+w)$ as the sum of $iF_{\hat{A}}\wedge w$ and $iF_{\hat{A}}\wedge ds\wedge\hat{a}$. To set the notation, fix $t_0\in\mathbb{R}/(\ell\mathbb{Z})$ such that $\Delta_{(s_0,t_0,z_0)}=\Delta_*$. Use Q^* to denote $Q^{(s_0,t_0,z_0)}$.

(7-10) If the integral of $iF_{\hat{A}}\wedge w$ over Q^* is less than $\frac{1}{2}\Delta_*$, then the integral of $i*B_{\hat{A}}\wedge ds\wedge\hat{a}$ over Q^* is greater than $\frac{1}{2}\Delta_*$.

This is so because $iF_{\hat{A}}\wedge ds\wedge\hat{a} = i*B_{\hat{A}}\wedge ds\wedge\hat{a}$.

Step 2 This step outlines the argument that is used to prove Lemma 7.2. Assume that $\Delta_*>\rho_{\mathrm{r}}^2$. It follows from (7-9) that the assertion of Lemma 7.2 is true if the integral of $iF_{\hat{A}}\wedge w$ over Q^* is greater than $\frac{1}{100}\Delta_*$, so assume that this is not the case. Use (7-10) to see that the integral of $i*B_{\hat{A}}\wedge ds\wedge\hat{a}$ over Q^* is greater than $\frac{1}{2}\Delta_*$.

The constant (s,t) slices of Q^* are disks that lie either in a cross-sectional sphere of $\mathcal{H}_0$ or a level set of f in M_δ. The former are compact surfaces without boundary, and so are most of the latter. The integral of $i*B_{\hat{A}}$ over a surface of this sort without boundary is 2π times the pairing of the first Chern class of E with the homology class defined by the surface and this no greater than $2\pi\mathrm{G}$. Thus, the integral of $i*B_{\hat{A}}$ over such a surface is a priori bounded by $2\pi\mathrm{G}$. If the integral of $i*B_{\hat{A}}$ over a disk in one of these surfaces is greater than this bound, then there must be other parts of the surface where the corresponding integral is negative.

The second bullet in (7-4) and (7-2) imply the following: the pullback of $i *B_{\hat{A}}$ to such a surface is no smaller than $-c_c \mathrm{r}^{1-1/q}$ times the area form (which is the pullback of w). Indeed, when this pullback is written as $\mathrm{B}w$, then the function B is what is depicted on the right-hand side of the second bullet of (7-4). Now B is zero where $|\alpha|^2$ is close to 1 because $\wp = 1$ there, so the issue is where $1-|\alpha|^2 > c_0^{-1}$. Since $\wp'$ is nonnegative, the function B can be negative only if $\sigma < 0$ or if the $\mathfrak{e}_B$ term is negative. Meanwhile, $|\mathfrak{e}_B| \le c_0$ (which is less than $\mathrm{r}^{1-1/q}$) and, as noted in (7-2), σ is greater than $-c_c \mathrm{r}^{-1/q}$ where $\wp \ne 1$. Thus, if B is negative, it is no smaller than $-c_c \mathrm{r}^{1-1/q}$. As explained in a later step, because B is greater than $-c_c \mathrm{r}^{1-1/q}$, the area where B is negative cannot be smaller than $c_c^{-1} \mathrm{r}^{-1+1/q}$. Now, Lemma 4.7 and (7-2) have a second implication which is this: the function B is negative at a point only if $*\big(ds \wedge \frac{\partial}{\partial s} A \wedge w\big)$ is of order r (because $\wp$ must be less than 1 and so $|\alpha| < 1 - c_0^{-1}$). As shown in a later step, this implies that $\big|\frac{\partial}{\partial s} A\big|^2$ is of order r^2 where $\mathrm{B} < 0$. Since the area where this happens is greater than $c_c^{-1} \mathrm{r}^{-1+1/q}$, the integral of $\big|\frac{\partial}{\partial s} A\big|^2$ on the surface is therefore greater than $c_c \mathrm{r}^{1+1/q}$. The extra factor of $\mathrm{r}^{1/q}$ is seen below to lead to a violation of Lemma 4.2 unless the number Δ_* is a priori bounded by $c_c \rho_{\mathrm{r}}^2$.

Step 3 Given $(s,t) \in \mathbb{R} \times \mathbb{R}/(\ell\mathbb{Z})$ with $|s-s_0| < 2\rho_{\mathrm{r}}$ and $|t-t_0| < 4\rho_{\mathrm{r}}$, introduce by way of notation $D_{(s,t)}$ to denote the slice at (s,t) of Q^*, and use $\mathrm{E}(s,t)$ to denote the integral of $\frac{i}{2\pi} *B_A$ over the disk $\{(s,t)\} \times D_{(s,t)}$. Given $n \in \{0,1,2,\dots\}$, let $U_n \subset \mathbb{R} \times \mathbb{R}/(\ell\mathbb{Z})$ denote the set of points with $|s-s_0| < 2\rho_{\mathrm{r}}$ and $|t-t_0| < 4\rho_{\mathrm{r}}$ and with one additional constraint: if $n = 0$, require that $0 \le \mathrm{E}(s,t) < \mathrm{G}$; if $n \ge 1$, require that $\mathrm{E}(s,t) \in [n\mathrm{G}, 2n\mathrm{G}]$. Use V_n to denote the measure of U_n.

Suppose for the remainder of this step that $\gamma(t_0)$ is either in $\mathcal{H}_0$ or in the part of M_δ where f is either less than $1-4\delta^2$, or between $1+2\delta^2$ and $2-2\delta^2$, or greater than $2+4\delta^2$. This assumption has the following implication: no point in Q^* is on a level set of f that enters a radius δ coordinate ball centered on either an index 1 or index 2 critical point of f in M. Keep this fact in mind.

Fix $n \in \{2,\dots\}$ such that $U_n \ne \varnothing$ and fix $(s,t) \in U_n$. The disk $\{(s,t)\} \times D_{(s,t)}$ lies in a compact surface in $\mathbb{R} \times (M_\delta \cup \mathcal{H}_0)$ whose tangent space is annihilated by $\hat{a}$. This surface is either in a component of a level set of f in M_δ or a cross-sectional sphere of $\mathcal{H}_0$. Use $S_{(s,t)}$ to denote this surface. The integral of $\frac{i}{2\pi} *B_{\hat{A}}$ over $S_{(s,t)}$ is equal to G if it is an f–level set with $f \in (1+\delta^2, 2-\delta^2)$ part of M_δ; it is equal to 0 otherwise.

Since $\mathrm{E}(s,t) \ge 2\mathrm{G}$, the integral of $\frac{i}{2\pi} *B_{\hat{A}}$ over $S_{(s,t)} - D_{(s,t)}$ must be less than $-\mathrm{G}$ $\big($because the integral of $\frac{i}{2\pi} *B_{\hat{A}}$ is at most $\mathrm{G}\big)$. To see what this entails, write the pullback of $\frac{i}{2\pi} *B_{\hat{A}}$ to $S_{(s,t)}$ as $\mathrm{B}w$; the function B is what appears on the right-hand side of (7-4). The latter function is no less than $-c_c((1-\wp)\mathrm{r}^{1-1/q}+\mathrm{r}^{-3/2})$. To explain: The term $(1-\wp)\sigma\mathrm{r}(1-|\alpha|^2)$ on the right-hand side of the second bullet of (7-4) is no smaller than $-c_c(1-\wp)\mathrm{r}^{1-1/q}$ because of (7-2) and the fact that $1-\wp$ is zero unless $|\alpha|$ is less than $1-c_0^{-1}$. The next term, $2\wp'|\nabla^{(1,0)}\alpha|^2$, is nonnegative because $\wp'$ is nonnegative. Finally, $|\mathfrak{e}_B| \le c_c((1-\wp)+\wp')$, which is smaller than $c_c((1-\wp)\mathrm{r}^{1-1/q}+\mathrm{r}^{-3/2})$ because $\wp' < c_0(1-\wp)^{3/4}$ and $(1-\wp)^{3/4} < c_0(\mathrm{r}^{1/2}(1-\wp)+\mathrm{r}^{-3/2})$.

The factor $-c_c\mathrm{r}^{-3/2}$ from the bound $\mathrm{B} \ge -c_c((1-\wp)\mathrm{r}^{1-1/q}+\mathrm{r}^{-3/2})$ contributes no more than $-c_c\mathrm{r}^{-3/2}$ to the integral of $\mathrm{B}w$ and this is no more than -10^{-6} if $\mathrm{r} > c_c$. Assume this to be the case.

If $\mathrm{E}(s,t) \ge -2\mathrm{G}$, then it follows from the bound $\mathrm{B} \ge -c_c((1-\wp)\mathrm{r}^{1-1/q}+\mathrm{r}^{-3/2})$ that the measure of the set in $S_{(s,t)}-D_{(s,t)}$ where B is such that $(1-\wp)\sigma < 0$ is no less than $c_c^{-1}(n-1)\mathrm{G}\mathrm{r}^{-1+1/q}$. Noting that $|\alpha| < \frac{3}{4}$ on this set, it follows from (5-7) (see also the first bullet in (7-4)) that $*\big(i\,ds \wedge \frac{\partial}{\partial s}A \wedge w\big) > c_c^{-1}\mathrm{r}$ on this same set. This implies in particular that $\big|\frac{\partial}{\partial s}A\big|^2 > c_c^{-1}\mathrm{r}^2$ on a set of measure greater than $c_c^{-1}(n-1)\mathrm{G}\mathrm{r}^{-1+1/q}$ in $S_{(s,t)}$. And, as a consequence, the integral of $\big|\frac{\partial}{\partial s}A\big|^2$ over $\big(\bigcup_{(s,t)\in U_n} S_{(s,t)}\big) \subset \mathbb{R}\times Y$ is no less than $c_c^{-1}\mathrm{r}^{1+1/q}(n-1)\mathrm{G}\mathrm{V}_n$.

Step 4 Suppose that $f(\gamma(t_0))$ is between $1-4\delta^2$ and $1+2\delta^2$ or else between $2-2\delta^2$ and $2+4\delta^2$. Fix $(s,t) \in \mathbb{R}\times\mathbb{R}/(\ell\mathbb{Z})$ with $|s-s_0| < 2\rho_{\mathrm{r}}$ and $|t-t_0| \le 4\rho_{\mathrm{r}}$. If $t > t_0$, use $T_{(s,t)}$ to denote the set of points in $\mathbb{R}\times T$ that have coordinates (s,τ,z) with $\tau \in [t, t_0+8\delta]$ and with z such that $|z-z_0|+|t-t_0| \le 4\rho_{\mathrm{r}}$. If $t \le t_0$, define $T_{(s,t)} \subset \mathbb{R}\times T$ to be the set that have coordinates (s,τ,z) with $\tau \in [t_0-8\delta, t]$ and with z as in the $t > t_0$ case. In either case, $T_{(s,t)}$ is a manifold with corners. In the $t > t_0$ case, there are three codimension 1 faces of $T_{(s,t)}$. These are the disks $D_{(s,t)}$ and $D_{(s,t_0+8\delta)}$, and third is the cylinder consisting of the points (s,τ,z) with $\tau \in [t, 8\delta+t_0]$ and $|z-z_0|+|t-t_0| = 4\rho_{\mathrm{r}}$. There is a similar story when $t < t_0$: the cylinder face of $T_{(s,t)}$ in this case is the set of points (s,τ,z) with $\tau \in [-8\delta+t_0, t]$ and with z as in the $t > t_0$ case. In either case, let $C_{(s,t)}$ denote the cylindrical face of $T_{(s,t)}$.

Use Stokes' theorem to see that the absolute value of the difference between the integral of $i*B_{\hat{A}}$ over the two disk faces of $T_{(s,t)}$ is equal to the absolute value of the integral of $i*B_{\hat{A}}$ over $C_{(s,t)}$. This integral involves only the $\mathfrak{r}-\mathfrak{X}$ part of (5-7)'s depiction of B_A. Meanwhile, the contribution from the term proportional to $\wp'$ in (6-25) is bounded by $c_0\wp'\big(|(\nabla^{(1,0)}\alpha)_0||(\nabla^{(1,0)}\alpha)_1|+1\big)$. With these last facts in mind, introduce by way of notation

$$\text{(7-11)}\quad \mathrm{N} = \int_{\mathcal{J}}\bigg(\int_{C_{(s,t)}}\big((1-\wp)(|\mathfrak{X}|+|\mathfrak{r}|)+\wp'(|(\nabla^{(1,0)}\alpha)_0||(\nabla^{(1,0)}\alpha)_1|+1)\big)\bigg)\,ds\wedge dt,$$

where the outer integral is over $\mathcal{J} = \{(s,t) \mid |s-s_0| < 2\rho_{\mathrm{r}}$ and $|t-t_0| < 4\rho_{\mathrm{r}}\}$. Let $Q^{\diamond}$ denote either $Q^{(s_0,t_0+8\delta,z_0)}$ or $Q^{(s_0,t_0-8\delta,z_0)}$. What was said above about Stokes' theorem implies that

$$\text{(7-12)} \qquad \mathrm{Q} = \left| \int_{Q^*} (i * B_{\hat{A}} \wedge ds \wedge \hat{a}) - \int_{Q^{\diamond}} (i * B_{\hat{A}} \wedge ds \wedge \hat{a}) \right| < c_0 \mathrm{N}.$$

Use T^* to denote the union of the $t \in [t_0 - 8\delta, t_0 + 8\delta]$ versions of $Q^{(s_0,t,z_0)}$. Fix for the moment $e > 1$. Change variables in the integration that defines N and use (5-8) with the fact that $\sigma > -c_c \mathrm{r}^{-1/q}$ and $(1-\sigma) > -c_c \mathrm{r}^{-1/q}$ to see that N is no greater than

$$\text{(7-13)} \qquad e^{-1} \int_{T^*} ds \wedge \hat{a} \wedge iF_{\hat{A}} + c_c e \int_{T^*} iF_{\hat{A}} \wedge w + c_c \mathrm{r}^{-1/(q+1)} \Delta_*.$$

The left-most integral in (7-13) is no greater than $c_0 \rho_{\mathrm{r}}^{-1} \Delta_*$ and so the left-most term in (7-13) is no greater than $c_0 e^{-1} \rho_{\mathrm{r}}^{-1} \Delta_*$. Therefore, if $e = 1000 c_0 \rho \mathrm{r}^{-1}$ and if $\mathrm{Q} > \frac{1}{100} \Delta_*$, then

$$\text{(7-14)} \qquad \int_{T^*} iF_{\hat{A}} \wedge w > c_c^{-1} \rho_{\mathrm{r}} \Delta_*$$

when $\mathrm{r} > c_c$. If (7-14) holds with $\mathrm{r} > c_c$, then what is said by Lemma 7.2 is true, this being a consequence of (7-8).

Assume that (7-14) does not hold. Then $\mathrm{Q} < \frac{1}{100} \Delta_*$ when $\mathrm{r} \geq c_c$ and so a repetition of Step 2 with (s_0, t_0, z_0) replaced by either $(s_0, t_0 + 8\delta, z_0)$ or with $(s_0, t_0 - 8\delta, z_0)$ supplies a lower bound for the integral of $\left|\frac{\partial}{\partial s} A\right|^2$ over $\left(\bigcup_{(s,t)\in U_n} S_{(s,t)}\right) \subset \mathbb{R} \times Y$ for each $n \in \{2, \dots\}$.

Step 5 Sum the bounds from Step 3 or Step 4 for the integral of $\left|\frac{\partial}{\partial s} A\right|^2$ over $\left(\bigcup_{(s,t)\in U_n} S_{(s,t)}\right) \subset \mathbb{R} \times Y$ for $n = 2, 3, \dots$ to see that the integral over $\mathbb{R} \times Y$ of $\left|\frac{\partial}{\partial s} A\right|^2$ is no less than $(c_c^{-1} \Delta_* - 2\mathrm{G}\mathrm{V}) \mathrm{r}^{1+1/q}$, where V is the upper bound for the various (s,t) versions of the sum $\mathrm{V}_0 + \mathrm{V}_1$. Since V is at most $16\pi\rho_{\mathrm{r}}^2$, this implies that

$$\text{(7-15)} \qquad \int_{\mathbb{R}\times Y} \left|\frac{\partial}{\partial s} A\right|^2 \geq (c_c^{-1} \Delta_* - c_0 \mathrm{G} \rho_{\mathrm{r}}^2) \mathrm{r}^{1+1/(2q)}.$$

What with Lemma 7.2's assumption about $\mathrm{A}_{\mathfrak{d}}$, this last inequality runs afoul of what is asserted by Lemma 4.2 if $\Delta_* > c_c \rho_{\mathrm{r}}^2$ and $\mathrm{r} > c_c$.

Part 3 The next lemma is an analog of Lemma 7.2 for pairs $(s_0, z_0) \in \mathbb{R} \times D_0$ whose corresponding Δ_* is small. Lemma 7.3 uses the following notation: Given $x \in \mathbb{R}$ and

$\rho \in (\mathrm{r}^{-1/2}, c_0^{-1})$, the lemma uses $\widehat{\mathrm{M}}_{(x,\rho)}$ to denote the integral of $iF_{\widehat{A}} \wedge (ds \wedge \hat{a} + w)$ over the ball of radius ρ in $\mathbb{R} \times Y$ centered at x. The lemma also introduces $T_{(s_0,z_0)}$ to denote the radius ρ_{r} tubular neighborhood of the $s = s_0$ and $z = z_0$ slice of $\mathbb{R} \times T$, this being the set of points of the form (s,t,z) with $(s-s_0)^2 + |z-z_0|^2 < \rho_{\mathrm{r}}^2$ and with $t \in \mathbb{R}/(\ell\mathbb{Z})$.

Lemma 7.3 *Given $c > 1$ there exists $\kappa_c > 1$ with the following significance: Fix $\mathrm{r} \geq \kappa_c$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 and let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to the (r, μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}} < c\mathrm{r}\ln\mathrm{r}$. Fix $(s_0, t_0) \in \mathbb{R} \times \mathbb{R}/(\ell\mathbb{Z})$ and a point $z_0 \in D_0$ with $|z_0|$ less than one fourth the radius of D_0. If $\widehat{\mathrm{M}}_{((s_0,t_0,z_0),\rho_{\mathrm{r}})} \neq 0$, then there exists $s_1 \in \mathbb{R}$ and $z_1 \in D_0$ with $|s_1 - s_0| < 8\rho_{\mathrm{r}}$ and $|z_1 - z_0| < 8\rho_{\mathrm{r}}$ with $\int_{T_{(s_1,z_1)}} iF_{\widehat{A}} \wedge w > \kappa_c^{-1}\rho_{\mathrm{r}}^4$.*

The proof of Lemma 7.3 invokes an $\widehat{A}$ analog of Lemma 4.10, this being:

Lemma 7.4 *There exists $\kappa \geq \pi$, and, given $q \geq 1$, there exists $\kappa_q \geq 1$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Let $\mathfrak{d} = (A, \psi)$ denote an instanton solution to the (r, μ) version of (4-1) with $\mathrm{A}_{\mathfrak{d}} < \mathrm{r}^2$ and $\sup_{s \in \mathbb{R}} \underline{\mathrm{M}}(s) \leq \mathrm{r}^{1-1/q}$. Suppose that $x \in \mathbb{R} \times Y$ is a point where $|\alpha| \leq \frac{3}{4}$.*

- *If $\rho_1 > \rho_0$ are in $(\kappa_q \mathrm{r}^{-1/2}, \kappa_q^{-1})$, then $\widehat{\mathrm{M}}_{(x,\rho_1)} \geq \kappa_q^{-1}\rho_1^2/\rho_0^2\widehat{\mathrm{M}}_{(x,\rho_0)}$.*
- *If $\rho \in (\kappa_q \mathrm{r}^{-1/2}, \kappa_q^{-1})$, then $\widehat{\mathrm{M}}_{(x,\rho)} \geq \kappa^{-1}\rho^2$.*

It follows from Lemmas 2.5, 4.1 and 5.2 that the assumptions of Lemma 7.3 are met using $q > 6$.

Proof of Lemma 7.4 Given Lemma 4.4 and the first bullet of Lemma 4.8 and the formula in (6-25) for $F_{\widehat{A}}$, the proof of Proposition 3.1 in [14] can be used with only cosmetic changes to prove the assertions. A second proof deduces Lemma 7.4 from Lemmas 4.8, 4.10 and 4.12 by proving the following assertion:

(7-16) There is a purely q–dependent constant, c_q, which is greater than 1 and is such that if $x \in \mathbb{R} \times Y$ is a point where $|\alpha| < \frac{1}{2}$ and $\rho \in (\mathrm{r}^{-1/2}, c_q^{-1})$, then $c_q^{-1}\mathrm{M}_{(x,\rho)} < \widehat{\mathrm{M}}_{(x,\rho)} < c_q\mathrm{M}_{(x,\rho)}$.

What follows is a sketch of the argument for (7-16). To start, fix $m > 100$ and use Lemma 4.12 with Proposition 4.11 to see that the contribution to $\mathrm{M}_{(x,\rho)}$ from the set of points where $|\alpha| \geq 1 - m^{-1}$ is greater than c_q^{-1} if r is greater than a purely m– and

q–dependent constant. This proves the upper bound in (7-16). To prove the lower bound, use Lemma 4.12 and Proposition 4.11 to see that the contribution to $\mathrm{M}_{(x,\rho)}$ from this same set is no less than $c_{m_q}\widehat{\mathrm{M}}_{(x,\rho)}$ if $\mathrm{r} > c_{m_q}$ with $c_{m_q} > 1$ being a constant that depends only on m and q. Meanwhile, the assertions in the second and fourth bullets of Lemma 4.8 can be used to prove the following: if $m \geq c_q$, then the contribution to $\mathrm{M}_{(x,\rho)}$ from the complement of the set where $|\alpha| \leq 1 - m^{-1}$ is no greater than c_{m_q} times the contribution to $\mathrm{M}_{(x,\rho)}$ from the set where $|\alpha| \geq 1 - m^{-1}$. □

The assertion of the second bullet of Lemma 7.3 follows from Lemma 4.12 and Proposition 4.11 as they imply that $|\alpha| < \frac{1}{2}$ on a ball in $\mathbb{R} \times Y$ of radius at least $c_0^{-1}\mathrm{r}^{-1/2}$ with distance at most $c_0\mathrm{r}^{-1/2}$ from x.

Part 4 This part of Proposition 7.1's proof supplies a proof of Lemma 7.3. By way of notation, the proof uses $\widehat{\mathrm{M}}_*$ to denote $\sup_{t\in\mathbb{R}/(\ell\mathbb{Z})} \widehat{\mathrm{M}}_{((s_0,t,z_0),\rho_\mathrm{r})}$. The proof uses the same conventions about c_c that are used in Lemma 7.2's proof; and it introduces one additional convention of the same sort: Given $m > 1$, the proof uses c_{cm} to denote a number that is greater than 1 and depends only on m, c, γ and the geometry of Y. In particular, this number does not depend on $\mathfrak{d}$ nor on r. The value of c_{cm} can be assumed to increase between successive appearances.

Proof of Lemma 7.3 The proof has five steps.

Step 1 There exists $N < c_0$ and a set of N points $\{(s_k, z_k)\}_{k=1,2\dots,N}$ with the following properties: First, $|s_0 - s_k| < 8\rho_\mathrm{r}$ and $|z_0 - z_k| < 8\rho_\mathrm{r}$ for each $k \in \{1,\dots,N\}$. Second, the union of the sets $\{T_{(s_k,z_k)}\}_{k=0,1,\dots,N}$ contains $Q_{(s_0,z_0)}$. This understood, invoke Lemma 7.2 to see that Lemma 7.3's assertion holds unless $\sup_{t\in\mathbb{R}/(\ell\mathbb{Z})} \Delta_{(s_0,t,z_0)} < c_c\rho_\mathrm{r}^2$. Indeed, if this isn't the case, then the integral of $iF_{\hat{A}} \wedge w$ over at least one $k \in \{0,1,\dots,N\}$ version of $T_{(s_k,z_k)}$ will be greater than $c_c^{-1}\rho_\mathrm{r}^3$.

Let $t_0 \in \mathbb{R}/(\ell\mathbb{Z})$ denote a point with $\widehat{\mathrm{M}}_{((s_0,t_0,z_0),\rho_\mathrm{r})} > 0$. If such is the case, then there must be a point in the radius ρ_r ball centered at (s_0,t_0,z_0) where $\wp < 1$ and so a point in this ball where $|\alpha| < \frac{3}{4}$. Let (s_1,t_1,z_1) denote such a point. The operative assumption that $\sup_{t\in\mathbb{R}(\ell\mathbb{Z})} \Delta_{(s_0,t,z_0)} < c_c\rho_\mathrm{r}^2$ requires that $\sup_{t\in\mathbb{R}/(\ell\mathbb{Z})} \widehat{\mathrm{M}}_{((s_1,t_1,z_1),\rho_\mathrm{r})} < c_c\rho_\mathrm{r}^2$ also. This being the case, it is enough to prove the following assertion:

(7-17) If $m > 10$, there is $c_{cm} > 1$ such that if $\mathrm{r} \geq c_{cm}$ and $\sup_{t\in\mathbb{R}/(\ell\mathbb{Z})} \widehat{\mathrm{M}}_{((s_1,t_1,z_1),\rho_\mathrm{r})} \in (0, m\rho_\mathrm{r}^2)$, then $\int_{T_{(s_1,z_1)}} iF_{\hat{A}} \wedge w > c_{cm}^{-1}\rho_\mathrm{r}^4$.

The remaining steps prove (7-17).

Step 2 Let $T_{1/4} \subset T_{(s_1,z_1)}$ denote the set of points whose (s,z)–coordinates are such that $|s-s_1|^2+|z-z_1|^2<\frac{1}{16}\rho_{\mathrm{r}}^2$. The assertion below summarizes the content of this step:

(7-18) If $\mathrm{r}>c_{cm}$ then either (7-17) holds or $F_{\widehat{A}}=0$ on $T_{1/4}\cap\mathcal{H}_0$.

The proof of (7-18) is given after a digression that follows directly. The proof invokes two key facts that are supplied by this digression.

To start the digression, let x' denote the point with (s,t,z) coordinates (s',t',z') with s' and z' constrained so that $|s'-s_1|^2+|z'-z_1|^2\le\frac{1}{4}\rho_{\mathrm{r}}^2$. The operative assumption in (7-17) requires that $\widehat{\mathrm{M}}_{(x',\rho_{\mathrm{r}}/2)}\le m\rho_{\mathrm{r}}^2$. Assume in addition that $|\alpha|<\frac{3}{4}$ at x'. The first bullet of Lemma 7.4 requires the bound $\widehat{\mathrm{M}}_{(x',\rho)}\le c_c m\rho^2$ for all $\rho\in(c_c\mathrm{r}^{-1/2},c_c^{-1})$ if $\mathrm{r}\ge c_c$. Fix $r>m^4$, $\varepsilon\in(0,m^{-4})$ and $k\in\{10,11,\dots\}$ to invoke Lemma 4.12 with the given q and value of m. As will be apparent in the proof of (7-18), choices for r, ε and k that depend only on c and m are sufficient. In any event, with r, ε and k chosen, Lemma 4.12 with the given bound on $\widehat{\mathrm{M}}_{(x',\rho)}$ will be invoked with it understood that r is greater than a constant that depends only on m, c and the chosen values for r, ε and k. Of particular interest here is the fact that the corresponding solution (A_0,α_0) of (4-7) is described by items (a) and (b) of the third bullet of Proposition 4.11. The assertions of these two items imply the following:

Fact 1 *There are zeros of α_0 with distance less than c_{cm} from the origin in $\mathbb{C}^2$ and so there are zeros of α with distance less than $c_{cm}\mathrm{r}^{-1/2}$ from x'.*

Fact 2 *Each zero of α_{r} with distance less than r from the origin has distance less than $c_{cm}\varepsilon$ from a zero of α_0, and each zero of $\alpha(0)$ with distance less than r from the origin has distance less than $c_{cm}\varepsilon$ from a zero of α_{r}.*

With regards to Fact 1, the assertion about the distance from origin of zeros of α_0 follows from three facts: the equations in (4-7) are elliptic modulo the action of $C^\infty(\mathbb{C}^2;S^1)$; the polynomial that defines the zero locus of α_0 has a priori bounded degree; and $|\alpha_0(0)|<\frac{3}{4}+\varepsilon$ because $|\alpha_{\mathrm{r}}(0)|<\frac{3}{4}$. Fact 2 follows from the a priori degree bound for the polynomial that defines the zeros of α_0. In particular, this fact has the following consequences: All but at most a finite set of affine lines in $\mathbb{C}^2$ intersect $\alpha_0^{-1}(0)$ in a finite set of points. Those that do not have this property are irreducible components of $\alpha_0^{-1}(0)$. Moreover, if a line intersects $\alpha_0^{-1}(0)$ in a finite set of points, then the local degree of each intersection point is positive and their sum is bounded by a purely m–dependent constant. Given that these zeros have positive local degree, each such intersection point must have distance less than $c_{cm}\varepsilon$ from an $\alpha_{\mathrm{r}} = 0$ point on the affine line if all points on the line at distance r from the origin have distance $c_{cm}\varepsilon r$ or more from all zeros of α_0.

With the digression now over, what follows is the proof of (7-18). To start the argument, suppose that $x \in T_{1/4} \cap \mathcal{H}_0$ and that $F_{\hat{A}} \neq 0$ at x. As $\wp < 1$ at x, so $|\alpha| < \frac{3}{4}$ at x. It follows from Fact 1 that the integral of $iF_{\hat{A}} \wedge (ds \wedge \hat{a} + w)$ over the radius $c_{cm}\mathrm{r}^{-1/2}$ ball centered at x is greater than $c_{cm}^{-1}\mathrm{r}^{-1}$ and so it follows from Lemma 7.4 that $\widehat{\mathrm{M}}_{(x,\rho_{\mathrm{r}}/4)}$ is greater than $c_{cm}^{-1}\rho_{\mathrm{r}}^2$. If the integral of $iF_{\hat{A}} \wedge w$ over the radius $\frac{1}{4}\rho_{\mathrm{r}}$ ball centered at x is greater than $\frac{1}{2}\widehat{\mathrm{M}}_{(x,\rho_{\mathrm{r}}/4)}$, then the conclusions of Lemma 7.3 follow because the integral of $iF_{\hat{A}} \wedge w$ over $T_{(s_0,x_0)}$ is no less than $-c_{cm}\mathrm{r}^{1/q}m^2\rho_{\mathrm{r}}^2$.

Granted what was just said, assume that the integral of $iF_{\hat{A}} \wedge w$ over the radius $\frac{1}{4}\rho_{\mathrm{r}}$ ball centered at x is less than $\frac{1}{2}\widehat{\mathrm{M}}_{x,\rho_{\mathrm{r}}/4}$. It then follows that the integral of $iF_{\hat{A}} \wedge (ds \wedge \hat{a})$ over this same ball is greater than $c_{cm}^{-1}\rho_{\mathrm{r}}^2$.

Let (s_x, t_x, z_x) denote the (s,t,z) coordinates of x. Introduce Q^x to denote the subset of $\mathbb{R} \times \mathbb{R}/(\ell\mathbb{Z}) \times D_0$ whose (s,t,z) coordinates obey $|s - s_x| < 2\rho_{\mathrm{r}}$ and $|t - t_x| + |z - z_x| < 4\rho_{\mathrm{r}}$. Given (s,t) with $|s - s_x| < 2\rho_{\mathrm{r}}$ and $|t - t_x| < 4\rho_{\mathrm{r}}$, use $D_{(s,t)}$ to denote the constant (s,t) slice of Q^x and use $\mathrm{E}(s,t)$ to denote the integral of $\frac{i}{2\pi} * B_{\hat{A}}$ over $D_{(s,t)}$. What follows is a direct consequence of Lemma 4.12 and Facts 1 and 2:

(7-19) If $\mathrm{E}(s,t)$ is greater than $c_{cm}\varepsilon$, then $\mathrm{E}(s,t)$ is greater than $1 - c_{cm}\varepsilon$.

Given $n \in \{1, \dots\}$, let $U_n \subset \mathbb{R} \times \mathbb{R}/(\ell\mathbb{Z})$ denote the set where the conditions $|s - s_x| < 2\rho_{\mathrm{r}}$, $|t - t_x| < 4\rho_{\mathrm{r}}$ and $\mathrm{E}(s,t) \in \left[n - \frac{1}{2}, n + \frac{1}{2}\right)$ hold. Use U_0 to denote the set of points with (s,t) such that $\mathrm{E}(s,t) \in \left[0, \frac{1}{2}\right)$. Use v_n to denote the measure of U_n.

Given (s,t) as just defined, let $S_{(s,t)}$ denote the slice of $\mathcal{H}_0 \cap M_\delta$ containing $D_{(s,t)}$. This is a J–holomorphic 2–sphere that has pairing 0 with the first Chern class of E. In particular, the integral of $\frac{i}{2\pi} * B_{\hat{A}}$ over $S_{(s,t)}$ is zero. This being the case, an almost verbatim copy of the arguments from Step 3 of the proof of Lemma 7.2 prove that the integral of $\left|\frac{\partial}{\partial s}A\right|^2$ over $\bigcup_{(s,t)\in U_n} S_{(s,t)}$ is no smaller than $c_c^{-1}\mathrm{r}^{1+1/q}\mathrm{v}_n$. What with (7-19), an almost verbatim repeat of the arguments in Step 5 of the proof of Lemma 7.2 proves that

$$\int_{\mathbb{R}\times Y} \left|\frac{\partial}{\partial s}A\right|^2 \geq (c_c^{-1} - c_{cm}\varepsilon)\rho_{\mathrm{r}}^2\mathrm{r}^{1+1/(2q)}. \tag{7-20}$$

This runs afoul of Lemma 4.2 if $\varepsilon < c_{cm}^{-1}$ and the latter happens if $\varepsilon < c_{cm}$ and $\mathrm{r} > c_{cm}$.

Step 3 This step states two observations that are used in the subsequent steps. To set the stage, fix $\tau \in \left[\frac{1}{4}, \frac{1}{8}\right]$ and introduce T_τ to denote the subset of points in $T_{(s_1,z_1)}$ whose (s,z) coordinates are such that $|s-s_1|^2+|z-z_1|^2 \le \tau^2\rho_{\mathrm{r}}^2$. The first observation here is that

$$c_c^{-1}\rho_{\mathrm{r}}^2 < \int_{T_\tau} iF_{\hat{A}} \wedge (ds \wedge a + w) < c_c m \rho_{\mathrm{r}}. \tag{7-21}$$

By way of an explanation, the lower bound follows from the version of the top bullet of Lemma 7.4 that takes $x = (s_1, t_1, z_1)$ and $\rho_0 = c_c \mathrm{r}^{-1/2}$ and $\rho_1 = \frac{1}{2}\rho_{\mathrm{r}}$. Meanwhile, the upper bound follows from the bound $\widehat{\mathrm{M}}_* < m\rho_{\mathrm{r}}^2$ and the fact that $T_{1/4}$ can be covered by $c_0\rho_{\mathrm{r}}^{-1}$ balls of radius $\frac{1}{16}\rho_{\mathrm{r}}$ with centers in $T_{1/4}$.

The second observation concerns the integral of $iF_{\hat{A}} \wedge w$. To say more, let U denote a given open subset of $T_{1/4}$. Then

$$\int_U iF_{\hat{A}} \wedge w \ge -c_c m \rho_{\mathrm{r}} \mathrm{r}^{-1/q}. \tag{7-22}$$

Given the upper bound in (7-21), this follows from the fact that the function $1-\sigma$ that appears in (5-7) is no less than $-c_c\mathrm{r}^{-1/q}$.

Step 4 Assume for this and the remaining steps that $F_{\hat{A}} = 0$ on $T_{1/4} \cap \mathcal{H}_0$. This being the case, the $\mathbb{R}/(\ell\mathbb{Z})$ parameter t on $T_{1/4}$ can be lifted to an $\mathbb{R}$–valued parameter on a neighborhood of the support of $|F_{\hat{A}}|$ and nothing is lost by assuming that the now $\mathbb{R}$–valued parameter t is constrained to an interval $I \subset \mathbb{R}$ of the form $[-c_0, c_0]$ at points in $T_{1/4}$ with distance 1 or less from the support of $|F_{\hat{A}}|$. Meanwhile, it follows from (7-1) that the 2–forms $ds \wedge \hat{a}$ and w on $\mathbb{R} \times I \times D$ can be written as

$$ds \wedge \hat{a} = d(-t\, ds) \quad \text{and} \quad w = \tfrac{i}{4} d(z\, d\bar{z} - \bar{z}\, dz + \cdots), \tag{7-23}$$

where the unwritten terms in the formula for w have no ds component and are bounded in absolute value by $|z|^2$.

Fix $\tau \in \left[\frac{1}{8}, \frac{1}{4}\right]$ and use Stokes' theorem with (7-21) to see that

$$\int_{T_\tau} iF_{\hat{A}} \wedge ds \wedge \hat{a} = \int_{\partial(I\times Y)\cap T_\tau} i * B_{\hat{A}} \wedge t\, ds. \tag{7-24}$$

Look at (6-25) and (5-7) to see that the absolute value of the right-hand side of (7-24) is no greater than

$$\int_{\partial(I\times Y)\cap T_\tau} \big((1-\wp)(|\mathfrak{X}| + |\mathfrak{r}|) + \wp'\big(|(\nabla^{(1,0)}\alpha)_0||(\nabla^{(1,0)}\alpha)_1| + 1\big)\big). \tag{7-25}$$

Fix $e > 64$ to be determined shortly and use (7-24) and (7-25) with (5-7) and (5-9) and the bound on $\widehat{\mathrm{M}}_*$ to see that

$$\text{(7-26)}\quad \int_{T_{1/4}-T_{1/8}} \Big(\int_{T_\tau} iF_{\hat{A}} \wedge ds \wedge \hat{a} \Big)\, d\tau \\ \leq c_0 e \int_{T_{1/4}-T_{1/8}} iF_{\hat{A}} \wedge w + e^{-1} \int_{T_{1/4}-T_{1/8}} iF_{\hat{A}} \wedge ds \wedge \hat{a} + c_0 \mathrm{r}^{-1/2}.$$

This last inequality is the input for the final step in the proof of (7-17).

Step 5 There are two cases to consider with regards to (7-26). The first is that when the integral on the left-hand side of (7-26) is less than

$$\text{(7-27)}\quad e^{-1/4} \int_{T_{1/4}-T_{1/8}} \Big(\int_{T_\tau} iF_{\hat{A}} \wedge (ds \wedge \hat{a} + w) \Big)\, d\tau.$$

If this is the case, then there exists $\tau \in \left[\frac{1}{8}, \frac{1}{4}\right]$ such that

$$\text{(7-28)}\quad \int_{T_\tau} iF_{\hat{A}} \wedge w > \frac{1}{2} \int_{T_\tau} iF_{\hat{A}} \wedge (ds \wedge \hat{a} + w).$$

Use the lower bound in (7-21) to see that the integral on the left-hand side of (7-28) is no less than $c_0^{-1} \rho_{\mathrm{r}}^2$. Thus $\int_{T_\tau} iF_{\hat{A}} \wedge w >\geq c_0^{-1} \rho_{\mathrm{r}}^2$. This with the bound in (7-22) implies what is asserted by (7-17).

The other case to consider is that where the left-hand side of (7-26) is greater than what is written in (7-28). It follows from the lower bound in (7-21) that what is written in (7-27) is greater than $c_0 e^{-1} \rho_{\mathrm{r}}^2$. Meanwhile, the term on the right-hand side of (7-26) that is proportional to the integral of $iF_{\hat{A}} \wedge ds \wedge \hat{a}$ is bounded by $c_0 e^{-1} \int_{T_{1/4}} iF_{\hat{A}} \wedge (ds \wedge \hat{a} + w)$. The upper bound in (7-21) implies that this last expression is no greater than $c_0 e^{-1} m \rho_{\mathrm{r}}$. Granted all of this, then (7-26) implies that

$$\text{(7-29)}\quad \int_{T_{1/4}-T_{1/8}} iF_{\hat{A}} \wedge w > c_0^{-1} e^{-5/4} \rho_{\mathrm{r}}^2 - c_0 e^{-2} m \rho_{\mathrm{r}}.$$

If $e = \rho_{\mathrm{r}}^{-8/5}$, then the right-hand side of (7-29) is greater than $c_0 \rho_{\mathrm{r}}^4$. This with (7-22) implies what is asserted in (7-17). □

Part 5 Fix (s_0, t_0, z_0) where the function q_{D} from Part 1 is nonzero and reintroduce Lemma 7.2's set $Q_{(s_0,t_0)}$ so as to consider the integral

$$\text{(7-30)}\quad \int_{Q_{(s_0,t_0)}} iF_{\hat{A}} \wedge q_{\mathrm{D}} w.$$

The assertion that follows summarizes what is said here about (7-30):

(7-31) Assume that r is greater than a constant that depends only on D, c, γ and the geometry of Y. Then the integral in (7-30) is nonnegative if $q_{\mathrm{D}}(z_0) > c_0 e^{-(\ln \mathrm{r})^2/c_0}$.

To explain (7-31), note first that the integral in (7-30) is zero if $\sup_{t\in\mathbb{R}/(\ell\mathbb{Z})} \Delta_{(s_0,t,z_0)}$ is zero, so assume that this is not the case. Use Δ_* to denote this supremum. Write q_{D} as $q_{\mathrm{D}}(z_0) + \mathfrak{q}$ with $\mathfrak{q}(z_0) = 0$. The integral in (7-30) has the corresponding decomposition as

$$q_{\mathrm{D}}(z_0) \int_{Q_{(s_0,t_0)}} iF_{\hat{A}} \wedge w + \int_{Q_{(s_0,t_0)}} iF_{\hat{A}} \wedge \mathfrak{q} w. \tag{7-32}$$

To see about the right-most term in (7-32), let $Q_- \subset Q_{(s_0,t_0)}$ denote the set of points where $iF_{\hat{A}} \wedge w$ is a negative multiple of the volume form. The inequality in (7-22) has its $Q_{(s_0,t_0)}$ analog, this being the lower bound

$$\int_{Q_-} iF_{\hat{A}} \wedge w \geq -c_c \rho_{\mathrm{r}}^{-1} \Delta_*. \tag{7-33}$$

The proof of (7-33) is identical to that of (7-22) with it understood that the $Q_{(s_0,t_0)}$ version of the upper bound in (7-21) replaces the integration domain with $Q_{(s_0,t_0)}$ and replaces the term $c_c m \rho_{\mathrm{r}}$ on the far right-hand side of (7-21) with $c_c \rho_{\mathrm{r}}^{-1} \Delta_*$. Granted (7-33), write $Q_{(s_0,t_0)}$ as $(Q_{(s_0,t_0)} - Q_-) \cup Q_-$ and then use Taylor's theorem with remainder to see that

$$\begin{aligned} \int_{Q_{(s_0,t_0)}} iF_{\hat{A}} \wedge \mathfrak{q} w \geq -\Big(\sup_{\{z \mid |z-z_0|<4\rho_{\mathrm{r}}\}} \Big| \frac{\partial}{\partial z} q_{\mathrm{D}} \Big| \Big) \rho_{\mathrm{r}} \int_{Q_{(s_0,t_0)}} iF_{\hat{A}} \wedge w \\ - c_c \Big(\sup_{\{z \mid |z-z_0|<4\rho_{\mathrm{r}}\}} \Big| \frac{\partial}{\partial z} q_{\mathrm{D}} \Big| \Big) \mathrm{r}^{-1/q} \Delta_*. \end{aligned} \tag{7-34}$$

Introduce $\varsigma(z_0)$ to denote

$$q_{\mathrm{D}}(z_0) - \Big(\sup_{\{z \mid |z-z_0|<4\rho_{\mathrm{r}}\}} \Big| \frac{\partial}{\partial z} q_{\mathrm{D}} \Big| \Big) (\rho_{\mathrm{r}} + c_c \mathrm{r}^{-1/q}). \tag{7-35}$$

If $\varsigma(z_0)$ is positive, then Lemmas 7.2 and 7.3 with (7-32) and (7-34) find

$$\int_{Q_{(s_0,t_0)}} iF_{\hat{A}} \wedge q_{\mathrm{D}} w \geq \varsigma(z_0) c_c^{-1} \rho_{\mathrm{r}}^2 \Delta_*, \tag{7-36}$$

which is positive. A look at (7-2) finds $\varsigma(z_0)$ to be negative only in the case that

$$2 - \mathrm{D}|z_0| \leq c_0 \rho_{\mathrm{r}}^{1/2}, \tag{7-37}$$

in which case q_{D} is less than $c_0 e^{-(\ln \mathrm{r})^2/c_0}$ because $\rho_{\mathrm{r}} = (\ln \mathrm{r})^4$.

Part 6 Define the function $\mathfrak{f}$ on $\mathbb{R}$ by the rule

$$s \mapsto \mathfrak{f}(s) = \int_{[s-2\rho_{\mathrm{r}}, s+2\rho_{\mathrm{r}}]\times Y} \frac{i}{2\pi} F_{\hat{A}} \wedge q_{\mathrm{D}} w. \tag{7-38}$$

It follows from what is said in (7-31) that $\mathfrak{f}(s) \ge -c_0 e^{-(\ln \mathrm{r})^2/c_0}$. Note here and for future reference that the function on $[0,1]$ given by the rule $x \mapsto e^{-(\ln x)^2/c_0}$ is bounded from above by $c_k x^{-k}$ for any $k > 0$ with c_k being a purely k–dependent constant.

With the lower bound on $\mathfrak{f}$ in mind, define the subset $\mathcal{W} \subset \mathbb{R}$ by the following rule: a point $s \in \mathcal{W}$ if and only if $\mathfrak{f}(s)$ is negative. The set $\mathcal{W}$ is open. More to the point, the fact that $\mathrm{X}_\gamma(A_+) - \mathrm{X}_\gamma(A_-)$ is an integer implies that $\mathrm{X}_\gamma(A_+) - \mathrm{X}_\gamma(A_-)$ is negative only if the measure of $\mathcal{W}$ is greater than $c_0^{-1} e^{(\ln \mathrm{r})^2/c_0}$. The paragraphs that follow explain why $\mathcal{W}$ is nowhere near this large.

Let $I \subset \mathbb{R}$ denote a closed interval of length 1 where the total change in the function $s \mapsto \mathfrak{a}(\mathfrak{d}|_s)$ across the interval is less than r^{-1}. Invoke Lemma 4.2 to see that

$$\int_{I\times Y} \Big(\Big|\frac{\partial}{\partial s} A\Big|^2 + |\mathfrak{B}_{(A,\psi)}|^2 + 2\mathrm{r}\Big(\Big|\frac{\partial}{\partial s}\psi\Big|^2 + |D_A \psi|^2 \Big)\Big) \le 2\mathrm{r}^{-1}. \tag{7-39}$$

This fact is used in a moment.

If $\mathcal{W}$ has total length greater than r^4, then it can be covered by no fewer than $c_0^{-1}\mathrm{r}^4$ closed intervals of length 1 with center at a point in $\mathcal{W}$ and such that no more than c_0^{-1} of these intervals contain any given point. Given that the total change along $\mathbb{R}$ of the function $s \mapsto \mathfrak{a}(\mathfrak{d}|_s)$ is bounded by $c\mathrm{r}\ln \mathrm{r}$, there are at least $c_0^{-1}\mathrm{r}^4$ intervals in the cover where (7-39) holds. Let I denote one of the latter and let $s_0 \in \mathcal{W}$ denote I's center point.

By assumption, $\mathfrak{f}(s)$ is negative, and so there exists $z_0 \in D_0$ with $\sup_{t\in\mathbb{R}/(\ell\mathbb{Z})} \Delta_{(s_0,t,z_0)}$ greater than zero. It then follows from Lemmas 7.2 and 7.3 that the integral of $iF_{\hat{A}} \wedge w$ over $Q_{(s_0,z_0)}$ is greater than $c_c^{-1}\rho_{\mathrm{r}}^2 \sup_{t\in\mathbb{R}/(\ell\mathbb{Z})} \Delta_{(s_0,t,z_0)}$. Given the formula in (6-25) and the bounds in Lemma 4.8 for $|\nabla_A \beta|$, this can occur only if the integral of $\big|\frac{\partial}{\partial s}A\big|^2 + \big|\frac{\partial}{\partial s}\psi\big|^2$ over $Q_{(s_0,z_0)}$ is likewise greater than $c_c^{-1}\rho_{\mathrm{r}}^2 \sup_{t\in\mathbb{R}/(\ell\mathbb{Z})} \Delta_{(s_0,t,z_0)}$. Since the latter is in any event greater than $c_c^{-1}\rho_{\mathrm{r}}^4$, it follows that the integral of $\big|\frac{\partial}{\partial s}A\big|^2 + \big|\frac{\partial}{\partial s}\psi\big|^2$ over $Q_{(s_0,z_0)}$ is greater than $c_c^{-1}\rho_{\mathrm{r}}^4$. This runs afoul of what is asserted by (7-39).

7.3 The proof of Proposition 1.4

The subsequent argument for Proposition 1.4 differs little from those used in [19] to prove corresponding assertions that concern the analogs of (1-14) and (1-20) in the case when $\hat{a}$ is replaced by a contact 1–form and w is replaced by the latter's exterior derivative.

Use D_1 to denote the data $(\mathrm{r}_1, \mu_1, \mathfrak{p}_1)$ and use D_2 to denote the corresponding set $(\mathrm{r}_2, \mu_2, \mathfrak{p}_2)$. Fix a smooth map, $s \mapsto \mathrm{r}_{(s)}$, from $\mathbb{R}$ to $[\mathrm{r}_1, \mathrm{r}_2]$ which is equal to r_1 for $s \ll -1$ and equal to r_2 for $s \gg 1$. Fix a smooth map, $s \mapsto \mu_{(s)}$, that is equal to μ_1 for $s \ll -1$ and equal to μ_2 for $s \gg 1$. Finally, fix a smooth map from $\mathbb{R}$ to $\mathcal{P}$ of the form $s \mapsto \mathfrak{p}_{(s)}$ such that $\mathfrak{p}_{(s)} = \mathfrak{p}_1$ for $s \ll -1$ and $\mathfrak{p}_{(s)} = \mathfrak{p}_2$ for $s \gg 1$. The data $(\mathrm{r}_{(s)}, \mu_{(s)}, \mathfrak{p}_{(s)})$ can be used to define a version of (1-20), this being a system of equations that requires a map from $\mathbb{R}$ to $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ to obey

$$\text{(7-40)} \qquad \begin{cases} \frac{\partial}{\partial s} A + B_A - \mathrm{r}_{(\cdot)}(\psi^\dagger \tau \psi - i\hat{a}) + \frac{1}{2} B_{A_K} - \mathfrak{T}^{(\cdot)}_{(A,\psi)} = 0, \\ \frac{\partial}{\partial s} \psi + D_A \psi - \mathfrak{S}^{(A,\psi)}_{(\cdot)} = 0, \end{cases}$$

where $\mathrm{r}_{(\cdot)}$ is the function $s \mapsto \mathrm{r}_{(s)}$ and $\mathfrak{T}^{(\cdot)}$ and $\mathfrak{S}^{(\cdot)}$ at any given $s \in \mathbb{R}$ are the gradients of $\mathfrak{e}_{\mu(s)} + \mathfrak{p}_{(s)}$ along the respective $\mathrm{Conn}(E)$ and $C^\infty(Y; \mathbb{S})$ factors of $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$. Of interest are the solutions to (7-40) with $s \mapsto -\infty$ and $s \to \infty$ limits that are solutions to the respective (r_1, μ_1) and (r_2, μ_2) versions of (1-13). Such a solution is called an instanton.

As explained in Chapter 25 of [7], if the map $\mathfrak{p}_{(\cdot)}$ is chosen from a suitable residual set, then there will be but a finite number of instantons of the form $s \mapsto \mathfrak{d}|_s$ with $\lim_{s \to \infty} \mathfrak{f}_s(\mathfrak{d}|_s) - \lim_{s \to -\infty} \mathfrak{f}_s(\mathfrak{d}|_s) = 0$. Chapters 25 and 23 of [7] explains how to use the latter set to define a homomorphism between the (r_1, μ_1) version of $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ to the (r_2, μ_2) version of $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ that preserves the relative $\mathbb{Z}/p_M\mathbb{Z}$–gradings and intertwines the endomorphisms that define the respective D_1 and D_2 differentials. This chapter also explains why the homomorphism intertwines the endomorphisms that define the actions of $\mathbb{Z}[\mathbb{U}] \otimes \left(\bigwedge^*(H_1(Y; \mathbb{Z})/\mathrm{tors})\right)$ on the respective D_1 and D_2 homologies.

The relevant homomorphism is defined by its action on the set of generators of the (r_1, μ_1) version of $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$. As such, it has the form

$$\text{(7-41)} \qquad [\mathfrak{c}] \mapsto \mathrm{w}_{[\mathfrak{c}'],[\mathfrak{c}]}[\mathfrak{c}'],$$

where the sum is indexed by the elements in the (r_2, μ_2) version of $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ with any given coefficient $\mathrm{w}_{[\mathfrak{c}'],[\mathfrak{c}]}$ being an integer. Only a finite set of these are nonzero. In particular, $\mathrm{w}_{[\mathfrak{c}'],[\mathfrak{c}]}$ is nonzero only if $\mathfrak{f}_s(\mathfrak{c}) = \mathfrak{f}_s(\mathfrak{c}')$. In the latter case, Chapter 23 in [7] defines $\mathrm{w}_{[\mathfrak{c}'],[\mathfrak{c}]}$ to be a certain ± 1 weighted count of the instanton solutions to (7-40) with $s \mapsto -\infty$ limit equal to $\mathfrak{c}$ and with $s \to \infty$ limit equal to $\mathfrak{c}'$.

What is said in Chapter 23 in [7] proves that the homomorphism in (7-41) induces an isomorphism between the corresponding D_1 and D_2 versions of $H^{\infty}_{\mathrm{SW,r}}$. Chapter 23.1 in [7] asserts that the respective isomorphisms that are defined by any two such $\mathbb{R}$–parametrized families are identical. The fact that any two such isomorphism are identical leads directly to the conclusion that there is a canonical isomorphism between the D_1 and D_2 versions of $H^{\infty}_{\mathrm{SW,r}}$.

The proof of the assertions in Proposition 1.4 that concern $H^{-}_{\mathrm{SW,r}}$, $H^{+}_{\mathrm{SW,r}}$, and the long exact sequence that relates the latter with $H^{\infty}_{\mathrm{SW,r}}$, has two parts.

Part 1 Let $\mathrm{r}_* = \frac{1}{2}(\mathrm{r}_1 + \mathrm{r}_2)$. Fix $\mu_* \in \Omega$ with $\mathcal{P}$–norm less than 1 and such that all instanton solutions to the (r_*, μ_*) version of (1-13) are nondegenerate. Fix in addition an element $\mathfrak{p}_* \in \mathcal{P}$ with small norm that obeys the (r_*, μ_*) version of (1-22). Use the data set $(\mathrm{r}_*, \mu_*, \mathfrak{p}_*)$ to define the corresponding versions of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$, $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ and $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})/\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$, the operator $\partial^{*}_{\mathrm{SW}}$ and then $H^{\infty}_{\mathrm{SW,r}}$, $H^{-}_{\mathrm{SW,r}}$ and $H^{+}_{\mathrm{SW,r}}$. Let $\mathbb{L}_{*1}$ denote a homomorphism of the sort described above from the (r_1, μ_1) version of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ to the (r_*, μ_*) version, and let $\mathbb{L}_{2*}$ denote a homomorphism of this sort from the (r_*, μ_*) version to the (r_2, μ_2) version.

Assume that $\mathbb{L}_{*1}$ maps the (r_1, μ_1) version of $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ to the (r_*, μ_*) version and that $\mathbb{L}_{2*}$ maps the (r_*, μ_*) version to the (r_2, μ_2) version; then the composition $\mathbb{L}_{2*}\mathbb{L}_{*1}$ will map the (r_1, μ_1) version of $\mathbb{Z}(H^{\infty}_{\mathrm{SW,r}})$ to the (r_2, μ_2) version of $H^{\infty}_{\mathrm{SW,r}}$ and will map the (r_1, μ_1) version of $H^{-}_{\mathrm{SW,r}}$ to its (r_2, μ_2) counterpart, and likewise define a homomorphism between the respective version of $H^{+}_{\mathrm{SW,r}}$. These homomorphisms will necessarily intertwine the homomorphisms in the two exact sequences.

Chapter 26 in [7] explains why the composition $\mathbb{L}_{2*}\mathbb{L}_{*1}$ induces the canonical isomorphism from the (r_1, μ_1) version of $H^{\infty}_{\mathrm{SW,r}}$ to the (r_2, μ_2) version. This understood, it follows as a consequence of what was said in the preceding paragraph that the canonical isomorphism from the (r_1, μ_1) version of $H^{\infty}_{\mathrm{SW,r}}$ to the (r_2, μ_2) version induces respective homomorphisms from the (r_1, μ_1) versions of $H^{+}_{\mathrm{SW,r}}$ and $H^{-}_{\mathrm{SW,r}}$ to their (r_2, μ_2) counterparts that intertwine the associated long exact sequence homomorphisms.

Part 2 Given the conclusions of Part 1, the assertions in Proposition 1.4 that concern $H^{-}_{\mathrm{SW,r}}$ and $H^{+}_{\mathrm{SW,r}}$ and the long exact sequence relating $H^{\infty}_{\mathrm{SW,r}}$, $H^{-}_{\mathrm{SW,r}}$ and $H^{+}_{\mathrm{SW,r}}$ follow as corollaries of the following lemma:

Lemma 7.5 *The versions of κ in Propositions 1.1–1.3 can be chosen so that what is said below is also true. Fix $\mathrm{R} > \kappa$; there exists $\kappa_{\mathrm{R}} > \kappa$ with the following significance:*

Fix $\mathrm{r}_1 \in (\kappa, \mathrm{R})$ *and an element* $\mu_1 \in \Omega$ *with* $\mathcal{P}$*–norm less than* 1 *such that all solutions to the* (r_1, μ_1) *version of* (1-13) *are nondegenerate. Fix* $\mathrm{r}_2 > \mathrm{r}_1$ *with* $|\mathrm{r}_1 - \mathrm{r}_2| < \kappa_{\mathrm{R}}^{-1}$ *and fix* $\mu_2 \in \Omega$ *with* $\mathcal{P}$*–norm less than* 1 *such that all solutions to the* (r_2, μ_2) *version of* (1-13) *are also nondegenerate. Require in addition that* $\mu_2 - \mu_1$ *have* $\mathcal{P}$*–norm less than* κ_{R}^{-1}*. Fix respective elements* $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *in* $\mathcal{P}$ *with small norm that obey the* $\mu = \mu_1$ *and* $\mu = \mu_2$ *versions of* (1-22)*. There are homomorphisms of the sort described at the outset from the* $(\mathrm{r}_1, \mu_1, \mathfrak{p}_1)$ *version of* $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ *to the* $(\mathrm{r}_2, \mu_2, \mathfrak{p}_2)$ *version that maps the* $(\mathrm{r}_1, \mu_1, \mathfrak{p}_1)$ *version of* $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ *to the* $(\mathrm{r}_2, \mu_2, \mathfrak{p}_2)$ *version of* $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$*.*

Proof Suppose that no such κ_{R} exists so as to derive nonsense. Given this contrarian assumption, there exist sequences $(\mathrm{r}_{n1}, \mu_{n1}, \mathfrak{p}_{n1})$ and $(\mathrm{r}_{n2}, \mu_{n2}, \mathfrak{p}_{n2})$ that obey the assumptions of the lemma but not the conclusions with $|\mathrm{r}_{n1} - \mathrm{r}_{n2}| < n^{-1}$, with $|\mu_{n1} - \mu_{n2}| < n^{-1}$ and with the $\mathcal{P}$–norms of $\mathfrak{p}_{n1}$ and $\mathfrak{p}_{n2}$ being less than n^{-1}. For each n, fix a corresponding $\mathbb{R}$–parametrized data set $(\mathrm{r}_n(\cdot), \mu_n(\cdot), \mathfrak{p}_n(\cdot))$ that gives a version of (7-40) with instanton solutions that can be used to define the homomorphism between the respective $(\mathrm{r}_{n1}, \mu_{n1}, \mathfrak{p}_{n1})$ and $(\mathrm{r}_{n2}, \mu_{n2}, \mathfrak{p}_{n2})$ versions of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ in the manner described in the paragraph that surrounds (7-41). Such a path can and should be chosen so that $|\mu_{n(\cdot)} - \mu_{n1}| < n^{-1}$ and such that $\mathfrak{p}_{n(\cdot)}$ has $\mathcal{P}$–norm less than n^{-1}. The resulting index n homomorphism will map the $(\mathrm{r}_{n1}, \mu_{n1}, \mathfrak{p}_{n1})$ version of $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ to the $(\mathrm{r}_{n2}, \mu_{n2}, \mathfrak{p}_{n2})$ version if the following is true: Let $\mathfrak{d}$ denote an instanton solution to the index n version of (7-40) with equal $s \to \infty$ and $s \to -\infty$ limits of $\mathfrak{f}_s(\mathfrak{d}_s)$. Then the $s \to \infty$ limit of $\mathrm{X}(\mathfrak{d}|_s)$ is no less than the $s \mapsto -\infty$ limit of $\mathrm{X}(\mathfrak{d}|_s)$.

Granted this point, there is for each n an instanton solution to be denoted by $\mathfrak{d}_n$ with equal $s \to \infty$ and $s \to -\infty$ limits of $\mathfrak{f}_s(\mathfrak{d}_n)$ and with

$$\lim_{s\to\infty} \mathrm{X}(\mathfrak{d}|_s) \le \lim_{s\to\infty} \mathrm{X}(\mathfrak{d}|_s) - 1.$$

An almost verbatim repetition of the final three paragraphs of Section 7.1 generates nonsense via Proposition 7.1 from the existence of the sequence $\{\mathfrak{d}_n\}_{n=1,2,\dots}$. □

7.4 Proof of Theorem 1.5, I

This subsection gives a proof of the first and second bullets of Theorem 1.5 and explains how the third bullet follows from an auxiliary proposition that is proved in the next subsection.

To start, fix r large and define Theorem 1.5's map $\hat{\Phi}^{\mathrm{r}}$ to be the L' version of the map supplied by Proposition 3.1. What is said by Proposition 3.1 implies that the resulting version of $\mathbb{L}^{\mathrm{r}}$ defines a $\mathbb{Z}$–module monomorphism from $\mathbb{Z}(\hat{\mathcal{Z}}^{L'}_{\mathrm{ech},M})$ to $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ obeying the first and second bullets of Theorem 1.5.

The upcoming Proposition 7.6 is used to prove Theorem 1.5's third bullet. To give some background for item (c) of the third bullet in this proposition, suppose for the moment that $\mathfrak{c}$ and $\mathfrak{c}'$ are nondegenerate solutions to some (r,μ) version of (1-13), and suppose that $\mathfrak{d}$ is a nondegenerate instanton solution to the corresponding version of (4-1) with $s\to-\infty$ limit $\mathfrak{c}'$ and $s\to\infty$ limit $\mathfrak{c}$. The corresponding operator in (1-21) is Fredholm, and this being the case, suppose that its index is 1 and that it has trivial cokernel. These properties are all that is needed to compute the ±1 weight that $\mathfrak{d}$ would contribute to the coefficient $\mathrm{w}_{[\mathfrak{c}'],[\mathfrak{c}]}$ in (1-24) were the pair $(\mathrm{r},\mathfrak{g}=\mathfrak{e}_\mu)$ suitable for defining the ∂^*_{SW} homology. This point is important for the following reason: if $(\mathrm{r},\mathfrak{g}=\mathfrak{e}_\mu+\mathfrak{p})$ with $\mathfrak{p}$ from $\mathcal{P}_\mu$ is suitable for defining the ∂^*_{SW} homology, then the (r,μ) instanton $\mathfrak{d}$ contributes to the $(\mathrm{r},\mathfrak{g}=\mathfrak{e}_\mu+\mathfrak{p})$ version of $\mathrm{w}_{[\mathfrak{c}'],[\mathfrak{c}]}$, and its contribution to the $(\mathrm{r},\mathfrak{g}=\mathfrak{e}_\mu+\mathfrak{p})$ version of $\mathrm{w}_{[\mathfrak{c}'],[\mathfrak{c}]}$ is this same ±1.

Proposition 7.6 refers to notation that is used in Section 1 to describe the endomorphisms that generate the respective actions of $\mathbb{Z}[\mathbb{U}]\otimes\bigwedge^* H_1(Y;\mathbb{Z})/\mathrm{tors}$ action on the ∂_{ech} homology and on the ∂^*_{SW} homology. By way of a reminder, from Part 4 of Section 1.2, the endomorphism of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{ech},M})$ that defines the action of $\mathbb{U}$ on the ∂_{ech} homology is defined as in (1-9) by a set of integers, $\{\mathrm{N}^{\mathbb{U}}_{\hat{\Theta}',\hat{\Theta}}\}_{\hat{\Theta}',\hat{\Theta}\in\hat{\mathcal{Z}}_{\mathrm{ech},M}}$, these being either 0 or 1. Part 7 of Section 1.3 describes an analogous set of integers, $\{\mathrm{w}^{\mathbb{U}}_{[\mathfrak{c}'],[\mathfrak{c}]}\}_{[\mathfrak{c}'],[\mathfrak{c}]\in\hat{\mathcal{Z}}_{\mathrm{SW,r}}}$, that appear in the version of (1-24) for the endomorphism that gives the action of $\mathbb{U}$ on the ∂^*_{SW} homology. Part 4 of Section 1.2 defines the set $\{\{[\gamma^{(z)}]\}_{z\in¥},\{\hat{\iota}_{\mathfrak{p}}\}_{\mathfrak{p}\in\Lambda}\}$, this being a set of 1–cycles that give a basis for $H_1(Y;\mathbb{Z})/\mathrm{tors}$. Let $\hat{\iota}$ denote a cycle from this set. This same Part 4 of Section 1.2 defines the endomorphism that gives the action of $\hat{\iota}$ on the ∂_{ech} homology. This endomorphism is a version of (1-9) whose coefficients are denoted by $\{\mathrm{N}^{\hat{\iota}}_{\hat{\Theta}',\hat{\Theta}}\}_{\hat{\Theta}',\hat{\Theta}\in\hat{\mathcal{Z}}_{\mathrm{ech},M}}$. Meanwhile, Part 7 of Section 1.3 defines the endomorphism that gives $\hat{\iota}$'s action on the ∂^*_{SW} homology by a version of (1-24) with the coefficients denoted by $\{\mathrm{w}^{\hat{\iota}}_{[\mathfrak{c}'],[\mathfrak{c}]}\}_{[\mathfrak{c}'],[\mathfrak{c}]\in\hat{\mathcal{Z}}_{\mathrm{SW,r}}}$.

Proposition 7.6 *Given $\Theta\in\mathcal{Z}_{\mathrm{ech},M}$, there exists $\kappa_\Theta\ge1$ with the following significance: Fix $\mathrm{r}\ge\kappa_\Theta$ and an element $\mu\in\Omega$ with $\mathcal{P}$–norm less than 1. Suppose that $\hat{\Theta}\in\hat{\mathcal{Z}}_{\mathrm{ech},M}$ is a lift of Θ.*

- *The element $\hat{\Theta}$ is in the domain of $\hat{\Phi}^{\mathrm{r}}$. Use $\mathfrak{c}$ to denote its image in $\hat{\mathcal{Z}}_{\mathrm{SW,r}}$.*
- *If $\mathfrak{c}'\in\hat{\mathcal{Z}}_{\mathrm{SW,r}}$ is such that $\mathcal{M}_1(\mathfrak{c}',\mathfrak{c})\neq\varnothing$, then $\mathfrak{c}'$ is in the image of $\hat{\Phi}^{\mathrm{r}}$.*

- *If* $\widehat{\Theta}' \in \widehat{\mathcal{Z}}_{\mathrm{ech},M}$ *is such that* $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta}) \neq \varnothing$, *then* $\widehat{\Theta}'$ *is in the domain of* $\widehat{\Phi}^{\mathrm{r}}$. *Granted that such is the case, use* $\mathfrak{c}'$ *to denote* $\widehat{\Phi}^{\mathrm{r}}(\widehat{\Theta}')$.
  - (a) *The space* $\mathfrak{M}_1(\mathfrak{c}', \mathfrak{c})$ *has only nondegenerate instantons.*
  - (b) *There exists a smooth,* $\mathbb{R}$*–equivariant,* 1–1, *onto map* $\Psi^{\mathrm{r}}\colon \mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta}) \to \mathfrak{M}_1(\mathfrak{c}', \mathfrak{c})$.
  - (c) *The* ± 1 *weight that any given element in* $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta})$ *contributes to the coefficient* $\mathrm{N}_{\widehat{\Theta}', \widehat{\Theta}}$ *in* (1-9)*'s formula for* $\partial_{\mathrm{ech}}\widehat{\Theta}$ *is the same as the weight that its* Ψ^{r}*–image would contribute to the coefficient* $\mathrm{W}_{[\mathfrak{c}'],[\mathfrak{c}]}$ *in* (1-24)*'s formula for* $\partial^*_{\mathrm{SW}}\mathfrak{c}$.
  - (d) *Let* $\hat{\iota}$ *denote a cycle from the set* $\{\{[\gamma^{(z)}]\}_{z \in \text{¥}}, \{\hat{\iota}_{\mathfrak{p}}\}_{\mathfrak{p} \in \Lambda}\}$. *The contribution of any given element in* $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta})$ *to the coefficient* $\mathrm{N}^{\hat{\iota}}_{\widehat{\Theta}', \widehat{\Theta}}$ *is the same as that of its* Ψ^{r} *image to the coefficient* $\mathrm{W}^{\hat{\iota}}_{[\mathfrak{c}'],[\mathfrak{c}]}$.
- *If* $\mathfrak{c}' \in \widehat{\mathcal{Z}}_{\mathrm{SW,r}}$ *is such that* $\mathfrak{M}_{2,p}(\mathfrak{c}', \mathfrak{c}) \neq \varnothing$, *then* $\mathfrak{c}' = \widehat{\Phi}^{\mathrm{r}}(\widehat{\Theta}')$ *with* $\widehat{\Theta}' \in \widehat{\mathcal{Z}}_{\mathrm{ech},M}$ *being the unique element that contributes to the* $\mathbb{U}$*–map coefficient* $\mathrm{N}^{\mathbb{U}}_{\widehat{\Theta}', \widehat{\Theta}}$. *The corresponding space* $\mathfrak{M}_{2,p}(\mathfrak{c}', \mathfrak{c})$ *contains a single instanton which is nondegenerate and the contribution of the latter to* $\mathrm{W}^{\mathbb{U}}_{[\mathfrak{c}'],[\mathfrak{c}]}$ *is* 1, *this being* $\mathrm{N}^{\mathbb{U}}_{\widehat{\Theta}', \widehat{\Theta}}$.

This proposition is proved in the next subsection. Accept it as true until then. What follows directly uses Proposition 7.6 to prove the third bullet of Theorem 1.5.

Fix L'. The corresponding set $\mathcal{Z}^{L'}_{\mathrm{ech},M}$ contains but a finite number of elements. This understood, introduce $\kappa_{L'}$ to denote the largest of the $\Theta \in \mathcal{Z}^{L'}_{\mathrm{ech},M}$ versions of the constant κ_Θ supplied by Proposition 7.6. Fix $\mathrm{r} > \kappa_{L'}$ and fix an element $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 and such that all solutions to the (r, μ) version of (1-13) are nondegenerate. Use these solutions to define the set $\mathcal{Z}_{\mathrm{SW,r}}$ and to define the $\mathbb{Z}$–module $\mathbb{Z}(\widehat{\mathcal{Z}}_{\mathrm{SW,r}})$. Fix an element $\mathfrak{p} \in \mathcal{P}_\mu$ with small $\mathcal{P}$–norm that can be used to define $\partial^*_{\mathrm{SW,r}}$.

The assertion of the third bullet in Theorem 1.5 to the effect that $\mathbb{L}^{\mathrm{r}}\partial_{\mathrm{ech},M} = \partial^*_{\mathrm{SW}}\mathbb{L}^{\mathrm{r}}$ follows directly from what is said by the second and third bullets of Proposition 7.6. Indeed, as elements in $\mathcal{P}_\mu$ vanish to second order on the images in $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ of nondegenerate instantons with $\mathrm{I}_{(\cdot)} = 1$, the second bullet of Proposition 7.6 with items (a) and (b) of the third bullet of Proposition 7.6 imply that any instanton that is used to compute the action of ∂^*_{SW} on the image of Φ^{r} is in the image of some version of Ψ^{r}. Granted that such is the case, item (c) of the third bullet of Proposition 7.6 guarantees that the contribution of such an instanton to ∂^*_{SW} is the same as the contribution of its Ψ^{r}–inverse image to ∂_{ech}.

The fact that $\mathbb{L}^{\mathrm{r}}$ intertwines the endomorphisms that define the respective actions of $\mathbb{Z}[\mathbb{U}] \otimes H_1(Y;\mathbb{Z})/\mathrm{tors}$ action on the ∂_{ech} homology and on the ∂_{SW}^* homology follows directly from the preceding paragraph with item (d) of the third bullet of Proposition 7.6 and the fourth bullet of Proposition 7.6.

7.5 Proof of Proposition 7.6

The proof of the proposition has seven parts.

Part 1 The first bullet of the proposition follows from Proposition 3.1. It is also the case that if r is large and $\hat{\Theta}'$ is such that $\mathcal{M}_1(\hat{\Theta}', \hat{\Theta}) \neq \varnothing$, then $\hat{\Theta}'$ is also in the domain of $\hat{\Phi}^{\mathrm{r}}$ when r is large, the reason being that there are but a finite number of such $\hat{\Theta}'$ in $\hat{\mathcal{Z}}_{\mathrm{ech},M}$, this a fact that is explained in Section II.A2.

The second bullet of Proposition 7.6 follows from Propositions 3.1 and 6.1, as does the assertion in the fourth bullet to the effect that if r is large, and if $\mathfrak{c}'$ is such that $\mathfrak{M}_{2,p}(\mathfrak{c}', \mathfrak{c}) \neq \varnothing$, then $\mathfrak{c}'$ is in the image of Φ^{r}. To explain how this comes about, introduce the function M on $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ from (1-30). Proposition 3.1 bounds $\mathrm{M}(\mathfrak{c}_+)$ by a multiple of $\sum_{\gamma\in\Theta} \ell_\gamma$. Suppose that $\mathfrak{c}'$ is a solution to the (r, μ) version of (1-13) and is such that $\mathfrak{M}_1(\mathfrak{c}', \mathfrak{c}) \neq \varnothing$ or $\mathfrak{M}_{2,p}(\mathfrak{c}', \mathfrak{c}) \neq \varnothing$. Let $\mathfrak{d}$ denote an instanton in either one of these spaces. Use Lemma 4.1 to see that $\mathrm{A}_{\mathfrak{d}} \leq c_0\mathrm{r}(\mathrm{M}(\mathfrak{c}) + 1)$ and thus it is bounded by $c_0\mathrm{r}\big(\sum_{\gamma\in\Theta} \ell_\gamma + 1\big)$. Granted this bound, use Proposition 6.1 to conclude that $\mathfrak{d}$'s version of the function $\underline{\mathrm{M}}$ is bounded by c_Θ with $c_\Theta > 1$ determined solely by Θ. It follows as a consequence that the $s \to -\infty$ limit of $\mathfrak{d}$'s version of $\underline{\mathrm{M}}$ is bounded by c_Θ and so $\mathrm{M}(\mathfrak{c}') < c_\Theta$. This being the case, Proposition 3.1 asserts that $\mathfrak{c}'$ is in the image of $\hat{\Phi}^{\mathrm{r}}$ if r is larger than a purely Θ–dependent constant.

Part 2 Keep in mind for what follows that the almost complex structure J is such that any $(\hat{\Theta}', \hat{\Theta})$ version of $\mathcal{M}_1(\hat{\Theta}', \hat{\Theta})$ has a finite set of $\mathbb{R}$ orbits and the Fredholm operator associated to each such orbit has trivial cokernel. The next remark is also important for what follows: Given $k \in \mathbb{Z}$, use $\hat{\Theta}_k$ for the moment to denote the translate of $\hat{\Theta}$ by the action of k on $\hat{\mathcal{Z}}_{\mathrm{ech},M}$ that comes about by viewing $\hat{\mathcal{Z}}_{\mathrm{ech},M}$ as a principal $\mathbb{Z}$–bundle over the set $\mathcal{Z}_{\mathrm{ech},M}$. Fix $\hat{\Theta}' \in \hat{\mathcal{Z}}_{\mathrm{ech},M}$. The respective sets of component subvarieties of $\mathcal{M}_1(\hat{\Theta}', \hat{\Theta})$ and of $\mathcal{M}_1(\hat{\Theta}'_k, \hat{\Theta}_k)$ are identical.

Granted these last facts, the construction that is described in Sections 4–7 of [20] can be employed with only minor alterations if r is large to construct an $\mathbb{R}$–equivariant, injective map from any given $\hat{\Theta}' \in \hat{\mathcal{Z}}_{\mathrm{ech},M}$ version of $\mathcal{M}_1(\hat{\Theta}', \hat{\Theta})$ to the corresponding space $\mathfrak{M}_1(\mathfrak{c}', \mathfrak{c})$. Denote this map by Ψ^{r}. The arguments in Section 2b of [23] can be used to construct Ψ^{r} to have the following property: Let C denote any given component subvariety in $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta})$. Write the instanton $\Psi^{\mathrm{r}}(C)$ as $(A, \psi = (\alpha, \beta))$. If $\hat{\imath} \in \{\{[\gamma^{(z)}]\}_{z \in ¥}, \{\hat{\imath}_{\mathrm{p}}\}_{\mathrm{p} \in \Lambda}\}$, then the intersections between C and $\mathbb{R} \times \hat{\imath}$ enjoy a 1–1 correspondence between those of $\alpha^{-1}(0)$ and $\mathbb{R} \times \hat{\imath}$ and this correspondence is such that partnered intersections have the same local intersection number. Note in this regard that J is such that C's intersections with $\mathbb{R} \times \hat{\imath}$ are finite in number and transversal. This is also the case for the intersections of $\alpha^{-1}(0)$ and $\mathbb{R} \times \hat{\imath}$. Note in addition that the distance between any given point in $C \cap (\mathbb{R} \times \hat{\imath})$ and its corresponding partner in $\alpha^{-1}(0) \cap (\mathbb{R} \times \hat{\imath})$ is bounded by a Θ–dependent multiple of $\mathrm{r}^{-1/2}$.

What follows is a parenthetical remark with regards to the use here of the constructions in Sections 4–7 of [20] and in Section 2b of [23]. The constructions here use the simplest versions of those in the latter references by virtue of three facts, the first being that all integral curves of v from all elements $\mathcal{Z}_{\mathrm{ech},M}$ are hyperbolic. The second is implied by the first: All subvarieties from any $\widehat{\Theta}', \widehat{\Theta} \in \widehat{\mathcal{Z}}_{\mathrm{ech},M}$ version of $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta})$ have the following property: Let C denote an element in $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta})$. If $|s| \gg 1$, then distinct components of the any constant s slice of C are in small radius tubular neighborhoods of distinct integral curves of v, and each such component is isotopic in this neighborhood to its core integral curve. The final fact constitutes what is asserted by Lemma 3.2.

Part 3 The arguments in Section 3 of [21] can be used with only very minor changes when r is large to prove the following: Fix $\widehat{\Theta}' \in \widehat{\mathcal{Z}}_{\mathrm{ech},M}$ with $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta}) \neq \varnothing$. The map Ψ^{r} restricts to each component of $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta})$ as an $\mathbb{R}$–equivariant diffeomorphism onto a smooth component of $\mathfrak{M}_1(\mathfrak{c}', \mathfrak{c})$ with only nondegenerate instantons. Moreover, the contribution of any given component of $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta})$ to the coefficient $\mathrm{N}_{\widehat{\Theta}', \widehat{\Theta}}$ is the same as that of its Ψ^{r} image to $\mathrm{W}_{[\mathfrak{c}'],[\mathfrak{c}]}$. Note in this regard that the assumptions in equation (1.14) of [21] are not needed, this being a consequence of the three facts that are stated in the final paragraph of Part 2.

Given what was said in Part 1 about intersections with $\hat{\imath} \in \{\{[\gamma^{(z)}]\}_{z \in ¥}, \{\hat{\imath}_{\mathrm{p}}\}_{\mathrm{p} \in \Lambda}\}$ versions of $\mathbb{R} \times \hat{\imath}$, the conclusions of the preceding paragraph lead directly to the following: if $\hat{\imath} \in \{\{[\gamma^{(z)}]\}_{z \in ¥}, \{\hat{\imath}_{\mathrm{p}}\}_{\mathrm{p} \in \Lambda}\}$, then the contribution of any given component of $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta})$ to $\mathrm{N}^{\hat{\imath}}_{\widehat{\Theta}', \widehat{\Theta}}$ is identical to that of its Ψ^{r} image to $\mathrm{W}^{\hat{\imath}}_{[\mathfrak{c}'],[\mathfrak{c}]}$.

Part 4 Suppose that $\widehat{\Theta}' \in \widehat{\mathcal{Z}}_{\mathrm{ech},M}$ is such that $\mathrm{N}^{\mathbb{U}}_{\widehat{\Theta}', \widehat{\Theta}} \neq \varnothing$. If r is large, then constructions in Sections 4–7 of [20] with those in Sections 2d and 4 of [23] construct a component of $\mathfrak{M}_{2,p}(\mathfrak{c}', \mathfrak{c})$ that contains a single instanton which is nondegenerate and suitable for use in the definition of the coefficient $w^{\mathbb{U}}_{[\mathfrak{c}'],[\mathfrak{c}]}$ and contributes $+1$ to this integer. Note again that only the simplest versions of what is done in [20; 22] are needed because only the J–holomorphic subvariety $\left(\prod_{\gamma\in\Theta}(\mathbb{R}\times\gamma)\right)\cup(\{0\}\times S)$ is used, and this is the union of disjoint product cylinders and a compact submanifold. Note also that there is no need to introduce the notion of a (δ, L) approximation to use the constructions in [22], this being yet another consequence of the three facts stated at the end of Part 2.

Part 5 It remains to prove that Ψ^{r} maps any given version of $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta})$ onto the corresponding version of $\mathfrak{M}_1(\mathfrak{c}, \mathfrak{c}')$ and to prove that $\mathfrak{M}_{2,p}(\mathfrak{c}', \mathfrak{c})$ has just the one component that is described in Part 4. The proofs that these assertions are true uses almost verbatim versions of arguments in Sections 4–7 of [22] and in Section 4e of [23]. Only the simplest cases of the arguments from [22; 23] are needed, this also a consequence of the three facts stated in the final paragraph of Part 2. What follows directly and in Parts 6 and 7 say more about the analogs here of the relevant parts of [22; 23].

Let $\mathfrak{c}'$ denote a solution to the (r, μ) version of (1-13) with either $\mathfrak{M}_1(\mathfrak{c}', \mathfrak{c})$ or $\mathfrak{M}_{2,p}(\mathfrak{c}', \mathfrak{c})$ nonempty. Let $\mathfrak{d}$ denote an instanton in one or the other of these spaces. The applications of the arguments from [22] require as input the bound $\underline{\mathrm{M}} < c_\Theta$ from Part 1 on $\mathfrak{d}$'s version of the function $\underline{\mathrm{M}}$. Keep in mind that such a bound exists.

Suppose that there exists for each $n \in \{1, 2, \dots\}$ a pair $\mathrm{r}_n > n$ and an element μ_n in Ω with $\mathcal{P}$–norm less than 1 such that the (r_n, μ_n) version of the map Ψ^{r} is not onto. If this is the case, there exists $\widehat{\Theta}' \in \widehat{\mathcal{Z}}^{L'}_{\mathrm{ech},M}$ and, for each n, either an instanton solution to the (r_n, μ_n) version of (4-1) in the corresponding version $\mathfrak{M}_1(\mathfrak{c}', \mathfrak{c})$ that is not in the image of the relevant version of Ψ^{r}, or an instanton in $\mathfrak{M}_{2,p}(\mathfrak{c}', \mathfrak{c})$ that is not the one from Part 4. Use $\mathfrak{d}_n$ to denote this instanton. The latter is written when needed as (A_n, ψ_n).

The rest of Part 5 and Parts 6 and 7 assume that the sequence $\{\mathfrak{d}_n\}_{n=1,2,\dots}$ contains an infinite subset from the corresponding version of $\mathfrak{M}_1(\mathfrak{c}', \mathfrak{c})$. Granted that this is the case, Lemma 6.1 in [22] has the following analog:

Lemma 7.7 *There is an element $C \subset \mathcal{M}_1(\Theta_-, \Theta_+)$, a subsequence of $\{\mathfrak{d}_n\}_{n=1,2,\dots}$* (hence renumbered consecutively from 1) *and a corresponding sequence of constant translations along the $\mathbb{R}$ factor of $\mathbb{R}\times Y$, all with the following property: For each n, write the translated version of ψ_n as a pair (α_n, β_n). The sequence whose n^{th} element is*

$$\sup_{z\in C} \mathrm{dist}(z, \alpha_n^{-1}(0)) + \sup_{z\in\alpha_n^{-1}(0)} \mathrm{dist}(C, z)$$

converges with limit zero. In addition, if $I \subset \mathbb{R}$ is an interval of length 1 and υ is a 2–form on $\mathbb{R} \times Y$ with $\|\upsilon\|_\infty = 1$ and support on $I \times Y$, then the sequence whose n^{th} element is $\frac{i}{2\pi} \int_{\mathbb{R}\times Y} \upsilon \wedge F_{\hat{A}_n} - \int_C \upsilon$ also converges with limit zero.

The proof of this lemma is given in Part 6; assume it to be true in the meantime. Lemma 7.7 leads to the analog of Lemma 6.2 in [22]; this has the identical assumptions and adds the following conclusion:

$$\lim_{n\to\infty} \mathrm{r}_n^{1/2}\Big(\sup_{z\in C} \operatorname{dist}(z, \alpha_n^{-1}(0)) + \sup_{z\in\alpha_n^{-1}(0)} \operatorname{dist}(C, z)\Big) = 0. \tag{7-42}$$

The proof of (7-42) is almost identical to that of Lemma 6.2 in [22] and so the reader is referred to Section 7 of [22] for the proof of the latter's Lemma 6.2. By way of a guide to the proof of Lemma 6.2 of [22], much of what is done in Section 7 of [22] is of no concern to (7-42) because of the three facts listed at the end of Part 2. In particular, the integer m that enters in Lemmas 7.2–7.5 and Lemma 7.7 of [22] can be set equal to 1. Moreover, most of the delicate estimates in Section 7d of [22] are not needed because distinct $s \gg 1$ slices, or distinct $s \ll -1$ slices, of any given subvariety from $\mathcal{M}_1(\hat{\Theta}', \hat{\Theta})$ are in tubular neighborhoods of distinct integral curves of υ and are isotopic in these neighborhoods to the core integral curve.

Given Lemma 7.7 and (7-42), the argument to prove that Ψ^{r} is onto is an almost verbatim copy of the arguments in Sections 6b–6e of [22]. Only the simplest cases of these arguments are needed by virtue of the facts listed in the final paragraph of Part 2. In any event, the modifications of the arguments in Sections 6b–6e of [22] are minimal and so nothing more will be said.

Part 6 The proof of Lemma 7.7 invokes an analog of Proposition 5.5 in [22], this constituting the lemma that follows:

Lemma 7.8 *Given $c \geq 1$, there exists $\kappa \geq 1$, and, given $m > \kappa$, there exists $\kappa_m \geq 1$ which, with κ, has the following significance: Fix $\mathrm{r} \geq \kappa_m$ and $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1. Suppose that $\mathfrak{d} = (A, \psi = (\alpha, \beta))$ is an instanton solution to* (4-1) *with $\mathrm{A}_{\mathfrak{d}} \leq c\mathrm{r}$ and with $\lim_{s\to\infty} \mathrm{M}(\mathfrak{d}|_s) < c$.*

- *Each point in $\mathbb{R} \times Y$ where $|\alpha| \leq 1 - \kappa^{-1}$ has distance at most $\kappa \mathrm{r}^{-1/2}$ from where $\alpha = 0$.*
- *Moreover, there exists:*
  (a) *A positive integer $N \leq \kappa$ and a cover of $\mathbb{R}$ as $\bigcup_{1\leq k\leq N} I_k$ by connected open sets of length at least $\frac{1}{2}m$. These are such that $I_k \cap I_{k'} = \varnothing$ if*

$|k-k'|>1$. *In addition, if* $|k-k'|=1$, *then* $I_k \cap I_{k'}$ *has length between* $\frac{1}{128}m$ *and* $\frac{1}{64}m$.

(b) *For each* $k \in \{1,2,\ldots,N\}$, *a set* ϑ_k *whose typical element is a pair* (C,m), *where* m *is a positive integer and where* $C \subset \mathbb{R}\times Y$ *is an irreducible, pseudoholomorphic subvariety. No two pairs from* ϑ_k *contain the same subvariety component and* $\sum_{(C,m)\in\vartheta_k} m\int_C w < \kappa$.

In addition, these sets $\{\vartheta_k\}_{k=1,\ldots,N}$ *are such that*:

(1) $$\sup_{z\,\in\,\bigcup_{(C,m)\in\vartheta_k} C \text{ and } s(z)\,\in\, I_k} \operatorname{dist}(z,\alpha^{-1}(0)) + \sup_{z\,\in\,\alpha^{-1}(0) \text{ and } s(z)\,\in\, I_k} \operatorname{dist}\Bigl(\bigcup_{(C,m)\in\vartheta_k} C, z\Bigr) < m^{-1}.$$

(2) *Let* $k\in\{1,\ldots,N\}$, *let* $I'\subset I_k$ *denote an interval of length* 1, *and let* υ *denote the restriction to* $I'\times Y$ *of a* 2*–form on* $\mathbb{R}\times Y$ *with* $\|\upsilon\|_\infty = 1$ *and* $\|\nabla\upsilon\|_\infty \le m$. *Then* $\bigl|\frac{i}{2\pi}\int_{I'\times Y}\upsilon\wedge F_{\hat{A}} - \sum_{(C,m)\in\vartheta} m\int_C \upsilon\bigr| \le m^{-1}$.

The arguments for this lemma are given in a moment. Assume it to be true for the subsequent proof of Lemma 7.7.

Proof of Lemma 7.8 The proof has three steps.

Step 1 Pass to a subsequence of $\{(\mathrm{r}_n,\mu_n),\mathfrak{d}_n\}_{n=1,2,\ldots}$ and renumber from 1 with the subsequence chosen so that Lemma 7.8 can be invoked with $m=n$ for each index n. Lemma 7.8 provides a corresponding sequence $\{\vartheta_{k,n}\}_{k=1,\ldots,N_n}$ with each N_n a priori bounded by Lemma 7.8's constant κ. Since the sequence $\{N_n\}_{n=1,2,\ldots}$ is bounded, the sequence $\{(\mathrm{r}_n,\mu_n),\mathfrak{d}_n\}_{n=1,2,\ldots}$ can be assumed to have the property that $N_n=N$ for all n.

A subsequence of $\{(\mathrm{r}_n,\mu_n),\mathfrak{d}_n\}_{n=1,2,\ldots}$ can be chosen and renumbered from 1 so that the sequence $\bigl\{\{\vartheta_{k,n}\}_{k=1,2,\ldots,N}\bigr\}_{n=1,2,\ldots}$ converges to what is said to be a broken pseudoholomorphic subvariety. This is a collection $\{\vartheta_k\}_{k=1,2,\ldots,N'}$ of sets with the properties that are listed next. First, each ϑ_k is a finite set of pairs with each pair having the form (C,m) with $C\in\mathbb{R}\times Y$ being an irreducible pseudoholomorphic subvariety and with m being a positive integer. Moreover, if $N'>1$, then each ϑ_k contains at least one pair whose subvariety is not $\mathbb{R}$–invariant.

The second property concerns the large s limits of the elements in each ϑ_k. In particular, the $s\to\infty$ limit of the constant s slices of ϑ_k determine a finite set, denoted by Θ_{k+}, whose elements are pairs of the form (γ,q) with γ being a closed integral curve of v and q being a positive integer. The manner by which ϑ_k determines

$\Theta_{k,+}$ is as follows: For $s \in \mathbb{R}$, let $C|_s \subset Y$ denote the constant s slice of C. View $\bigcup_{(C,m)\in\vartheta_k} mC|_s$ as a current. This current converges as $s \to \infty$ and the limit is the current $\bigcup_{(\gamma,q)\in\Theta_{k+}} q\gamma$. By the same token, the $s \to -\infty$ limit of $\bigcup_{(C,m)\in\vartheta_k} mC|_s$ determines a second set, $\Theta_{k,-}$, this having the same form as Θ_{k+}. The collection $\{(\Theta_{k,-}, \Theta_{k,+})\}_{k=1,2,\dots,N'}$ are constrained by the requirement

$$\text{(7-43)}\quad \Theta_{1-} = \Theta', \qquad \Theta_{k,+} = \Theta_{k+1,-} \quad \text{for } k = 1, \dots, N'-1, \qquad \Theta_{N',+} = \Theta.$$

Here and in what follows, Θ' denotes the image of $\widehat{\Theta}'$ via the projection to $\mathcal{Z}_{\mathrm{ech},M}$. The convergence of $\big\{\{\vartheta_{k,n}\}_{k=1,2,\dots,N}\big\}_{n=1,2,\dots}$ to $\{\vartheta_{k'}\}_{k'=1,2,\dots,N'}$ is analogous to that described in the paragraph surrounding equation (5.38) in [22], with the only salient modification being the replacement of $d\mathrm{a}$ in this equation in [22] with w.

Step 2 The constraints on the first Chern class of E, what is said in (7-43) and what is said in Section II.3 about pseudoholomorphic subvarieties in $\mathbb{R} \times Y$ place extra constraints on the pairs from $\{\vartheta_k\}_{k=1,\dots,N'}$. The first constraint involves the integer components of these pairs: if $(C,m) \in \bigcup_{k=1,\dots,N'} \vartheta_k$, then $m = 1$ unless either C is compact or all components of its constant s slices converge as $|s| \to \infty$ to closed integral curves of v in $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p}}$. The remaining constraints involve the sets $\big\{\{\Theta_{k-}, \Theta_{k+}\}\big\}_{k=1,\dots,N'}$:

(7-44)
- If γ comes from a pair in $\bigcup_{k=1,\dots,N'}(\Theta_{k-} \cup \Theta_{k+})$, then γ is disjoint from $\mathcal{H}_0$ and as a consequence, γ lies entirely in the union of the $f \in (1,2)$ part of M_δ with $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p}}$.
- Each $(\gamma, q) \in \bigcup_{k=1,\dots,N'}(\Theta_{k-} \cup \Theta_{k+})$ has $q = 1$ unless $\gamma \subset \bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p}}$.
- Fix $k \in \{1, \dots, N'\}$ and let $\Theta_{k,*}$ denote either $\Theta_{k,-}$ or $\Theta_{k,+}$. Then $\big(\bigcup_{(\gamma,1)\in\Theta_k}, \gamma\big) \cap M_\delta$ consists of G arcs that pair the index 1 and index 2 critical points of f in M in the sense that distinct arcs start on the respective boundaries of the radius δ coordinate balls about distinct index 1 critical points of f and end on the respective boundaries of the radius δ coordinate balls about distinct index 2 critical points of f.

The proof given below that these constraints must be satisfied uses the following observation: if Θ is any element from the set $\{\Theta_{k,-}, \Theta_{k,+}\}_{k=1,2,\dots,N'}$, then the homology class $[\Theta] = \sum_{(\gamma,q)\in\Theta} q[\gamma]$ is Poincaré dual to the first Chern class of E. This is proved using backwards induction on the integer k, starting from $k = N'$. It holds in this case because $\Theta_{N',+} = \Theta$. Supposing that it holds for any $\Theta_{k,+}$, then it holds for the corresponding $\Theta_{k,-}$ because $\Theta_{k,-}$ and $\Theta_{k,+}$ are homologous. (The class $[\Theta_{k,+}]-[\Theta_{k,-}]$ is the pushforward to Y via the projection from $\mathbb{R}\times Y$ of the relative homology class $\sum_{(C,m)\in\vartheta_k} m[C]$.) And, it holds for $\Theta_{k-1,+}$ if it holds for $\Theta_{k,-}$ because these two sets are identical.

The proof that the constraints in (7-41) are obeyed uses backwards induction on k also. Start with $\Theta_{N'}$ to see that the constraints on the integer components of its pairs are forced by the condition that $\Theta_{N',+}=\Theta$. The constraints on $\Theta_{N',-}$ are then forced by the first Chern class considerations and what is said in Section II.2 about the closed, integral curves of v. For example, a loop γ from a pair in $\Theta_{N',-}$ cannot intersect $\mathcal{H}_0$ because, as explained in Section II.3, it would then have positive intersection number with any cross-sectional 2–sphere in $\mathcal{H}_0$. This would mean that the first Chern class of E has positive pairing with such spheres, which is not the case by assumption. The fact that $(\gamma,q)\in\Theta_{N',-}$ has $q=1$ unless $\gamma\in\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}$ follows for a similar reason: If $q>1$ and γ is not entirely in some $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$, then the class $[\Theta_{N',-}]$ would have intersection number at least q with a cross-sectional sphere in some $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$. (According to Section II.2, all integral curves of v intersecting $\mathcal{H}_{\mathfrak{p}}$ have positive intersection number with such spheres except $\gamma_{\mathfrak{p}}^{+}$ and $\gamma_{\mathfrak{p}}^{-}$.) This can't happen because the first Chern class of E has pairing 1 with each such cross-sectional sphere. The condition in the third bullet of (7-41) must hold because the first Chern class of E has pairing G with the $f=\frac{2}{3}$ level set in M (and because of what Section II.3 says about the integral curves of v in the M_δ part of Y being arcs on which f is monotonic).

The constraints on $\Theta_{N'-1,+}$ are obeyed because this is the same set as $\Theta_{N',-}$. Then, exactly the same considerations (except with N' changed to $N'-1$) as in the preceding paragraph shows that the constraints in (7-41) must hold for $\Theta_{N'-1,-}$. Continuing in this vein proves that the constraints hold for all of the $\Theta_{k,+}$ and $\Theta_{k,-}$.

Step 3 Fix $k\in\{1,\dots,N'\}$ and use Z_k to denote the 2–cycle in Y given by the pushforward via the projection of $\sum_{(C,m)\in\vartheta_k} m[C]$ with $[C]$ here denoting the non-compact cycle in Y that is carried by the fundamental class of C. The boundary of Z_k is the 1–cycle $\sum_{(\gamma,q)\in\Theta_{k,+}} q[\gamma]-\sum_{(\gamma,q)\in\Theta_{k,-}} q[\gamma]$.

Definition 2.14 in [2] uses $[Z_k]$ to define the embedded contact homology index, this being an integer that is denoted here by $I(\Theta_{k-},\Theta_{k+},Z_k)$. Let Z denote $\sum_{1\le k\le N'} Z_k$. Given Remark 2.16 in [2], what is said by (7-43) implies that

$$I(\Theta',\Theta,Z)=\sum_{1\le k\le N'} I(\Theta_{k,-},\Theta_{k,+},Z_k). \tag{7-45}$$

The argument in Part 2 of the proof of Lemma 6.1 in [22] for the nontorsion case can be copied here with only minor changes to see that $I(\Theta', \Theta, Z) = 1$. This argument uses what is said in Lemma 7.8 about the large n versions of $\alpha_n^{-1}(0)$ and the fact that the instanton $\mathfrak{d}_n$ is in the (r_n, μ_n) version of $\mathfrak{M}_1(\mathfrak{c}', \mathfrak{c})$.

It follows from Hutchings' Definition 2.14 in [2], from the description in Propositions II.3.1–II.3.4 of the pseudoholomorphic subvarieties in $\mathbb{R} \times Y$, and from (7-41) that $I(\Theta', \Theta, Z) = 1$ if and only if $N' = 1$, in which case ϑ_1 defines an element in $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta})$. The argument for this uses the inherently positive intersection numbers between pseudoholomorphic subvarieties in much the same way as used in the proof of Lemma III.8.3 to more than offset negative contributions to the sum in (7-45) that come from any given pair $(C, m) \in \bigcup_{k=1,\dots,N'} \vartheta_k$. This is illustrated in the proof of Lemma III.8.3 by the formula (III.8-6). □

Proof of Lemma 7.7 Given that $N' = 1$ and that ϑ_1 defines an element in $\mathcal{M}_1(\widehat{\Theta}', \widehat{\Theta})$, then what is said in Lemma 7.7 follows directly from the conclusions of Lemma 7.8. □

Part 7 This part contains the:

Proof of Lemma 7.8 The arguments are much like the simplest versions of those used to prove Proposition 5.5 in [22]. The six steps that follow describe what is needed from [22] and what parts of these arguments need more than purely cosmetic changes.

Step 1 Given the bound in Proposition 6.1 on $\underline{\mathrm{M}}$, Proposition 4.5 in [22] has a simpler analog here also. This analog is a slightly weaker version of Lemma 7.8 that differs from Lemma 7.8 only to the extent that it does not make the claim that the pseudoholomorphic subvarieties in any given $k \in \{1, \dots, N\}$ version of ϑ_k are defined on the whole of $\mathbb{R} \times Y$. The weaker version claims instead that the ϑ_k subvarieties are defined on a neighborhood of $I_k \times Y$.

The argument that derives Lemma 7.8 from its weak version amounts to little more than a standard application of a local form of the Gromov compactness theorem for pseudoholomorphic subvarieties. This argument differs little from the compactness theorems in [1]. Given the a priori bound on $\underline{\mathrm{M}}$ from Proposition 6.1, the derivation of Lemma 7.8 from its weak analog differs only in notation from what is said in [22] to deduce Proposition 5.5 in [22] from Proposition 5.1 in [22].

Step 2 What follows here and in the subsequent steps proves the weak version of Lemma 7.8 using a modified version of the argument for Proposition 4.5 in [22]. The modified version of this proposition is stated by the next lemma:

Lemma 7.9 *Given* $c \geq 1$, *there exists* $\kappa \geq 1$, *and, given* $m > \kappa$, *there exists* $\kappa_m \geq 1$ *which, with* κ, *has the following significance: Fix* $\mathrm{r} \geq \kappa_m$ *and* $\mu \in \Omega$ *with* $\mathcal{P}$*–norm less than* 1. *Suppose that* $\mathfrak{d} = (A, \psi = (\alpha, \beta))$ *is an instanton solution to* (4-1) *with* $\mathrm{A}_{\mathfrak{d}} \leq c\mathrm{r}$ *and with* $\lim_{s\to\infty} \mathrm{M}(\mathfrak{d}|_s) < c$. *Let* $I \subset \mathbb{R}$ *denote an interval of length at least* $2m$.

- *Each point in* $I \times Y$ *where* $|\alpha| \leq 1 - \kappa^{-1}$ *has distance* $\kappa \mathrm{r}^{-1/2}$ *or less from a zero of* α.
- *There exists a finite set,* ϑ, *whose components are pairs of the form* (C, m) *where* C *is a closed, irreducible pseudoholomorphic subvariety in a neighborhood of the closure of* $I \times Y$ *and where* m *is a positive integer. Moreover, no two pairs in* ϑ *share the same subvariety component. This set is such that:*
  (a) $$\sup_{z \in \bigcup_{(C,m)\in\vartheta} C,\, s(z)\in I} \mathrm{dist}(z, \alpha^{-1}(0)) + \sup_{z\in\alpha^{-1}(0),\, s(z)\in I} \mathrm{dist}\Big(\bigcup_{(C,m)\in\vartheta} C, z\Big) < m^{-1}.$$
  (b) *Let* υ *denote a smooth* 2*–form on* $I \times Y$ *with compact support, with* $\|\upsilon\|_\infty = 1$ *and with* $\|\nabla\upsilon\|_\infty \leq m$. *Then*
  $$\left| \frac{i}{2\pi} \int_{I\times Y} \upsilon \wedge F_{\hat{A}} - \sum_{(C,m)\in\vartheta} m \int_C \upsilon \right| \leq m^{-1}.$$
  (c) $\sum_{(C,m)\in\vartheta} m \int_C w \leq \kappa$.

The proof of this lemma is given in a moment. The next lemma plays a central role in the proof, and in subsequent arguments in this section:

Lemma 7.10 *Given* $c \geq 1$, *there exists* $\kappa_c \geq 1$ *with the following significance: Fix* $\mathrm{r} \geq \kappa_c$ *and* $\mu \in \Omega$ *with* $\mathcal{P}$*–norm less than* 1. *Suppose that* $\mathfrak{d} = (A, \psi = (\alpha, \beta))$ *is an instanton solution to* (4-1) *with* $\mathrm{A}_{\mathfrak{d}} \leq c\mathrm{r}$ *and with* $\lim_{s\to\infty} \mathrm{M}(\mathfrak{d}|_s) < c$. *Let* $I \subset \mathbb{R}$ *denote an open interval. Then* $-\kappa_c \mathrm{r}^{-1/3} < \int_{I\times Y} iF_{\hat{A}} \wedge w < \kappa_c$.

Proof of Lemma 7.9 Given the bound in Proposition 6.1 on $\underline{\mathrm{M}}$, what is asserted by Lemma 7.9 follows directly from the $Y_* = Y$ version of Proposition 6.3 with the help of Lemma 7.10. The latter is needed to deduce item (c) of the second bullet of Lemma 7.9 from the second bullet of Proposition 6.3. □

Proof of Lemma 7.10 Invoke Proposition 6.1 to bound $\underline{\mathrm{M}}$ by c_c with c_c denoting here and in what follows a purely c–dependent constant which is greater than 1. Its value can be assumed to increase between successive appearances.

Consider first the upper bound. To this end, write I as (s_1, s_2). Fix $s \in (0, 1)$ and suppose that $m > 0$ is such that the integral of $iF_{\hat{A}} \wedge w$ over $[s_1 - s, s_2 + s] \times Y$ is bounded from above by m. As explained directly, this implies that the integral of $iF_{\hat{A}} \wedge w$ over the smaller region $I \times Y$ is bounded from above by $m + c_c \mathrm{r}^{-1}$. To see why this is, use the bound on $\underline{\mathrm{M}}$ to invoke Lemma 4.9. The depiction of $\frac{\partial}{\partial s} A \pm B_A$ by this lemma implies that the function σ in (5-7) must obey $1 - \sigma \geq -c_c \mathrm{r}^{-1}$. With this bound available, then use the top bullet in (7-4) to see that $*(iF_{\hat{A}} \wedge w) \geq -c_c(1-\wp)^{3/4}$. (Keep in mind that $|\mathfrak{e}_A| \leq c_c((1-\wp) + \wp')$ and that $\wp' \leq c_0(1-\wp)^{3/4}$.) Since $\wp < 1$ only where $|\alpha|^2$ is less than $\frac{9}{16}$, putting a factor of $(1 - |\alpha|^2)$ here (and making c_c bigger) gives the bound $*(iF_{\hat{A}} \wedge w) \geq -c_c(1 - |\alpha|^2)$. Therefore, the integral of $*(iF_{\hat{A}} \wedge w)$ over the domain in question is no smaller than that of $-c_c(1 - |\alpha|^2)$. Meanwhile, the integral of $\mathrm{r}(1 - |\alpha|^2)$ over $[s_1 - s, s_1] \times Y$ is at most $\underline{\mathrm{M}}$ (which is less than c_c), and likewise the integral over $[s_2, s_2 + s] \times Y$; and so the integral of $*(iF_{\hat{A}} \wedge w)$ over either of these domains is no smaller than $-c_c \mathrm{r}^{-1}$. Thus, the integral of $*(iF_{\hat{A}} \wedge w)$ over $I \times Y$ is at most $m + c_c \mathrm{r}^{-1}$ because its integral over the larger domain $[s_1 - s, s_2 + s] \times Y$ is bounded by m (by assumption).

Since $\underline{\mathrm{M}}$ is bounded by c_c, there exists $s \in (0, 1)$ such that $\mathrm{M}(\mathfrak{d}|_{s_1 - s})$ and $\mathrm{M}(\mathfrak{d}|_{s_2 + s})$ are both bounded by c_c. Given what is said in the preceding paragraph, it is sufficient to bound the integral of $iF_{\hat{A}} \wedge w$ over $[s_1 - s, s_2 + s] \times Y$. This is done by comparing this integral to the integral of $iF_A \wedge w$ over $[s_1 - s, s_2 + s] \times Y$. The comparison is made in a moment. What follows directly studies the integral of $iF_A \wedge w$ over $[s_1 - s, s_2 + s] \times Y$.

Write $A = A_E + \hat{a}_A$ and integrate by parts to see that

$$(7\text{-}46) \qquad \int_{[s_1 - s, s_2 + s] \times Y} iF_A \wedge w = i \int_{\{s_2 + s\} \times Y} \hat{a}_A \wedge w - i \int_{\{s_1 - s\} \times Y} \hat{a}_A \wedge w.$$

Meanwhile, use (1-26)–(1-28) and the fact that $\frac{i}{2\pi}\big(F_{A_E} + \frac{1}{2} F_{A_K}\big)$ can be written as $w + d\mathfrak{b}$ to see that

$$(7\text{-}47) \quad \mathrm{r}^{-1}(\mathfrak{a}(\mathfrak{d}|_{s_1 - s}) - \mathfrak{a}(\mathfrak{d}|_{s_2 - s})) = (1 - 2\mathrm{r}^{-1})\bigg(i \int_{\{s_2 + s\} \times Y} \hat{a}_A \wedge w - i \int_{\{s_1 + s\} \times Y} \hat{a}_A \wedge w\bigg) + \mathfrak{e},$$

where $\mathfrak{e}$ has absolute value bounded by $c_c \mathrm{r}^{-1/3}$. To see why this is, note first that $\mathfrak{e}$ is bounded by c_0 times a sum of two terms. The first is itself a sum,

$$(7\text{-}48) \quad \mathrm{r}^{-1}\bigg|\int_{\{s_2 + s\} \times Y} \hat{a}_A \wedge d\hat{a}_A\bigg| + \mathrm{r}^{-1}\bigg|\int_{\{s_2 + s\} \times Y} \mathfrak{b} \wedge d\hat{a}_A\bigg| + \mathrm{r}^{-1}\bigg|\int_{\{s_2 + s\} \times Y} \mu \wedge B_A\bigg| + \mathrm{r}^{-1}\bigg|\int_{\{s_2 + s\} \times Y} \psi^\dagger D_A \psi\bigg|,$$

and the second has the same form but for the replacement of s_2+s with s_1-s. The terms in (7-48) are bounded as follows: starting from left to right, the integral of $\hat{a}_A \wedge d\hat{a}_A$ is bounded as in (6-16), thus by $\mathrm{r}^{2/3}\mathrm{M}^{4/3}$. (In this paragraph, $\mathrm{M} = \mathrm{M}(\mathfrak{d}|_{s_2+s})$.) Thus, with the extra r^{-1} factor, the left-most term in (7-48) contributes $c_c\mathrm{r}^{-1/3}$ or less to $\mathfrak{e}$. The two middle terms contribute at most $c_c\mathrm{r}^{-1}\mathrm{M}$ to $\mathfrak{e}$, which is to say at most $\mathrm{r}^{-1}c_c$. This is because $*d\hat{a}_A = B_A - B_{A_E}$ and because Lemma 4.7 (with Lemma 5.2) bounds the norm of B_A by $c_0\big(\mathrm{r}\big|1-|\alpha|^2\big|+1\big)$. Finally, the right-most term in (7-48) is bounded by $c_0(\mathrm{M}^{1/2}+\mathrm{r}^{-1/2})$ and thus by c_c. (This bound follows from the top bullet of Lemma 4.8 and Lemma 4.4 since $|\psi^\dagger D_A\psi|$ is no greater than $c_0(|\nabla_A\alpha| + |\nabla_A\beta| + |\beta|)$.)

To finish the story on (7-46), note that $\mathfrak{a}(\mathfrak{d}|_{s_1-s}) - \mathfrak{a}(\mathfrak{d}|_{s_2-s})$ is nonnegative and, in any event, no greater than $\mathrm{A}_{\mathfrak{d}}$, and thus no greater than $\mathrm{r}c$. Therefore, because $\mathfrak{e} \le c_c$, the right-hand side of (7-46) is likewise less than c_c.

Write $\hat{A}$ as $A_E + \hat{a}_{\hat{A}}$ and use integration by parts to write the $\hat{A}$ analog of the formula in (7-46). The latter has the integral of $iF_{\hat{A}} \wedge w$ over $[s_1 - s, s_2 + s] \times Y$ on the left-hand side and has the same right-hand side as the original, A version but for the replacement of $\hat{a}_A$ by $\hat{a}_{\hat{A}}$. This being the case, (1-15) with the bounds in the top bullet of Lemma 4.8 for $|\nabla_A\alpha|$ can be used to see that the absolute value of the difference between right-hand sides of the respective $\hat{a}_A$ and $\hat{a}_{\hat{A}}$ versions of (7-46) is at most $c_c(\mathrm{M}^{1/2} + \mathrm{r}^{-1/2})$. This is less than c_c because $\mathrm{M} \le c_c$.

To prove Lemma 7.10's lower bound assertion, remember from what was said previously that $*(iF_{\hat{A}} \wedge w) \ge -c_c(1-|\alpha|^2)$. Since the integral of $\mathrm{r}(1-|\alpha|^2)$ over an interval of length 2 (centered at any given $s \in \mathbb{R}$) is bounded by $\underline{\mathrm{M}}(s)$ (which is less than c_c by assumption), it follows that the integral of $*(iF_{\hat{A}}\wedge w)$ over $I \times Y$ is no less than $-c_c\mathrm{r}^{-1}$ when I has length 2 or less. This understood, assume henceforth that the length of I is greater than 2. Write $I = [s_1, s_2]$ and suppose that $s \in (0,1)$ and $m > 0$ are such that the integral of $iF_{\hat{A}} \wedge w$ over $[s_1 + s, s_2 - s] \times Y$ is greater than $-m$. Then, the integral of $iF_{\hat{A}} \wedge w$ over the larger domain $[s_1, s_2] \times Y$ is no less than $-(m + c_c\mathrm{r}^{-1})$ because the extra regions tacked on are of the form $I' \times Y$ with length $I' = s$ which is less than 1 (and, as just established, the integral of $*(iF_{\hat{A}} \wedge w)$ over such domains is not less than $-c_c\mathrm{r}^{-1}$.) With the preceding understood, use the fact that $\underline{\mathrm{M}}$ is bounded by c_c to choose s so that both $\mathrm{M}(\mathfrak{d}_{s_1+s})$ and $\mathrm{M}(\mathfrak{d}_{s_2-s})$ are bounded by c_c. The plan is to compare the integral of $iF_{\hat{A}} \wedge w$ over $[s_1 + s, s_2 - s] \times Y$ with that of $iF_A \wedge w$ over $[s_1 + s, s_2 - s] \times Y$. Use (7-46)–(7-48) with s replaced by $-s$ to see that the latter integral is no less than $-c_c\mathrm{r}^{-1/3}$. Meanwhile, the right-hand side of this $-s$ version of (7-46) differs from the right-hand side of its $\hat{A}$ counterpart by at most $c_c\mathrm{r}^{-1/2}$. The argument for this is identical but for the change $s \mapsto -s$ as that given in the preceding paragraph.

Step 3 The $Y_* = Y$ versions of Lemmas 6.4 and 6.5 play the role here of that played by Lemma 4.6 in [22] and Corollary 4.7 in [22]. The next lemma is a replacement for Lemma 4.8 in [22]:

Lemma 7.11 *Given $m > 1$, there exists $\kappa_m > 1$, and, given $\varepsilon > 0$, there exists $\mathrm{R}_\varepsilon > 16$; and these have the following significance: Let $\mathbb{I} \subset \mathbb{R}$ denote an interval of length at least $2\mathrm{R}_\varepsilon$, and suppose that C is a closed, irreducible, pseudoholomorphic subvariety in a neighborhood of $\mathbb{I} \times Y$ with $\int_{C\cap(I'\times Y)} w < \kappa_m^{-1}$ and $\int_{C\cap(I'\times Y)} ds \wedge \hat{a} \le m$ for all intervals $I' \subset \mathbb{I}$ of length 1. Assume in addition that C has intersection number zero with all submanifolds in $\mathbb{R} \times Y$ of the form $\{s\} \times S$ with S being a cross-sectional sphere in $\mathcal{H}_0$. Let $I \subset \mathbb{I}$ denote the subset with distance at least R_ε from any boundary point of $\mathbb{I}$. There exists a finite set Θ consisting of pairs (γ, q) with γ a closed, integral curve of v and q a positive integer. The set Θ is such that no two pairs share the same closed integral curve. Moreover:*

- $\sum_{(\gamma,q)\in\Theta} q\ell_\gamma \le m$.
- *Each point of $C|_s$ for $s \in I$ has distance along Y less than ε from $\bigcup_{(\gamma,q)\in\Theta} \gamma$. Conversely, each point in $\bigcup_{(\gamma,q)\in\Theta} \gamma$ has distance no greater than ε from $C|_s$.*
- *If υ is a smooth 2–form on $I \times Y$ with $\|\upsilon\|_\infty = 1$ and $\|\nabla\upsilon\|_\infty \le \varepsilon^{-1}$, then*

$$\left| \int_{C\cap(I\times Y)} \upsilon - \sum_{(\gamma,q)\in\Theta} q \int_{I\times\gamma} \upsilon \right| < \varepsilon.$$

Proof The proof of Lemma 4.8 in [22] can be copied with only the replacement of M with Y and with the references to Corollary 4.7 in [22] replaced by references to Lemma 6.5. □

The lower bound in Lemma 7.10 for integrals of $iF_{\hat{A}} \wedge w$ serves as a replacement for Lemma 4.9 in [22].

Step 4 The remaining arguments for the weak version of Lemma 7.8 are similar in most respects, and simpler, than those given in Parts 4 and 5 of Section 4d in [22] to prove Proposition 4.5 in [22].

To complete the proof of the weak version of Lemma 7.8, fix $\varepsilon' > 0$ and define the subset $\mathcal{I} \subset \mathbb{Z}$ by the rule that places a given integer k in $\mathcal{I}$ if and only if the integral of $iF_{\hat{A}} \wedge w$ over $[k, k+1] \times Y$ is greater than ε'. It follows from the asserted upper bound from Lemma 7.10 in the case $I = \mathbb{R}$ and from the lower bound as applied to the components of $\mathbb{R} - \big(\bigcup_{k \in \mathcal{I}} [k, k+1]\big)$ that $\mathcal{I}$ is a finite set with the number of components bounded by a constant that depends solely on c and ε'. Use $n_{\varepsilon'}$ to denote this number.

Introduce the number $\mathrm{R}_{\varepsilon'}$ from Lemma 7.11. There is a set, $\mathcal{V}$, of at most $n_{\varepsilon'}$ intervals in $\mathbb{R}$ and a pair of numbers, $c_{m_{*\varepsilon'}}$ and $c_{m_{\varepsilon'}}$, with the properties listed below:

(7-49)
- $c_{m_{*\varepsilon'}}$ and $c_{m_{\varepsilon'}}$ are determined solely by $n_{\varepsilon'}$ and m. In any event, $c_{m_{*\varepsilon'}} > c_{m_{\varepsilon'}} > 100 n_{\varepsilon'}$.
- $\bigcup_{I \in \mathcal{V}} I$ contains $\bigcup_{k \in \mathcal{I}} [k, k+1]$.
- Suppose that $I \in \mathcal{V}$.
  - (a) I has length greater than $c_{m_{\varepsilon'}}(m + \mathrm{R}_{\varepsilon'})$ but less than $c_{m_{*\varepsilon'}}(m + \mathrm{R}_{\varepsilon'})$.
  - (b) I contains at least one $k \in \mathcal{I}$ version of $[k, k+1]$.
  - (c) $I \cap \big(\bigcup_{k \in \mathcal{I}} [k, k+1]\big)$ has distance at least $10\mathrm{R}_{\varepsilon'}$ from I's boundary.
- If I and I' are distinct intervals from $\mathcal{V}$ with nonempty intersection, then $I \cap I'$ has length greater than $\frac{1}{128} m$.
- Each component of $\mathbb{R} - \big(\bigcup_{I \in \mathcal{V}} I\big)$ has length greater than $4m$.

Given $I \subset \mathcal{V}$, use $I_* \subset I$ to denote the set of points with distance $\frac{1}{64} m$ or greater from any boundary point of I. The assertion of the weak version of Lemma 7.8 follows directly by using Step 2's analog of Proposition 4.5 in [22] (ie Lemma 7.9) for its interval $\mathbb{I}$ taken in turn to be the intervals from $\mathcal{V}$ and using Lemma 7.11 for each component of $\mathbb{R} - \big(\bigcup_{I \in \mathcal{V}} I_*\big)$ with the constant in both replaced by ε' and with the latter being a suitable function of m. □

7.6 Proof of Theorem 1.5, II

The five parts of this subsection complete the proof of Theorem 1.5. Part 1 proves the assertion at the very end of the theorem about the versions of $\mathbb{L}^{\mathrm{r}}$ that are defined by distinct data sets. Part 2 of the subsection talks about some points in the proof given in Part 1 that are used implicitly in Parts 3–5. Parts 3 and 4 of the subsection prove the fifth bullet of Theorem 1.5, and Part 5 uses what is done in Parts 3 and 4 to prove the fourth bullet of Theorem 1.5.

Part 1 A proof is given in a moment for the final assertion of Theorem 1.5. What follows directly spells out what need proving. Fix $L' > 1$ and suppose that $(\mathrm{r}, \mu, \mathfrak{p})$ and $(\mathrm{r}', \mu', \mathfrak{p}')$ are data sets that satisfy the conditions demanded by Theorem 1.5. This is to say that the solutions to the respective (r, μ) and (r', μ') versions of (1-13) are nondegenerate and holonomy nondegenerate, and that the respective instanton solutions to the $(\mathrm{r}, \mathfrak{g} = \mathfrak{e}_\mu + \mathfrak{p})$ and $(\mathrm{r}', \mathfrak{g} = \mathfrak{e}'_\mu + \mathfrak{p}')$ versions of (1-20) are also nondegenerate. Granted these assumptions, the pair (r, μ) can be used to define $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ and $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$, and $(\mathrm{r}, \mu, \mathfrak{p})$ can be used to define the endomorphism ∂^*_{SW}. By the same token, (r', μ') can be used to define $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r'}})$ and $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r'}})$, and with $\mathfrak{p}'$ they define the corresponding version of ∂^*_{SW}. As noted in Proposition 1.4, there is a canonical homomorphism between $H^{\infty}_{\mathrm{SW,r}}$, $H^{-}_{\mathrm{SW,r}}$ and $H^{+}_{\mathrm{SW,r}}$ and the corresponding primed triad that intertwines the respective long exact sequences. Now suppose in addition that r and r′ are such that Proposition 3.1 can be used to define the $\hat{\Phi}^{\mathrm{r}}$ and $\hat{\Phi}^{\mathrm{r}'}$ on $\mathcal{Z}^{L'}_{\mathrm{ech},M}$. Theorem 1.5 asserts that this canonical homomorphism between homology groups can be lifted to a chain complex homomorphism between the respective $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ and $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r'}})$ which has the properties demanded by Proposition 1.4 and also intertwines the two versions of $\mathbb{L}^{\mathrm{r}}$.

To prove this assertion of Theorem 1.5, return for the moment to Section 7.3. The existence of a lift of Proposition 1.4's canonical homology homomorphism to a homomorphism from $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ to $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r'}})$ that satisfied Proposition 1.4's requirements is proved in Section 7.3. In particular, it follows from what is said in Section 7.3 that such a lift can be found that factors as $\hat{\imath}_N \circ \hat{\imath}_{N-1} \circ \cdots \circ \hat{\imath}_1$ with $\hat{\imath}_1$ mapping the $(\mathrm{r}, \mu, \mathfrak{p})$ version of $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ to an $(\mathrm{r}_1, \mu_1, \mathfrak{p}_1)$ version, with $\hat{\imath}_2$ mapping the latter version to an $(\mathrm{r}_2, \mu_2, \mathfrak{p}_2)$ version, and so on, and with $\hat{\imath}_N$ mapping an $(\mathrm{r}_{n-1}, \mu_{n-1}, \mathfrak{p}_{N-1})$ version to the $(\mathrm{r}', \mu', \mathfrak{p}')$ version. These various data sets are such that each $k \in \{1, \dots, N-1\}$ version of $|\mathrm{r}_{k+1} - \mathrm{r}_k|$ is very small as are the $\mathcal{P}$–norms of $\mu_{k+1} - \mu_k$ and $\mathfrak{p}_{k+1} - \mathfrak{p}_k$. Likewise, both $|\mathrm{r}_1 - \mathrm{r}|$ and $|\mathrm{r}' - \mathrm{r}_{n-1}|$ are small, as are the $\mathcal{P}$–norms of $\mu_1 - \mu$ and $\mathfrak{p}_1 - \mathfrak{p}$, and also $\mu' - \mu_{N-1}$ and $\mathfrak{p}' - \mathfrak{p}_{N-1}$. Note that "small" here means as small as desired (but not zero) at the expense of increasing N.

Of particular import is that the sequence $\{\mathrm{r}_k\}_{k=1,2,\dots}$ can be chosen so that each element obeys the requirements set forth by Proposition 3.1 to define the corresponding version of $\hat{\Phi}^{(\cdot)}$ on $\hat{\mathcal{Z}}_{\mathrm{ech},M}$. According to Proposition 3.1, each such version of $\hat{\Phi}^{(\cdot)}$ maps to nondegenerate and holonomy nondegenerate solutions to the appropriate version of (1-13). It is also the case that each element can be assumed to obey the conditions set forth in Proposition 7.6 for all pairs $(\Theta, \Theta') \in \mathcal{Z}^{L'}_{\mathrm{ech},M}$. Proposition 7.6 supplies for each such pair a corresponding map $\Psi^{(\cdot)}$ and of particular import is that the image of the latter consists of nondegenerate solutions to (4-1). Granted these last remarks, the final assertion of Theorem 1.5 follows from what is said in Parts 3–5 of Section 3h in [19]. Part 2 of this subsection says more about these parts of [19].

Part 2 The appeal to Parts 3–5 of Section 3h in [19] uses only the fact that sufficiently large versions of $\hat{\Phi}^{(\cdot)}$ map to $\mathcal{G}_{M_\Lambda}$–orbits of nondegenerate solutions to the relevant version of (1-13). What follows elaborates on what nondegeneracy implies. Let $\mathfrak{c}$ denote a nondegenerate solution to a given (r, μ) version of (1-13). The nondegeneracy assumption is used in two related ways. The first uses $\mathfrak{c}$'s nondegeneracy with the implicit function theorem to build a smooth map from a neighborhood of r in $(\pi, \infty)\times\Omega$ into $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ with two salient properties: The map sends (r, μ) to $\mathfrak{c}$ and it maps any (r', μ') in its domain to a solution to the (r', μ') version of (1-13). In addition, if $\mathfrak{c}'$ is a solution to the (r', μ') version of (1-13) and if the $C^\infty(Y; S^1)$ orbit of $\mathfrak{c}'$ is sufficiently close to $\mathfrak{c}$, then the image of (r', μ') via the map lies on this orbit. This map is denoted by $\hat{\mathfrak{c}}_{\mathfrak{c}}$ in what follows.

The second use of the nondegeneracy assumption concerns instanton solutions to (7-40). To say more, suppose that (r, μ) and (r', μ') are very close in $(\pi, \infty) \times \Omega$, and that $\mathfrak{p} \in \mathcal{P}_\mu$ and $\mathfrak{p}' \in \mathcal{P}_{\mu'}$ are likewise very near each other in $\mathcal{P}$. Consider (7-40) when the data set $(\mathrm{r}_{(\cdot)}, \mu_{(\cdot)}, \mathfrak{p}_{(\cdot)})$ has $s \to -\infty$ limit given by $(\mathrm{r}, \mu, \mathfrak{p})$ and $s \to \infty$ limit given by $(\mathrm{r}', \mu', \mathfrak{p}')$, and when it is such that $(\mathrm{r}_{(\cdot)}, \mu_{(\cdot)}, \mathfrak{p}_{(\cdot)})$ is nearly constant as s varies in $\mathbb{R}$. Because $\mathfrak{c}$ is nondegenerate, standard perturbative techniques will prove the following: There is a unique instanton solution to (7-40) with Fredholm index equal to 0 whose $s \to -\infty$ limit is $\mathfrak{c}$. This instanton is very close to $\mathfrak{c}$ at each $s \in \mathbb{R}$ and its $s \to \infty$ limit is a solution to the (r', μ') version of (1-13) that is very close to $\mathfrak{c}$. In particular, this limit is the translate of $\hat{\mathfrak{c}}_{\mathfrak{c}}(\mathrm{r}', \mu')$ by a map from Y to S^1 that is very close to the constant map to $1 \in S^1$. Let $\mathfrak{c}'$ denote this limit.

Looking ahead to Parts 3–5, the fact that the map from Y to S^1 is almost the constant map has the following implications: The values on $\mathfrak{c}'$ of the functions $\mathfrak{cs}$ and W in (1-26) and (1-27) are identical to their values on (r', μ'). Likewise, the value on $\mathfrak{c}'$ of the (r', μ') version of (1-28)'s function $\mathfrak{a}$ is the same as its value on (r', μ'). The r' version of the function M automatically has the same values on $\mathfrak{c}'$ and $\hat{\mathfrak{c}}_{\mathfrak{c}}(\mathrm{r}', \mu')$.

Part 3 The proof of the fifth bullet of Theorem 1.5 is given here and in Part 4. To start the proof, note that the residual set that is described by the second bullet in (1-18) can be chosen so as to have the properties listed below:

There is a countable, nonaccumulating bad set in (π, ∞) such that if r avoids it, then:

(7-50) • The corresponding $(\mathrm{r}, \mathfrak{g} = \mathfrak{e}_\mu)$ version of $\mathcal{Z}_{\mathrm{SW,r}}$ is a finite set of $C^\infty(Y; S^1)$ orbits in $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$.

• Each solution to (1-13) is nondegenerate and holonomy nondegenerate.

• If $\mathfrak{c}$ and $\mathfrak{c}'$ are solutions to (1-13) in distinct $C^\infty(Y; S^1)$ orbits, then $\mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}) \neq \mathfrak{a}^{\mathfrak{f}}(\mathfrak{c}')$.

The arguments in Sections 7.2 and 7.3 of [17] can be used almost verbatim to prove this.

Fix an element $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 that is described by (7-47). The latter describes a certain countable, nonaccumulating subset of (π, ∞). Denote this set by $\mathcal{U}$. If $\mathrm{r} > \pi$ and is not in $\mathcal{U}$, then the solutions to the (r, μ) version of (1-13) are suitable for defining the $\mathbb{Z}$–module $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$. Fix $\mathrm{r} > \pi$ in the complement of $\mathcal{U}$ and choose a suitably generic element $\mathfrak{p} \in \mathcal{P}_\mu$ to define the differential ∂^*_{SW} on $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$. Let I denote a given class in either $\mathbb{H}^\infty$, $\mathbb{H}^-$ or $\mathbb{H}^+$. The class I is then represented by a ∂^*_{SW} cycle in $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$. This is to say that any given representative of I can be written as

(7-51)
$$\mathfrak{z} = \sum_{[\mathfrak{c}] \in \hat{\mathcal{Z}}_{\mathrm{SW,r}}} z_{[\mathfrak{c}]} [\mathfrak{c}],$$

where each $z_{[\mathfrak{c}]} \in \mathbb{Z}$ and where only finitely many of these integers are nonzero. Associate to such a representative cocycle the number

(7-52)
$$\mathfrak{a}^{\mathfrak{f}}[\mathfrak{z}, \mathrm{r}] = \inf_{[\mathfrak{c}] \in \hat{\mathcal{Z}}_{\mathrm{SW,r}},\, z_{\mathfrak{c}} \neq 0} \{\mathfrak{a}^{\mathfrak{f}}[\mathfrak{c}]\},$$

and associate to the class I the number

(7-53)
$$\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}[\mathrm{r}] = \sup\{\mathfrak{a}^{\mathfrak{f}}[\mathfrak{z}, \mathrm{r}] \mid \mathfrak{z} \in \mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}}) \text{ represents } \mathrm{I}\}.$$

There are but a finite number of $C^\infty(Y; S^1)$–equivalence classes of solutions to (1-13) and so there is at least one cycle in $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ that represents I with $\mathfrak{a}^{\mathfrak{f}}[\cdot, \mathrm{r}]$ equal to $\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}[\mathrm{r}]$. Proposition 2.7 finds $\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}[\mathrm{r}] < c_0 \mathrm{r}$ and Proposition 2.7 with Lemma 2.5 find $\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}[\mathrm{r}] > -c_0 \mathrm{r} \ln \mathrm{r}$.

Propositions 4.7 and 4.8 in [19] have the following analog:

Lemma 7.12 *Choose $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 and described by* (7-47). *Denote by $\mathcal{U} \subset (\pi, \infty)$ the bad set. Given $\mathrm{r} \in (\pi, \infty) - \mathcal{U}$, use the solutions to the (r, μ) version of* (1-13) *to define $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$. There is a smooth map, $\mathrm{r} \mapsto \mathfrak{p}_{\mathrm{r}}$, from*

$(\pi,\infty)-\mathcal{U}$ *to* $\mathcal{P}$ *such that*:

- *For each* $r\in(\pi,\infty)-\mathcal{U}$, *the element* $\mathfrak{p}_r$ *vanishes to second order on all solutions to the* (r,μ) *version of* (1-13).
- *The pair* $(r,\mathfrak{g}=\mathfrak{e}_\mu+\mathfrak{p}_r)$ *is suitable for defining* ∂^*_{SW} *on* $\mathbb{Z}(\hat{\mathcal{Z}}_{SW,r})$ *if* $r\in(\pi,\infty)-\mathcal{U}$ *is chosen from the complement of a discrete set*, $\mathcal{V}$, *that accumulates only on the points in* $\mathcal{U}$.
- *Proposition 1.4's canonical isomorphism between the various* $r\in(\pi,\infty)-(\mathcal{U}\cup\mathcal{V})$ *versions of the* ∂^*_{SW} *homology groups* $H^\infty_{SW,r}$, $H^-_{SW,r}$ *and* $H^+_{SW,r}$ *is such that following is true*: *if* I *is any given nonzero homology class in* H^∞_{SW}, H^-_{SW} *or* H^+_{SW}, *then the assignment* $r\mapsto\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}[r]$ *as defined above for* $r\in(\pi,\infty)-(\mathcal{U}\cup\mathcal{V})$ *is the restriction of a continuous and piecewise differentiable function on* (π,∞).

Proof But for notation and interchanging min with max, the proof mimics the arguments for Proposition 2.5 in [18] and for Proposition 4.2 in [17]. Note in this regard that the arguments in these papers use a homomorphism between the $(r,\mu,\mathfrak{p}_r)$ and $(r',\mu,\mathfrak{p}_{r'})$ versions of the ∂^*_{SW} homology that is not obviously the canonical isomorphism. Even so, what is said in Part 2 of this subsection with arguments much like those in Section 7.3 can be used with the arguments in [18; 17] to obtain a proof of Lemma 7.12's assertion about the canonical isomorphism. See also Proposition 10.7 in [4] and its proof. □

Let I denote a given class in either H^∞_{SW}, H^-_{SW} or H^+_{SW}. The function $\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}$ is important only to the extent that it can be used to analyze a second function of r, this denoted by $\mathrm{M}_{\mathrm{I}}(\cdot)$. To define the latter, fix $\mathfrak{p}_{(\cdot)}$ as in Lemma 7.12 so as to define ∂^*_{SW} on $\mathbb{Z}(\hat{\mathcal{Z}}_{SW,r})$ when $r\in(\pi,\infty)-(\mathcal{U}\cup\mathcal{V})$. Given such a value for r, let $\mathfrak{z}$ denote a ∂^*_{SW} cycle that represents I. Write $\mathfrak{z}$ as in (7-47) and define $\mathrm{M}[\mathfrak{z},r]=\sup_{[\mathfrak{c}]\in\hat{\mathcal{Z}}_{SW,r},\, z_{\mathfrak{c}}\neq 0}\mathrm{M}(\mathfrak{c})$. Define

(7-54) $$\mathrm{M}_{\mathrm{I}}[r]=\inf\{\mathrm{M}[z,r]\mid \mathfrak{z}\in\mathbb{Z}(\hat{\mathcal{Z}}_{SW,r})\text{ represents }\mathrm{I}\text{ and }\mathfrak{a}^{\mathfrak{f}}[z,r]=\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}\}.$$

It follows from what is said in Part 2 of this section that M_{I} is a priori a smooth function on $(\pi,\infty)-(\mathcal{U}\cup\mathcal{V})$.

Part 4 Theorem 1.5's fifth bullet follows from Proposition 3.1 if the following assertion is true:

(7-55) Fix a class I in H^∞_{SW}, H^-_{SW} or H^+_{SW}. Then the corresponding function $\mathrm{M}_{\mathrm{I}}(\cdot)$ is bounded.

Of course, (7-55) makes sense only when μ is described by (7-47); but Theorem 1.5 follows in any event using the fact that the set described in (7-47) is dense in Ω and what is said in Part 2 of this subsection.

To see about (7-55), fix an interval component of $(\pi,\infty)-(\mathcal{U}\cup\mathcal{V})$ and differentiate the expression in (1-28) on this interval to see that

$$\frac{d}{d\mathrm{r}}\Big(-\frac{2}{\mathrm{r}}\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}\Big)=\frac{1}{\mathrm{r}^2}\mathfrak{cs}^{\mathfrak{f}}(\mathfrak{c}), \tag{7-56}$$

with $\mathfrak{c}$ a particular solution to (1-13) whose equivalence class has nonzero coefficient in some representative cycle for I with $\mathfrak{a}^{\mathfrak{f}}[\cdot,\mathrm{r}]=\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}$. Use Lemma 2.5 and Proposition 2.7 to see that the right-hand side of (7-56) is no greater than $c_0\mathrm{r}^{-4/3}(\ln\mathrm{r})^{4/3}$. This being the case, integrate (7-56) on the components of $(\pi,\infty)-(\mathcal{U}\cup\mathcal{V})$ and use the fact that $\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}[\cdot]$ is continuous to see that $-\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}[\mathrm{r}]\le c_{\mathrm{I}}\mathrm{r}+\mathrm{r}^{1/3}(\ln\mathrm{r})^{4/3}$ with c_{I} being a constant that depends on I but is independent of r. This last bound plus the bound implied by Lemma 2.5 and Proposition 2.7 for $|\mathfrak{cs}^{\mathfrak{f}}|$ requires $\mathrm{W}^{\mathfrak{f}}[\mathfrak{c}]\le c_{\mathrm{I}}+\mathrm{r}^{-2/3}(\ln\mathrm{r})^{4/3}$ with $\mathfrak{c}$ being a particular solution to (1-13) whose equivalence class has nonzero coefficient in some representative cycle for I with $\mathfrak{a}^{\mathfrak{f}}[\cdot,\mathrm{r}]=\mathfrak{a}^{\mathfrak{f}}_{\mathrm{I}}$. Thus, $\mathrm{W}^{\mathfrak{f}}[\mathfrak{c}]$ is bounded by an r–independent constant determined by the class I. This understood, (7-55) follows directly from the second bullet of Proposition 2.7.

Part 5 This part proves the fourth bullet of Theorem 1.5 in four steps.

Step 1 Fix μ from the set described by the second bullet of (1-18) and use $\mathcal{U}$ to denote the corresponding countable, nonaccumulating set in (π,∞). Suppose that $\mathrm{r}\in(\pi,\infty)-\mathcal{U}$ is sufficiently large. In particular, require that r with μ and a suitably generic, small-normed element $\mathfrak{p}$ from $\mathcal{P}$ can be used to define $\mathbb{L}^{\mathrm{r}}$ on $\mathcal{Z}^{L}_{\mathrm{ech},M}$ so as to satisfy the first three bullets of Theorem 1.5 and the fifth bullet. Require in addition that the final assertion of Theorem 1.5 hold for $(\mathrm{r},\mu,\mathfrak{p})$ and data sets $(\mathrm{r}',\mu,\mathfrak{p}')$ with $\mathrm{r}'\ge\mathrm{r}$.

Step 2 Introduce $\mathbb{Q}^{L}_{\mathrm{ech}}$ to denote either

$$\mathbb{Z}(\hat{\mathcal{Z}}^{L}_{\mathrm{ech},M}),\quad \mathbb{Z}(\hat{\mathcal{Z}}^{L,<}_{\mathrm{ech},M})\quad\text{or}\quad \mathbb{Z}(\hat{\mathcal{Z}}^{L}_{\mathrm{ech},M})/\mathbb{Z}(\hat{\mathcal{Z}}^{L,<}_{\mathrm{ech},M})$$

and let $\mathbb{Q}_{\mathrm{SW}}$ denote the corresponding $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$, $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$ or $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})/\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$. Let ς denote an element in $\mathbb{Q}^{L}_{\mathrm{ech}}$ such that $\mathbb{L}^{\mathrm{r}}\varsigma=\partial^{*}_{\mathrm{SW}}\mathfrak{z}$ with $\mathfrak{z}$ being an element in $\mathbb{Q}_{\mathrm{SW}}$. If the fourth bullet of Theorem 1.5 holds for ς, then it holds for $\varsigma+\partial_{\mathrm{ech}}\varsigma'$ for any $\varsigma'\in\mathbb{Q}^{L}_{\mathrm{ech}}$, this being a consequence of the second and third bullets of Theorem 1.5 and the fact that $\mathbb{L}^{\mathrm{r}}$ is a monomorphism.

Since $(\partial_{\mathrm{SW}}^*)^2 = 0$, the second and third bullets of Theorem 1.5 require $\partial_{\mathrm{ech},M}\varsigma = 0$. Granted this, then ς defines a class in the homology of the chain complex $(\mathbb{Q}_{\mathrm{ech}}^L, \partial_{\mathrm{ech}})$. Use I_ς to denote this class. What is said in the preceding paragraph implies that the question of whether the fourth bullet of Theorem 1.5 holds for a given element ς depends only on the class I_ς.

Step 3 The map $\mathbb{L}^{\mathrm{r}}$ induces a homomorphism from the $(\mathbb{Q}_{\mathrm{ech}}^L, \partial_{\mathrm{ech}})$ homology to the $(\mathbb{Q}_{\mathrm{SW}}, \partial_{\mathrm{SW}}^*)$ homology. The conclusions of Step 2 mean that the fourth bullet of Theorem 1.5 is asking about the kernel of this map. Let $\mathbb{K}_L$ denote this kernel. As explained next, the $\mathbb{Z}$–module $\mathbb{K}_L$ does not depend on the value of r if sufficiently large.

To start the explanation, suppose that $\mathrm{I}_\varsigma \in \mathbb{K}_L$. This is to say that $\mathbb{L}^{\mathrm{r}}\varsigma = \partial_{\mathrm{SW}}^*\mathfrak{z}$ with $\mathfrak{z} \in \mathbb{Q}_{\mathrm{SW}}$. As $\partial_{\mathrm{ech}}\varsigma = 0$, the chain $\mathbb{L}^{\mathrm{r}'}\varsigma$ is annihilated by the $(\mathrm{r}', \mu, \mathfrak{p}')$ version of ∂_{SW}^* when $\mathrm{r}' \geq \mathrm{r}$ is disjoint from $\mathcal{U}$. It follows as a consequence that $\mathbb{L}^{\mathrm{r}'}\varsigma$ defines a class in either $H_{\mathrm{SW},\mathrm{r}'}^{\infty}$, $H_{\mathrm{SW},\mathrm{r}'}^{-}$ or $H_{\mathrm{SW},\mathrm{r}'}^{+}$ as the case may be. It therefore defines a class in H_{SW}^{∞}, H_{SW}^{-} or H_{SW}^{+}. Given what is said in the last assertion of Theorem 1.5, this class in $H_{\mathrm{SW},\mathrm{r}'}^{\infty}$, $H_{\mathrm{SW},\mathrm{r}'}^{-}$ or $H_{\mathrm{SW},\mathrm{r}'}^{+}$ corresponds to the class in $H_{\mathrm{SW},\mathrm{r}}^{\infty}$, $H_{\mathrm{SW},\mathrm{r}}^{-}$ or $H_{\mathrm{SW},\mathrm{r}}^{+}$ that is defined by $\mathbb{L}^{\mathrm{r}}\varsigma$.

The homology of $(\mathbb{Q}_{\mathrm{ech}}^L, \partial_{\mathrm{ech}})$ is finitely generated, and so $\mathbb{K}_L$ is finitely generated. This understood, the assertion made by the fourth bullet of Theorem 1.5 holds for all ς with $\mathbb{L}^{\mathrm{r}}\varsigma \subset \mathrm{Image}(\partial_{\mathrm{SW}}^*)$ if it holds for a judiciously chosen finite set of such elements. Therefore, the fourth bullet of Theorem 1.5 holds if the following assertion is true:

(7-57) Fix $\varsigma \in \mathbb{Q}_{\mathrm{ech}}^L$ with $\mathbb{L}^{\mathrm{r}}(\varsigma) = \partial_{\mathrm{SW}}^*\mathfrak{z}$ for some $\mathfrak{z} \in \mathbb{Q}_{\mathrm{SW}}$. There exists $L' > L$ such that $\varsigma = \partial_{\mathrm{ech}}\varsigma'$ for some $\varsigma' \in \mathbb{Q}_{\mathrm{ech}}^{L'}$.

The proof of this assertion is given in Step 4.

Step 4 Arguments much like those that prove Lemma 7.12 prove the following:

Lemma 7.13 *Choose $\mu \in \Omega$ with $\mathcal{P}$–norm less than 1 and described by (7-47). Denote by $\mathcal{U} \subset (\pi, \infty)$ the associated accumulation set. Given $\mathrm{r} \in (\pi, \infty) - \mathcal{U}$, use the solutions to the (r, μ) version of (1-13) to define the $\mathbb{Z}(\widehat{\mathcal{Z}}_{\mathrm{SW},\mathrm{r}})$. There is a smooth map, $\mathrm{r} \mapsto \mathfrak{p}_{\mathrm{r}}$, from $(\pi, \infty) - \mathcal{U}$ to $\mathcal{P}$ such that:*

- *For each $\mathrm{r} \in (\pi, \infty) - \mathcal{U}$, the element $\mathfrak{p}_{\mathrm{r}}$ vanishes to second order on all solutions to the (r, μ) version of (1-13).*
- *The data $(\mathrm{r}, \mathfrak{g} = \mathfrak{e}_\mu + \mathfrak{p}_{\mathrm{r}})$ is suitable for defining ∂_{SW}^* on $\mathbb{Z}(\widehat{\mathcal{Z}}_{\mathrm{SW},\mathrm{r}})$ if $\mathrm{r} \in (\pi, \infty) - \mathcal{U}$ is chosen from the complement of a discrete set, $\mathcal{V}$, with accumulation only at points in $\mathcal{U}$.*

- *The assignment* $\mathrm{r} \mapsto \mathfrak{a}^{\mathfrak{f}}_{\varsigma}[\mathrm{r}]$ *as defined above for* $\mathrm{r} \in (\pi, \infty) - (\mathcal{U} \cup \mathcal{V})$ *is the restriction of a continuous and piecewise differentiable function on* (π, ∞).

Fix $\mathfrak{p}_{(\cdot)}$ as in Lemma 7.13 so as to define ∂^*_{SW} when $\mathrm{r} \in (\pi, \infty) - (\mathcal{U} \cup \mathcal{V})$. Given such a value for r, let $\mathfrak{z}$ denote an element which is in either $\mathbb{Z}(\hat{\mathcal{Z}}_{\mathrm{SW,r}})$ or $\mathbb{Z}(\hat{\mathcal{Z}}^{<}_{\mathrm{SW,r}})$, with $\mathfrak{z}$ in the latter when $\mathbb{Q}^{L}_{\mathrm{ech}}$ is $\mathbb{Z}(\hat{\mathcal{Z}}^{L,<}_{\mathrm{ech},M})$. Assume that $\mathfrak{z}$ obeys $\partial^*_{\mathrm{SW}}\mathfrak{z} = \mathbb{L}^{\mathrm{r}}\varsigma$ in $\mathbb{Q}_{\mathrm{SW}}$. Write $\mathfrak{z}$ as in (7-51) and set $\mathrm{M}[\mathfrak{z}, \mathrm{r}] = \sup_{[\mathfrak{c}] \in \hat{\mathcal{Z}}_{\mathrm{SW,r}},\, \mathrm{z}_{\mathfrak{c}} \neq 0} \mathrm{M}(\mathfrak{c})$. Now define

$$\mathrm{M}_{\varsigma}[\mathrm{r}] = \inf\{\mathrm{M}[\mathfrak{z}, \mathrm{r}] \mid \mathfrak{z} \text{ is such that } \partial^*_{\mathrm{SW}}\mathfrak{z} = \mathbb{L}^{\mathrm{r}}\varsigma \text{ in } \mathbb{Q}_{\mathrm{SW}} \text{ and } \mathfrak{a}^{\mathfrak{f}}[\mathfrak{z}, \mathrm{r}] = \mathfrak{a}^{\mathfrak{f}}_{\varsigma}\}. \tag{7-58}$$

An almost verbatim copy of the argument in Part 4 of this subsection proves that $\mathrm{M}_{\varsigma}[\mathrm{r}]$ is bounded. This being the case, (7-57) follows from what is said in Proposition 3.1 about the map $\hat{\Phi}^{\mathrm{r}}$. □

Appendix The proof of Proposition 2.6

This appendix supplies a proof for Proposition 2.6. Much of what is done here mirrors similar constructions in Section 3 of [20] and Section 2 of [21]. Even so, the reworking can be justified for two reasons. First, the spectral flow function is not invariant under the action on $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ of the whole of the group $C^\infty(Y; \mathbb{S})$, and so care must be taken so as to not introduce a spurious gauge transformation in any given step of the proof. Care must also be taken so as not to introduce spurious factors of $\ln \mathrm{r}$ in any given step. Such factors are easy to come by because there are solutions to (1-13) with (1-30)'s function M being greater than $c_0^{-1} \ln \mathrm{r}$. The need to avoid spurious gauge transformations and spurious factors of $\ln \mathrm{r}$ accounts for the much of the length of the proof.

This appendix has three sections.

A The eigenvalue equation $\mathfrak{L}_{\mathfrak{c},\mathrm{r}}\mathfrak{b} = \lambda\,\mathfrak{b}$

This section of the appendix supplies some necessary background for the proof of Proposition 2.6. Much of what is done here borrows heavily from Sections 3a–3c of [20] and Section 2a in [21].

Aa Pairs in $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ and solutions to the vortex equation

This subsection uses solutions to (2-8) to construct pairs of connection on E and section of $\mathbb{S}$ over the complement in Y of tubular neighborhoods of a chosen subset of curves from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^{+}\cup\hat{\gamma}_{\mathfrak{p}}^{-})$. These constructions mimic those in Section 3 of [20]. There are six parts to what follows.

Part 1 The input from the vortex equations includes first a pair (A_0,α_0) that obeys (2-8) and is such that $\frac{1}{2\pi}(1-|\alpha_0|^2)$ is integrable. As noted in (3-1), the integral of this function is a nonnegative integer and of interest here is the case when the integer in question is 1. This is to say that (A_0,α_0) define a point in the vortex moduli space $\mathfrak{C}_1$. Use the pair (A_0,α_0) to construct the square-integrable solution ς to (3-27). Meanwhile, let y denote the square-integrable, real-valued function on $\mathbb{C}$ that solves the equation

$$-\partial\bar{\partial}y+\tfrac{1}{2}|\alpha_0|^2y=-2^{-1/2}(1-|\alpha_0|^2). \tag{A-1}$$

The pair (y,ς) can be written explicitly in terms of α_0 and its covariant derivative in the given case when (A_0,α_0) determine an element in $\mathfrak{C}_1$. For example, if $\alpha_0^{-1}(0)=0$, then

$$\varsigma=-\bar{z}\bar{\alpha}_0^{-1}(1-|\alpha_0|^2)\quad\text{and}\quad y=2^{1/2}z\alpha_0^{-1}\partial_{A_0}\alpha_0. \tag{A-2}$$

In general, if $m\geq 1$ and if (A_0,α_0) defines a point in $\mathfrak{C}_m$, then there is a unique square-integrable solution to (A-1) and a unique, square-integrable solution to (3-27). This pair (y,ς) obeys $|y|+|\varsigma|\leq c_m e^{-\operatorname{dist}(\cdot,\alpha_0^{-1}(0))/c_0}$.

Suppose that (A_0,α_0) defines an element in $\mathfrak{C}_1$ and is such that $\alpha_0^{-1}(0)$ is the origin in $\mathbb{C}$. Any two such solutions differ by the action of $C^\infty(\mathbb{C};S^1)$ as they correspond to a single point in $\mathfrak{C}_1$; this is the point with σ_1 in (3-2) equal to zero. This point in $\mathfrak{C}_1$ is called the symmetric vortex as it is the unique fixed point in $\mathfrak{C}_1$ of the action by S^1 that is induced by the latter's action on $\mathbb{C}$ as the group of rotations about the origin. There is a unique solution to (2-8) that maps to the symmetric vortex in $\mathfrak{C}_1$ and is such that

$$A_0=\theta_0-a_0\cdot\tfrac{1}{2}(z^{-1}\,dz-\bar{z}^{-1}\,d\bar{z})\quad\text{and}\quad\alpha_0=|\alpha_0|\frac{z}{|z|}, \tag{A-3}$$

where the notation has θ_0 denoting the product flat connection on the product line bundle over $\mathbb{C}$ and a_0 denoting a real-valued function on $\mathbb{C}$. Lemma 3.3 can be used to prove that a_0 and $|\alpha_0|$ obey

$$|1-a_0|\leq c_0(1-|\alpha_0|^2)\quad\text{and}\quad 1-|\alpha_0|\leq c_0e^{-|z|/c_0}. \tag{A-4}$$

Note that if $m > 1$, then the point in $\mathfrak{C}_m$ with (3-2)'s coordinate functions all zero also corresponds to solutions (A_0, α_0) with $\alpha_0^{-1}(0) = 0$. There is in this case a unique solution with $\alpha_0^{-1}(0) = 0$ that has $\alpha_0 = |\alpha_0|(z/|z|)^m$.

Part 2 This part of the subsection defines various terms and notions that are employed in the subsequent parts.

The constructions that are described below require the a priori choice of constants $c_v \geq 10^6$, $z \geq c_v^6$ and $\rho_* \geq c_v^2 \mathrm{z}^{-1/2}$. The lower bound for c_v is increased to some $c_0 \geq 10^6$ in the applications to follow. The constants c_v and ρ_* are also constrained so that $c_v^2 \rho_* \leq c_0^{-1}$ and, in particular, $c_v^2 \rho_*$ is smaller than $\frac{1}{100}$ times the maximum allowed radius of any transverse disk.

Proposition 3.1's map Φ^{r} uses the pairs constructed below with $\mathrm{z} = \mathrm{r}$, with c_v on the order of 1, with ρ_* constrained to be greater than $\mathrm{r}^{-1/2+\delta}$ for some fixed $\delta > 0$. The proof of Proposition 2.6 uses versions of the constants with $\mathrm{z} \in (c_0, \mathrm{r})$, with $c_v = c_0$, and with ρ_* no larger than $c_0 \mathrm{z}^{-1/2}$ (but ρ_* here is still greater than $c_v^2 \mathrm{z}^{-1/2}$, as required in the preceding paragraph). Thus, different versions of ρ_* are used by these propositions:

Let $Y_{*\Lambda}$ denote the subset of Y with distance at least $c_v^2 \rho_*$ from $\bigcup_{\mathrm{p} \in \Lambda} (\hat{\gamma}_{\mathrm{p}}^+ \cup \hat{\gamma}_{\mathrm{p}}^-)$. Given that $c_v^2 \rho_* \leq c_0^{-1}$, this subset $Y_{*\Lambda}$ is a smooth manifold with boundary whose boundary components are tori and whose complement is a disjoint union of solid tori tubular neighborhoods of the curves from the set $\{\hat{\gamma}_{\mathrm{p}}^+ \cup \hat{\gamma}_{\mathrm{p}}^-\}_{\mathrm{p} \in \Lambda}$.

Part 3 A union of components of $Y - Y_{*\Lambda}$ must be specified in advance before starting the construction. This can be the empty set. The chosen subset of $Y - Y_{*\Lambda}$ is denoted in what follows by $T_{*\Lambda}$ The constructions that follow define a connection on E's restriction to $Y_{*\Lambda} \cup T_{*\Lambda}$ and a section of $\mathbb{S}$ over $Y_{*\Lambda} \cup T_{*\Lambda}$. Proposition 3.1's map Φ^{r} uses only the case when $T_{*\Lambda} = Y - Y_{*\Lambda}$. The proof of Proposition 2.6 uses all possible versions of $T_{*\Lambda}$.

The construction of a connection and section of $\mathbb{S}$ over $Y_{*\Lambda} \cup T_{*\Lambda}$ requires the choice of a finite set Θ whose elements are described below:

(A-5)
- Each element in Θ is either a curve from $\{\hat{\gamma}_{\mathrm{p}}^+ \cup \hat{\gamma}_{\mathrm{p}}^-\}_{\mathrm{p} \in \Lambda}$ that lies in $T_{*\Lambda}$ or a properly embedded, 1–dimensional, oriented submanifold in $Y_{*\Lambda}$. The curves from Θ are distinct.
- The $T_{*\Lambda}$ boundary components of $Y - Y_{*\Lambda}$ are disjoint from the $Y_{*\Lambda}$ curves from Θ. Any other boundary component of $Y_{*\Lambda}$ contains no more

than two endpoints of arcs from Θ, and if two, then one has $u < 0$ while the other has $u > 0$.

- Suppose that γ is a curve from Θ in $Y_{*\Lambda}$.
  (a) The unit-length, oriented tangent vector to γ has distance at most $c_v \mathrm{z}^{-1/2}$ from v.
  (b) The curve γ intersects any $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ where $1 - 3\cos^2\theta > 0$.
  (c) If γ is disjoint from a given boundary torus of $Y_{*\Lambda}$, then it has distance greater than $3c_v\rho_*$ from this torus.
  (d) If γ intersects a boundary torus of $Y_{*\Lambda}$, then it does so only at its endpoints. These intersections are transversal. Moreover, one endpoint of γ lies where $u < 0$ on some boundary component of $Y_{*\Lambda}$ and the other where $u > 0$ on some boundary component of $Y_{*\Lambda}$.
- The intersection of $\bigcup_{\gamma\in\Theta}\gamma$ with M_δ sits in the $f^{-1}(1,2)$ part of M_δ. This intersection consists of G properly embedded segments that pair the index 1 and index 2 critical points of f in the sense that distinct segments start on the boundary of the radius δ coordinate balls about distinct index 1 critical points of f and end on the boundary of the radius δ coordinate balls about distinct index 2 critical points.

The proof of Proposition 2.6 uses only versions of Θ that lack curves from the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$. Versions with curves from this set are needed to define Proposition 3.1's map Φ_r.

Let γ denote a 1–manifold in $Y_{*\Lambda}$ from an element in Θ. Introduce U_γ to denote the union of the radius $4\rho_*$ transverse disks centered at the points in γ. Use $U'_\gamma \subset U_\gamma$ to denote the union of the radius ρ_* transverse disks centered at the points in γ. If γ is a $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$ curve from Θ, use U_γ to denote the union of the radius $4\rho_{*\Lambda}$ transverse disks centered on γ and use $U'_\gamma \subset U_\gamma$ to denote the union of the concentric radius $\rho_{*\Lambda}$ transverse disks. Keep in mind the following consequence of the formula in Section 1.1 for v: If $c_v \geq c_0$, and if $\gamma \subset Y_{*\Lambda}$ is from Θ, then U_γ is an open solid torus with γ the core circle. Moreover, if γ and γ' are in $Y_{*\Lambda}$ and come from distinct elements in Θ, then $U_\gamma \cap U_{\gamma'} = \varnothing$. It is assumed in what follows that c_v is such as to guarantee this.

Part 4 This part of the subsection describes a certain set of preferred coordinates for the various versions of U_γ. Each element in this set is determined in a canonical fashion by an isometric isomorphism from $K^{-1}|_\gamma$ to $\gamma \times \mathbb{C}$.

To define these coordinates, introduce T_γ to denote the union of the transverse disks of radius c_0^{-1} centered at the points of γ. Choose this radius so that the union of the transverse disks with centers on any length less than c_0^{-1} segment of γ is a solid, embedded cylinder with the segment as the core arc. The desired coordinates for U_γ obtained by restricting the domain of a set of functions on T_γ that define local coordinates on each such solid cylinder.

Let ℓ_γ denote the length of γ. The first of these functions is a parameter, denoted by t, with values in $\mathbb{R}/(\ell_\gamma\mathbb{Z})$ when γ is a closed loop and with values in an interval of length ℓ_γ otherwise. The coordinate t is constant along each transverse disk in T_γ with center on γ. The other coordinate is denoted by z; it is a $\mathbb{C}$–valued function that identifies each transverse disk with the radius c_0^{-1} disk in $\mathbb{C}$ centered at the origin. The coordinate identification is such that the origin in $\mathbb{C}$ corresponds to γ.

Definition Fix a $\mathbb{C}$–linear isomorphism between $K^{-1}|_\gamma$ and $\gamma\times\mathbb{C}$. This defines an orthonormal, oriented frame for the kernel of $\hat{a}$ along γ. Use this isomorphism with the metric's exponential map to identify a tubular neighborhood of γ with $\gamma\times\mathbb{C}$. Use t to denote an affine coordinate long γ with the property that the corresponding tangent vector field has unit length and positive pairing with $\hat{a}$. The coordinate t and the standard complex coordinate z for $\mathbb{C}$ are the desired coordinate functions.

These coordinates are such that the $z=0$ locus is γ and ∂_z along γ is in the kernel of $\hat{a}$ and has norm $2^{-1/2}$. The first-order Taylor's expansion writes v and w as

$$\text{(A-6)}\qquad \begin{aligned} v &= \frac{\partial}{\partial t} + 2i(\nu z + \mu\bar{z} - x_\gamma)\frac{\partial}{\partial z} - 2i(\nu\bar{z} + \bar{\mu} z - \bar{x}_\gamma)\frac{\partial}{\partial \bar{z}} + \cdots, \\ w &= \tfrac{i}{2}\, dz\wedge d\bar{z} - (\nu z + \mu\bar{z} - x_\gamma)\, d\bar{z} - (\nu\bar{z} + \bar{\mu} z - \bar{x}_\gamma)\, dz\wedge dt + \cdots, \end{aligned}$$

where ν is a real-valued function of t while μ and x_γ are $\mathbb{C}$–valued functions of t with x_γ such that $|x_\gamma| \le c_v \mathrm{z}^{-1/2}$. The unwritten terms are bounded by

$$c_0(|\nu| + |\mu|)|z|(\mathrm{z}^{-1/2} + |z|)$$

with c_0 here dependent on ν and μ. Note that ν must be real so as to have $dw = 0$. What is said in item (a) of the third bullet of (A-5) leads to the asserted bound on $|x_\gamma|$.

Changing the isomorphism between $K^{-1}|_\gamma$ and $\gamma\times\mathbb{C}$ writes z as $z = u(t)z'$ with u being a smooth map from the domain of t to S^1. The resulting version of (A-6) replaces ν with $\nu' = \nu + \frac{i}{2}u^{-1}\frac{d}{dt}u$, it replaces μ with $\mu' = u^{-2}\mu$ and it has x_γ replaced by $x'_\gamma = u^{-1}x_\gamma$.

As is explained in a moment, this last observation has the following important consequence: coordinates of the sort just described can be found with the property that the functions ν and μ in (A-6) obey $|\nu|+|\mu|\le c_0$. To see why this is, fix a point p in the interior of γ and a unitary frame for $K^{-1}|_p$. Parallel transport this frame along a small length interval in γ containing p. Use the latter frame with the exponential map to define the coordinates (t,z) for a solid cylinder with this interval as the core arc. Use T_p to denote this cylinder. The Lie derivative of w by $\partial/\partial\bar{z}$ is bounded by c_0 at p because the covariant derivative of $\partial/\partial\bar{z}$ is zero at p. Use the (t,z) version of (A-6) to see that this Lie derivative at p is $-\nu\, dz\wedge dt-\mu\, d\bar{z}\wedge dt$. This implies in particular that $|\mu|\le c_0$ at p. The fact that $|\mu|$ is independent of the chosen orthonormal frame for $K^{-1}|_\gamma$ implies that $|\mu|\le c_0$ along the whole of γ no matter what frame is used to define the coordinates. Meanwhile, the freedom to change ν to $\nu-\frac{i}{2}u^{-1}\frac{d}{dt}u$ can be exploited to obtain a version of the coordinates with the function ν such that $|\nu|$ is also bounded along γ. Indeed, if γ is not closed, then this equation can be integrated so as to obtain a version with $\nu=0$. If γ is closed, then a version can be found with ν constant and less than $c_0\ell_\gamma^{-1}$.

Unless told otherwise, assume in here and in Appendices B and C that any chosen coordinate system of the sort described above has $|\nu|+|\mu|<c_0$. A second convention with regards to these coordinates concerns the case when γ is an integral curve from the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^{+}\cup\hat{\gamma}_{\mathfrak{p}}^{-})$. As explained in Part 5 of Section 3.3 there is a version of these coordinates with both ν and μ constant, with μ real and such that $\mu>|\nu|$ These constant values for ν and μ are denoted at times by ν_0 and μ_0. This (ν_0,μ_0) version of the coordinates should be assumed in what follows when $\gamma\in\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^{+}\cup\hat{\gamma}_{\mathfrak{p}}^{-})$.

A coordinate system of the sort described above should be chosen for each set from the collection $\{U_\gamma\}_{\gamma\in\Theta}$. These chosen coordinate systems are used in what follows.

Part 5 Fix a set Θ as described in Part 3. The corresponding pair of connection and spinor is defined with the help of the open cover of $Y_{*\Lambda}\cup T_{*\Lambda}$ that consists of the collection $\{U_\gamma\cap Y_\gamma\}_{\gamma\in\Theta}$ and the set $U_0=(Y_{*\Lambda}\cup T_{*\Lambda})-\big(\bigcup_{\gamma\in\Theta}(U'_\gamma\cap Y_{*\Lambda})\big)$. Use $\mathfrak{U}$ to denote the collection of sets consisting of U_0 and $\{U_\gamma\}_{\gamma\in\Theta}$. For each $U\in\mathfrak{U}$, an isometric isomorphism must first be chosen to identify $E|_U$ with $U\times\mathbb{C}$. The bundle E can be reconstructed from these isomorphisms using the corresponding transition functions.

The pair (A,ψ) on any given set U from the cover is written using the isomorphism between $E|_U$ and $U\times\mathbb{C}$ as $A=\theta_0+a_U$, where θ_0 denotes the product connection on $U \times \mathbb{C}$ and where a_U is an $i\mathbb{R}$–valued 1–form on U. Meanwhile, ψ is written as (α_U, β_U), where α_U and β_U denote a respective $\mathbb{C}$–valued function and section of K^{-1} on U. With regards to the section β_U, the $U \neq U_0$ versions of $K^{-1}|_U$ come with an isomorphism $K^{-1}|_U = U \times \mathbb{C}$ that is defined using the chosen coordinates for U. This isomorphism is defined so that its inverse maps the constant section 1 of $U \times \mathbb{C}$ to the section that can written as $(1+\mathfrak{r})(\partial_z + \mathfrak{q})$ with $\mathfrak{r}$ being real and $\mathfrak{q}$ being orthogonal to ∂_z. Moreover, $|\mathfrak{r}| + |\mathfrak{q}| \leq c_0|z|$. Granted this isomorphism, the $U \neq U_0$ versions of β_U are also viewed as functions on U.

What follows specifies the various $U \in \mathfrak{U}$ versions of $(a_U, (\alpha_U, \beta_U))$ on the complement in U of $\bigcup_{U' \in \mathfrak{U} - \{U\}} U \cap U'$.

(A-7) • **When $U = U_0$** Fix an isomorphism $E|_{U_0} = U_0 \times \mathbb{C}$. Set $a_{U_0} = 0$, $\alpha_{U_0} = 1$ and $\beta_{U_0} = 0$.

The definition of $(a_U, (\alpha_U, \beta_U))$ for $U \in \mathfrak{U} - \{U_0\}$ requires first the introduction of the rescaling map from $\mathbb{C}$ to itself that multiplies the coordinates by $\mathrm{z}^{1/2}$. The latter map is denoted here by $\mathfrak{r}_{\mathrm{z}}$. The definition refers to the functions y and ς on $\mathbb{C}$ that are depicted in (A-2) and the function $\mathfrak{a}_0$ on $\mathbb{C}$ given in (A-3).

(A-8) • **When $U = U_\gamma$** Fix an isomorphism between $E|_U$ and $U \times \mathbb{C}$. Use Part 4's coordinates to identify U with the product of either S^1 or the appropriate interval with the radius ρ_* concentric disk in D_0. Use (A_0, α_0) to denote the symmetric solution to (2-8) from $\mathfrak{C}_1$ with $\alpha_0 = |\alpha_0| z/|z|$. Set $a_U = i2^{1/2} \nu \mathfrak{r}_{\mathrm{z}}^* y\, dt - \frac{1}{2} \mathfrak{r}_{\mathrm{z}}^* \mathfrak{a}_0 (z^{-1} dz - \bar{z}^{-1} d\bar{z})$, $\alpha_U = \mathfrak{r}_{\mathrm{z}}^* \alpha_0$ and $\beta_U = i\mu \mathrm{z}^{-1/2} \mathfrak{r}_{\mathrm{z}}^* \varsigma$.

Part 6 This part describes each $U \in \mathfrak{U}$ version of $(a_U, (\alpha_U, \beta_U))$ on the intersection between U and $\bigcup_{U' \in \mathfrak{U}} (U \cap U')$. The transition function between a given set $U \in \mathfrak{U} - \{U_0\}$ and U_0 are as follows:

(A-9) • Suppose that $\gamma \subset Y_{*\Lambda}$. The transition function for $U_0 \cap U_\gamma$ identifies the constant section 1 of the bundle $U_0 \times \mathbb{C}$ with the section $z/|z|$ of the bundle $U_\gamma \times \mathbb{C}$.

• Suppose that $\gamma \in \bigcup_{\mathrm{p} \in \Lambda} (\hat{\gamma}_{\mathrm{p}}^+ \cup \hat{\gamma}_{\mathrm{p}}^-)$. The transition function for $U_0 \cap U_\gamma$ on the part where $|z| \leq 2c_\upsilon \rho_*$ maps the section 1 of $U_0 \times \mathbb{C}$ to the section $z/|z|$ of $U_\gamma \times \mathbb{C}$.

What is said in (A-9) is all that is needed for Proposition 3.1's map Φ^{r} because the latter has no cases where three sets from the cover intersect.

Suppose that $U \in \mathfrak{U} - \{U_0\}$. Use $\hat{U}$ to denote the part of $U \cap U_0$ that is considered in (A-9). The definition of $(a_U, (\alpha_U, \beta_U))$ on $\hat{U}$ follows. The definition introduces $\chi_{\hat{U}}$ to denote $\chi(\rho_*^{-1}|z| - 1)$.

(A-10)
- $a_U = \nu \chi_{\hat{U}} i 2^{1/2} r_{\mathrm{z}}^* y\, dt - \frac{1}{2}(1 - \chi_{\hat{U}} + \chi_{\hat{U}} r_{\mathrm{z}}^* a_0)(z^{-1} dz - \bar{z}^{-1} d\bar{z})$.
- $\alpha_U = (1 - \chi_{\hat{U}}(1 - r_{\mathrm{z}}^* |\alpha_0|)) z/|z|$.
- $\beta_U = i\mu \mathrm{z}^{-1/2} \chi_{\hat{U}} r_{\mathrm{z}} * \varsigma$.

Items (A-7)–(A-10) together define a smooth pair of connection on E's restriction to $U_0 \cup \big(\bigcup_{\gamma\in\Theta} U_\gamma\big)$ and section of $\mathbb{S}$ over this same set. In particular, they define a pair of connections on E's restriction to $Y_{*\Lambda} \cup T_{*\Lambda}$ and section of $\mathbb{S}$ over $Y_{*\Lambda} \cup T_{*\Lambda}$.

Ab Constraints

The operator in (1-17) will be analyzed in the case where the relevant version of (A, ψ) is assumed to have five properties that are given in a moment. In particular, these properties are satisfied by solutions to a given (r, μ) version of (1-13). The upcoming Lemma A.1 asserts that these properties are also satisfied by the pairs that are constructed in Section Aa.

The upcoming properties refer to constants $c_0 \geq 100$ and $\mathrm{z} \geq c_0^{10}$, These lower bounds are increased in subsequent subsections. The properties refer to a given pair $(A, \psi) \in \mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$. By way of a look ahead, a pair (A, ψ) with these properties looks much like a solution to the $\mathrm{r} = \mathrm{z}$ and μ version of (1-13) with $\mu \in \Omega$ having $\mathcal{P}$–norm bounded by 1. The properties listed below are such that (A, ψ) resembles such a solution inasmuch as

$$\mathrm{z}^{-1/2} |B_A - \mathrm{z}(\psi^\dagger \tau \psi - \hat{a})| + |D_A \psi| \leq c_0^{-6} \mathrm{z}^{1/2} + c_0. \tag{A-11}$$

The list of properties follows directly:

Property 1 *The section $\psi = (\alpha, \beta)$ is such that*:

- $|\alpha| \leq 1 + c_0 \mathrm{z}^{-1}$ *and* $|\beta| \leq c_0 \mathrm{z}^{-1/2}$.
- $|\nabla_A \alpha|^2 \leq c_0 \big(\mathrm{z} \big|1 - |\alpha|^2\big| + 1\big)$.
- $|\nabla_A \beta| \leq c_0$.

The second property introduces the following notation: the section $D_A\psi$ of $\mathbb{S}$ is written as $([D_A\psi]_0, [D_A\psi]_1)$ with respect to the splitting $\mathbb{S} = E \oplus EK^{-1}$. As always, the Hodge dual of the curvature 2–form of A is denoted by B_A.

Property 2 *The* 1*–form* B_A *and the section* $D_A\psi$ *of* $\mathbb{S}$ *are such that*:

- $\big|\langle\hat{a}, B_A\rangle + i\,\mathrm{z}(1-|\alpha|^2)\big| \le c_0^{-20}\mathrm{z} + c_0$.
- $|\hat{a}\wedge B_A| \le c_0^6\mathrm{z}^{1/2}\big|1-|\alpha|^2\big|^{1/2} + c_0$.
- $|[D_A\psi]_0| \le c_0^6$.
- $|[D_A\psi]_1| \le c_0^{-10}\mathrm{z}^{1/2} + c_0$.

Note for future reference that the third bullets of Properties 1 and 2 have the following consequence: Let $(\nabla_A\alpha)_v$ denote the section of E that is obtained by pairing $\nabla_A\alpha$ with v. The norm of this section $(\nabla_A\alpha)_v$ is bounded by c_0c_0 because $[D_A\psi]_0$ is the sum of $i(\nabla_A\alpha)_v$ with a linear combination of covariant derivatives of β.

The third property introduces $Y_{\diamond\mathrm{z}}$ to denote the subset of Y with distance at least $c_0^4\mathrm{z}^{-1/2}$ from $\gamma\in\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+\cup\hat{\gamma}_{\mathfrak{p}}^-)$. This subset $Y_{\diamond\mathrm{z}}$ is a smooth manifold with boundary whose boundary components are tori. The third property also refers to the (A,ψ) version of the connection $\widehat{A}$ that is defined in (1-15).

Property 3 *The zero locus of* α *in* $Y_{\diamond\mathrm{z}}$ *is transversal and it consists of the disjoint union of at most* G *components with each a properly embedded arc or circle. The zero locus of* α *in* $Y_{\diamond\mathrm{z}}$ *has the following additional properties*:

- *Any given boundary component of* $Y_{\diamond\mathrm{z}}$ *contains either zero or two endpoints of the arc components of* α*'s zero locus in* $Y_{\diamond\mathrm{z}}$. *If two, then the distance between them is at least* $100c_0^2\mathrm{z}^{-1/2}$. *Moreover,* $u<0$ *on one and* $u>0$ *on the other.*
- *Suppose that* γ *is a component of the zero locus of* α *in* $Y_{\diamond\mathrm{z}}$.
  - (a) *The unit-length, oriented tangent vector to* γ *has distance at most* $c_0\mathrm{z}^{-1/2}$ *from* v.
  - (b) *The curve* γ *intersects any given* $\mathfrak{p}\in\Lambda$ *version of* $\mathcal{H}_{\mathfrak{p}}$ *where* $1-3\cos^2\theta>0$.
  - (c) *If* γ *is disjoint from a given boundary torus of* $Y_{\diamond\mathrm{z}}$, *then it has distance greater than* $3c_0^3\mathrm{z}^{-1/2}$ *from this torus.*
  - (d) *If* γ *intersects a boundary torus of* $Y_{\diamond\mathrm{z}}$, *then it does so only at its endpoints and these intersections are transversal.*
- *The intersection of* α*'s zero locus with* M_δ *lies in the* $f^{-1}(1,2)$ *part of* M_δ. *This intersection consists of* G *properly embedded segments that pair the index* 1 *and index* 2 *critical points of* f *in the sense that distinct segments start on the boundary of the radius* δ *coordinate balls about distinct index* 1 *critical points of* f *and end on the boundary of the radius* δ *coordinate balls about distinct index* 2 *critical points.*

- *The 2–form* $\frac{i}{2\pi}F_{\hat{A}}$ *has compact support and integral* 1 *on any transverse disk in* Y *with radius* $c_0 z^{-1/2}$ *and center at a zero of* α *in* $Y_{\diamond z}$.

The fourth property constrains α away from its zero locus.

Property 4 *The absolute value of* $1-|\alpha|^2$ *is less than* c_0^{-10} *at all points with distance greater than* $c_0 z^{-1/2}$ *from the zero locus of* α *in* Y.

The final property is not strictly speaking required; it is imposed solely to avoid some extra effort. To set the notation, let $p \in Y$ denote a given point. Fix a $\mathbb{C}$–linear isometry between $\mathbb{C}$ and the kernel of $\hat{a}$ at p. With z given, use $\varphi_z\colon \mathbb{C} \to Y$ to denote the composition of first multiplication on $\mathbb{C} = \mathrm{Ker}(\hat{a})$ by $z^{-1/2}$ followed by the metric's exponential map. With $(A, \psi = (\alpha, \beta)) \in \mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ given, use (A_z, α_z) to denote the pullback of (A, α) using φ_z.

Property 5 *Fix* $p \in Y$. *The pair* (A_z, α_z) *has distance at most* c_0^{-10} *in the* C^4–*topology from a solution to the vortex equations when restricted to the disk of radius* c_0 *with center at the origin in* $\mathbb{C}$.

To see an example of a pair with these properties, fix $r \geq c_0$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1. Every solution to the corresponding (r, μ) version of (1-13) has Properties 1–5 if z is set equal to r with r large, and if c_0 is chosen less than $r^{1/6}$ and chosen to avoid at most G intervals of a priori bounded length. By way of an explanation why this is so, the top three bullets in Property 1 are asserted by Lemma 2.1 and the fourth bullet has zero on its right-hand side. Meanwhile (1-13) guarantees Property 2, Lemma 2.3 guarantees Property 4 and Lemma 2.9 guarantees Property 5. The first bullet of Proposition 2.4 gives item (a) of the second bullet of Property 3, its second bullet guarantees item (b) of Property 3, its third bullet guarantees the third bullet of Property 3 and its fifth bullet guarantees the fourth bullet of Property 3. If the first bullet of Property 3 or items (c) or (d) of the second bullet of Property 3 are not obeyed for a given choice of c_0, then at most G replacements of the form $c_0 \mapsto c_0 + c_0$ will satisfy all of them. That this is so follows directly from the first three bullets of Proposition 2.4 and the formula for v in (1-3). This is explained in the next paragraph.

To elaborate on the last assertion: Fix $\mathfrak{p} \in \Lambda$ and consider the boundary of $Y_{\diamond z}$ near the circle $\hat{\gamma}_{\mathfrak{p}}^+$. This circle is the locus in $\mathcal{H}_{\mathfrak{p}}$ where the coordinate $u = 0$ and the coordinate θ is such that $\cos\theta = \frac{1}{\sqrt{3}}$. The "outside" of this boundary torus is a solid torus neighborhood of $\hat\gamma_{\mathrm{p}}^+$. Now, the key point is that the unit tangent vector to γ differs by at most $c_0 \mathrm{z}^{-1/2}$ from v. This implies two key facts: First, if v has inner product greater than $c_0 \mathrm{z}^{-1/2}$ with the normal vector to a given intersection point between γ and the boundary torus, then γ will be transversal to the boundary torus at that point. It also implies that γ will have distance at most $c_0 c_0^5 \mathrm{z}^{-1}$ inside the corresponding solid torus from the integral curve of v through that boundary point. (Note that since $\mathrm{z} > c_0^{10}$, this distance is at most $c_0 c_0 \mathrm{z}^{-1/2}$.) Because of these facts about γ and v's integral curves, conclusions about γ's intersections with the boundary torus follow from properties of v's integral curves.

Keeping the preceding in mind, here are three facts that follow from (1-3)'s depiction of v: First, v is transversal to the boundary torus except at the two $u = 0$ loci on this torus $\big($one where $\cos^2\theta < \frac{1}{3}$ and the other where $\cos^2\theta > \frac{1}{3}\big)$. These loci are circles that are ϕ–invariant. Second, v's inner product with the normal vector to the boundary torus has norm greater than $c_0 \mathrm{z}^{-1/2}$ where the distance to these $u = 0$ circles is greater than $c_0 c_0^{-1}$. Third, the distance from the $u = 0$ loci where an integral curve of v enters the solid torus is the same as where it exits the solid torus.

Granted these facts, then there exists c_* (with $c_* \le c_0$) with the following significance: the conditions in the second bullet of Property 3 and in items (c) and (d) of the third bullet are guaranteed to be satisfied if γ's distance from the $u = 0$ points on the boundary torus is greater than $c_* c_0^3 \mathrm{z}^{-1/2}$ if c_0 is large and z is very much greater than c_0.

With the last paragraph understood, suppose for the sake of argument that the distance between γ and the $u = 0$ points on the boundary torus is less than $100 c_* c_0^3 \mathrm{z}^{-1/2}$. Fix $\mathfrak{e} > 0$ for the moment and consider a bigger solid torus neighborhood of $\hat\gamma_{\mathrm{p}}^+$, one with radius $(c_0 + \mathfrak{e})^4 \mathrm{z}^{-1/2}$. Then, the distance between γ and the $u = 0$ points on this torus will be greater than $(c_0 \mathfrak{e} - 100 c_*)(c_0 + \mathfrak{e})^3 \mathrm{z}^{-1/2}$, which will be greater than $3(c_0 + \mathfrak{e})^3 \mathrm{z}^{-1/2}$ if $\mathfrak{e} > c_0$.

The lemma that follows asserts that certain versions of the pairs (A, ψ) that are described in Section Aa also have the five properties listed above.

Lemma A.1 *There exists $\kappa \ge 100$ with the following significance: Fix parameters $c_v \ge \kappa$ and $\mathrm{z} \ge \kappa c_v^{10}$, and then set $\rho_* = c_v^2 \mathrm{z}^{-1/2}$. Fix a set $T_{*\Lambda}$ and then a set Θ as described by (A-5) which obeys the first and second bullets of the $(\mathrm{z}, c_0 = c_v)$ version of Property 3. Suppose that $(A, \psi) \in \mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ is given by the $(\mathrm{z}, c_v, \rho_*)$*

version of (A-7)–(A-10) *on* $Y_{*\Lambda} \cup T_{*\Lambda}$ *and that the* $(\mathrm{z}, c_0 = c_v)$ *version of Properties 1, 2, 4 and 5 hold on* $Y - (Y_{*\Lambda} \cup T_{*\Lambda})$. *Then* (A, ψ) *obeys the* $(\mathrm{z}, c_0 = c_v)$ *version of Properties 1–5 on the whole of* Y.

Proof This is a straightforward calculation given what is said in Section 3.3 and (3-3) about solutions to (2-8). Much the same calculation is done in Sections 2e and 2f of [15]. The specifics of the calculation are omitted. □

Ac Bounds on eigenvectors

Suppose in what follows that $\mathfrak{c} = (A, \psi)$ satisfies Properties 1–5 in the previous subsection. The two lemmas in this subsection give some preliminary information about eigenvectors of the associated version of the operator $\mathfrak{L}_{\mathfrak{c},\mathrm{z}}$, this being the $\mathrm{z} = \mathrm{r}$ version of the operator that is depicted in (1-17). The notation is such that $\nabla\mathfrak{b}$ is used to denote the covariant derivative of a given $\mathfrak{b} \in C^\infty(Y; iT^*Y \oplus \mathbb{S} \oplus i\mathbb{R})$ that is defined by the Levi-Civita covariant derivative on the iT^*Y summand, the covariant derivative on sections of $\mathbb{S}$ that is defined by the Levi-Civita connection with the connection A, and the exterior derivative on the sections of $i\mathbb{R}$. These lemmas also use $\|\cdot\|_2$ to denote the L^2–norm on Y.

Lemma A.2 *There exists* $\kappa \geq 100$ *and, given* $c_0 \geq \kappa$, *there exists* $\kappa_{c_0} \geq \kappa$ *with the following significance: Fix* $\mathrm{z} \geq \kappa_{c_0} c_0^{10}$ *and suppose that* $\mathfrak{c} = (A, \psi) \in \mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ *obeys the* (c_0, z) *version of Properties 1 and 2 in Section Ab.*

- *Let* $\mathfrak{b} = (b, \eta, \phi)$ *denote an eigenvector of* $\mathfrak{L}_{\mathfrak{c},\mathrm{z}}$. *Use* λ *to denote the corresponding eigenvalue. Then* $\|\nabla\mathfrak{b}\|_2 \leq \kappa(\lambda + c_0 \mathrm{z}^{1/2})\|\mathfrak{b}\|_2$.
- *Suppose in addition that* $\mathfrak{c}$ *obeys Property 4 in Section Ab and that* $|\lambda| \leq c_0^{-\kappa}\mathrm{z}^{1/2}$. *Fix* $m > 2c_0$. *The* L^2*–norm of* $\mathfrak{b}$ *over the subset in* Y *with distance greater than* $m\mathrm{z}^{-1/2}$ *from* $\alpha^{-1}(0)$ *is no greater than* κm^{-1}.

To set the notation for the next lemma suppose that $\mathfrak{b} = (b, \eta, \phi)$ is a section of $iT^*Y \oplus \mathbb{S} \oplus i\mathbb{R}$. The lemma writes b as $b = b_0\hat{a} + b^\perp$, where $b^\perp$ annihilates v, and it writes η with respect to the splitting $\mathbb{S} = E \oplus EK^{-1}$ as $\eta = (\eta_0, \eta_1)$. Lemma A.3 also uses $(\nabla b^\perp)_v$ and $(\nabla_A \eta_0)_v$ to denote the directional covariant derivatives along the vector field v.

Lemma A.3 *There exists* $\kappa \geq 100$, *and, given* c_0, *there exists* $\kappa_{c_0} \geq \kappa$ *with the following significance: Fix* $\mathrm{z} \geq \kappa_{c_0} c_0^{10}$ *and let* $\mathfrak{c} = (A, \psi) \in \mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$

denote a pair that obeys the (c_0, z) *version of Properties 1–4 in Section Ab. Suppose that* $\mathfrak{b} = (b, \eta, \phi)$ *is an eigenvector of* $\mathfrak{L}_{\mathfrak{c},z}$ *with* L^2*–norm equal to* 1*, and use* λ *to denote* $\mathfrak{b}$*'s eigenvalue. Assume that* $|\lambda| \le c_0^{-\kappa} z^{1/2}$*.*

- *The* L^2*–norms of* b_0*,* η_1 *and* ϕ *are bounded by* $c_0^{\kappa} z^{-1/2}$ *and the* L^2*–norms of their covariant derivatives are bounded by* κ_{c_0}*.*
- *The* L^2*–norms of* $(\nabla b^{\perp})_v$ *and* $(\nabla \eta_0)_v$ *are bounded by* $\kappa_{c_0} + \kappa|\lambda|$*.*

The proofs of Lemmas A.2 and A.3 are given in a moment. A Bochner–Weitzenbock formula for $L^2_{\mathfrak{c},r}$ plays the central role in the arguments for these lemmas. To state this formula, fix $z \ge 1$ and let $\mathfrak{c} = (A, \psi)$ denote a pair in $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ and let $\mathfrak{L}_{\mathfrak{c},z}$ denote the corresponding version of (1-17). The respective iT^*Y, $\mathbb{S}$ and $i\mathbb{R}$ components of $\mathfrak{L}^2_{\mathfrak{c},z}\mathfrak{b}$ are:

(A-12)
- $\nabla^\dagger \nabla b + 2z|\psi|^2 b - 2^{1/2} z^{1/2} (\nabla\psi^\dagger \eta - \eta^\dagger \nabla\psi) + 2^{-1/2} z^{1/2} ((D\psi)^\dagger \tau\eta + \eta^\dagger \tau D\psi) + \mathrm{Ric}(b),$
- $D_A^2 \eta + z[(\psi^\dagger\eta - \eta^\dagger\psi)\psi - \mathrm{cl}(\psi^\dagger\tau\eta + \eta^\dagger\tau\psi)\psi] - 2^{3/2} z^{1/2} \langle b, \nabla\psi\rangle - 2^{1/2} z^{1/2} (\mathrm{cl}(b) + \phi) D\psi,$
- $d^\dagger d\phi + 2z|\psi|^2\phi + 2^{-1/2} z^{1/2} ((D\psi)^\dagger \eta - \eta^\dagger D\psi).$

To explain the notation, $\mathrm{Ric}(\cdot)$ denotes the endomorphism of T^*Y defined by the Ricci tensor. Meanwhile, $\langle\ ,\ \rangle$ is defined as follows: Let $V \to Y$ denote any given vector bundle. Given V, then $\langle\ ,\ \rangle$ is the homomorphism from the bundle $T^*Y \otimes (V \otimes T^*Y)$ to V that is defined by the Riemannian metric.

Proof of Lemma A.2 To prove the first bullet, take the inner product between $\mathfrak{b}$ and $\mathfrak{L}^2_{\mathfrak{c},z}\mathfrak{b}$ and integrate the result over Y. Use (A-11) and (A-12) with an integration by parts to obtain the asserted bound. Note in this regard that the bounds on $|\psi|$ by c_0 and on $|\nabla\psi|$ by $c_0 z^{1/2}$ are needed.

To prepare the stage for the proof of the second bullet, let $c \in \mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ denote any given element. What is written in (A-12) can be depicted schematically as

$$\mathfrak{L}^2_{\mathfrak{c},z}\mathfrak{b} = \nabla^\dagger\nabla\mathfrak{b} + 2z\mathfrak{b} + \mathfrak{e}(\mathfrak{b}), \tag{A-13}$$

where the endomorphism $\mathfrak{e}$ obeys $|\mathfrak{e}| \le c_0(|B_A| + z^{1/2}|\nabla\psi| + c_0)$. If $\mathfrak{b}$ is an eigenvector of $\mathfrak{L}_{\mathfrak{c},z}$ with eigenvalue λ, then (A-13) leads to the inequality

$$d^\dagger d|\mathfrak{b}| + 2z|\mathfrak{b}| - |\mathfrak{e}||\mathfrak{b}| \le \lambda^2|\mathfrak{b}|. \tag{A-14}$$

Now suppose that $\mathfrak{b}$ is as described in Lemma A.2's second bullet. The assumption that $\lambda \le c_0^{-1} z^{1/2}$ and what is said in Properties 1, 2 and 4 in Section Ab have the following consequence: the inequality in (A-14) implies the more straightforward inequality

$$d^\dagger d|\mathfrak{b}| + z|\mathfrak{b}| \le 0 \tag{A-15}$$

at points with distance $c_0 z^{-1/2}$ or more from $\alpha^{-1}(0)$. To exploit this inequality, let $\Delta_\alpha(\cdot)$ denote for a moment the function $\mathrm{dist}(\cdot, \alpha^{-1}(0))$. Given $m \ge 2c_0$, convolve the function $\chi(2 - 2m^{-1} z^{1/2} \Delta_\alpha(\cdot))$ with a suitably chosen smoothing kernel to construct a nonnegative function on Y with the following properties: Let g_m denote this function. Then $g_m = 1$ where the distance to $\alpha^{-1}(0)$ is greater than $m z^{-1/2}$ and $g_m = 0$ where the distance to Y is less than $\frac{1}{2} m z^{-1/2}$. Furthermore, $|dg_m| \le c_0 m^{-1} z^{1/2}$. Multiply both sides of (A-15) by $g_m^2 |\mathfrak{b}|$ and integrate by parts. The resulting inequality implies the bound $z\|g_m \mathfrak{b}\|_2 \le c_0 m^{-1} z \|\mathfrak{b}\|_2$. Divide both sides of this by z to obtain what is asserted by Lemma A.2. □

Proof of Lemma A.3 The bounds in the second bullet follow from those in the first from the form of $\mathfrak{L}_{\mathfrak{c},z}$. Indeed, the relevant version of the equation $\mathfrak{L}_{\mathfrak{c},z}\mathfrak{b} = \lambda \mathfrak{b}$ equates $(\nabla \mathfrak{b}^\perp)_v$ and $(\nabla \eta_0)_v$ with linear combinations of the following: first, covariant derivatives of b_0, η_1 and ϕ; second, linear combinations of $z^{1/2} b_0$, $z^{1/2}\eta_1$ and $z^{1/2}\phi$ times factors of α or its complex conjugate; third, linear combinations of factors of $z^{1/2} b^\perp$ and $z^{1/2}\eta_0$ times factors of β or its complex conjugate; finally, components of $\lambda\mathfrak{b}$. This property of $\mathfrak{L}_{\mathfrak{c},z}$ is directly evident from its depiction in the upcoming (A-16) and (A-17).

The proof of the first bullet has six steps.

Step 1 The asserted bounds are proved with the help of (A-12). The bounds for ϕ will use the formula in (A-12) for the $i\mathbb{R}$ component of $\mathfrak{L}_{\mathfrak{c},z}\mathfrak{b}$. Those for b_0 are obtained with the help of the formula in (A-12) for the iT^*Y component of $\mathfrak{L}_{\mathfrak{c},z}^2\mathfrak{b}$ by projecting the latter onto the span of $\hat{a}$. Those for η_1 are obtained using the formula in (A-12) for the $\mathbb{S}$ component of $\mathfrak{L}_{\mathfrak{c},z}\mathfrak{b}$ by projecting the latter onto the EK^{-1} summand of $\mathbb{S}$. In this regard, the projection of the iT^*Y component of $\mathfrak{L}_{\mathfrak{c},z}\mathfrak{b}$ to the span of $\hat{a}$ can be written as

$$\begin{aligned} d^\dagger d b_0 + 2z|\psi|^2 b_0 - 2^{1/2} z^{1/2}((\nabla\psi)_v^\dagger \eta - \eta^\dagger (\nabla\psi)_v) \\ + 2^{-1/2} z^{1/2}((D\psi)^\dagger \,\mathrm{cl}(\hat{a})\eta + \eta^\dagger \,\mathrm{cl}(\hat{a}) D\psi) + \mathfrak{R}_0(\nabla b) + \langle \hat{a}, \mathrm{Ric}(b)\rangle, \end{aligned} \tag{A-16}$$

where $(\nabla\psi)_v$ denotes the directional covariant derivative along v and where $\mathfrak{R}_0$ denotes a linear form on $T^*Y \otimes T^*Y$ that is defined by the covariant derivatives of $\hat{a}$.

In particular, the latter is bounded in absolute value by c_0. Meanwhile, the projection of the $\mathbb{S}$ component of $\mathfrak{L}^2_{\mathfrak{c},\mathrm{z}}\mathfrak{b}$ to the EK^{-1} summand of $\mathbb{S}$ can be written as

$$\text{(A-17)}\quad \nabla_A^{\dagger}\nabla_A\eta_1 + i\langle\hat{a}, B_A\rangle\eta_1 + 2\mathrm{z}(|\alpha|^2+|\beta|^2)\eta_1 + \mathrm{cl}(B_A^{\perp})\eta_0 - 2^{3/2}\mathrm{z}^{1/2}\langle b,\nabla\beta\rangle - 2^{1/2}\mathrm{z}^{1/2}[(\mathrm{cl}(b)+\phi)D\psi]_1 + \mathfrak{R}_1(\nabla\eta) + \mathfrak{r}_1(\eta),$$

where the notation use $B_A^{\perp}$ to denote $B_A - \hat{a}\langle\hat{a}, B_A\rangle$, it uses $[\cdot]_1$ to denote the EK^{-1} component of the given section of $\mathbb{S}$, and it uses $\mathfrak{R}_1$ and $\mathfrak{r}_1$ to denote endomorphisms that depend only on the Riemannian metric.

Step 2 The notation that follows uses ξ to denote (b_0, η_1, ϕ) and it uses $\nabla\xi$ to denote the 3–tuple whose first and third entries are db_0 and $d\phi$, and whose second entry is $\nabla_A\eta_1$. Fix a constant $m_\Lambda > 8c_0^4$ to be determined shortly. Suppose in this step that the L^2–norm of ξ over the part of $Y_{\diamond\mathrm{z}}$ with distance greater than $m_\Lambda\mathrm{z}^{-1/2}$ from the boundary of $Y_{\diamond\mathrm{z}}$ is less than $m_\Lambda^{-1/4}\|\xi\|_2$.

Introduce θ_Λ to denote the characteristic function for the set of points in $Y_{\diamond\mathrm{z}}$ with distance at least $m_\Lambda\mathrm{z}^{-1/2}$ from the boundary of $Y_{\diamond\mathrm{z}}$. Meanwhile, use the function χ to construct a smooth, nonnegative function which is 1 where the distance to $Y - Y_{\diamond\mathrm{z}}$ is less than $2m_\Lambda\mathrm{z}^{-1/2}$ and zero where the distance to this set is greater than $4m_\Lambda\mathrm{z}^{-1/2}$. Use χ_Λ to denote this function. The function χ_Λ can and should be constructed so that its differential obeys $|d\chi_\Lambda| \le 16m_\Lambda^{-1}\mathrm{z}^{1/2}$. Note that $|d\chi_\Lambda|$ has support where θ_Λ is equal to 1.

Take the L^2 inner product of the components of $\chi_\Lambda^2\xi$ with the relevant parts of the eigenvalue equation $\mathfrak{L}^2_{\mathfrak{c},\mathrm{z}}\mathfrak{b} = \lambda\mathfrak{b}$. Use the third bullet of (A-12), (A-16) and (A-17) with an integration by parts to derive the inequality

$$\text{(A-18)}\quad \|\nabla(\chi_\Lambda\xi)\|_2^2 \le c_0\lambda^2\|\chi_\Lambda\xi\|_2^2 + c_0 m_\Lambda^{-2}\mathrm{z}\|\theta_\Lambda\xi\|_2^2 + c_0((c_0^{-10}\mathrm{z}+c_0)\|\chi_\Lambda\xi\|_2^2 + c_0^6\mathrm{z}^{1/2}\|\chi_\Lambda\xi\|_2\|\mathfrak{b}\|_2 + \|\mathfrak{b}\|_2^2).$$

The rest of this step explains how the various terms in this inequality come about.

The term $\|\nabla(\chi_\Lambda\xi)\|_2^2$ on the left-hand side and the term $c_0 m_\Lambda^{-2}\mathrm{z}\|\theta_\Lambda\xi\|_2^2$ on the right-hand side arise from the integration by parts that rewrites the L^2 inner product between $\chi_\Lambda^2\xi$ and $\nabla^{\dagger}\nabla\xi$ as the square of the L^2–norm of $\nabla(\chi_\Lambda\xi)$ and a term with derivatives of χ_Λ. The former accounts for the term on the left-hand side of (A-18) and the latter accounts for the appearance of $c_0 m_\Lambda^{-2}\mathrm{z}\|\theta_\Lambda\xi\|_2^2$ on the right-hand side of (A-18). These two terms with the term $c_0\|\mathfrak{b}\|_2^2$ on the right-hand side of (A-18) also account for the L^2 inner product between $\chi_\Lambda^2\xi$ and the $\mathfrak{R}_0(\nabla\mathfrak{b})$ and $\mathfrak{R}_1(\nabla\mathfrak{b})$ terms in (A-12) and (A-13). The term $\lambda^2\|\chi_\Lambda\xi\|_2^2$ comes from the L^2 inner product between $\chi_\Lambda^2\xi$ and $\lambda\xi$.

The terms $(c_0^{-10}\mathrm{z}+c_0)\|\chi_\Lambda\xi\|_2^2$ and $c_0^6\mathrm{z}^{1/2}\|\chi_\Lambda\xi\|_2\|\mathfrak{b}\|_2$ and $\|\mathfrak{b}\|_2^2$ on the right-hand side of (A-18) account for the L^2 inner product between components of $\chi_\Lambda^2\xi$ and the various terms in the third bullet of (A-12), (A-16) and (A-17) that lack covariant derivatives of components of ξ. To elaborate, there are, first of all, the terms that have $2\mathrm{z}|\psi|^2$ multiplying ϕ in (A-12) and b_0 in (A-16). These are discarded when writing (A-18) as they contribute nonpositive terms to the right-hand side of (A-18). There is also a nonpositive contribution to the right-hand side of (A-18) from the $2\mathrm{z}(|\alpha|^2+|\beta|^2)\eta_1$ term in (A-17) and from $i\langle\hat{a}, B_A\rangle\eta_1$. Properties 1 and 2 are used to rewrite this last term as $\mathrm{z}(1-|\alpha|^2)\eta_1$ plus a remainder term that is bounded by $(c_0^{-10}\mathrm{z}+c_0)|\eta_1|$. The remainder term is accounted for by a part of the $(c_0^{-10}\mathrm{z}+c_0)\|\chi_\Lambda\xi\|_2^2$ term on the right-hand side of (A-18).

The other terms without covariant derivatives of ξ in the third bullet of (A-12), (A-16) and (A-17) are bounded by either

$$c_0|[D_A\psi]_1||\xi| \quad \text{or} \quad c_0\big(|B_A^\perp|+|(\nabla\psi)_v|+|\nabla\beta|+|[D_A\psi]_0|+1\big)|\mathfrak{b}|. \tag{A-19}$$

With (A-19) understood, what follows is a consequence of Properties 1 and 2: The terms without covariant derivatives of ξ that are bounded by $c_0|[D_A\psi]_1||\xi|$ are accounted for by the term $(c_0^{-10}\mathrm{z}+c_0)\|\chi_\Lambda\xi\|_2^2$ on the right-hand side of (A-18). Meanwhile, the terms without covariant derivatives of ξ that are bounded by the right-most expression in (A-19) are accounted for by the term $c_0^6\mathrm{z}^{1/2}\|\chi_\Lambda\xi\|_2\|\mathfrak{b}\|_2+\|\mathfrak{b}\|_2^2)$ in (A-18). Note in this regard that Properties 1 and 2 imply the bound $|(\nabla\psi)_v|\le c_0c_0^6$. This is stated explicitly with regards to $(\nabla\beta)_v$; and $(\nabla_A\alpha)_v|\le c_0c_0^6$ because $[D_A\psi]_0$ is a sum of $i(\nabla_A\alpha)_v$ and linear combinations of covariant derivatives of β.

Step 3 Fix $m_\Lambda=100c_0^4$ and use the assumption $\|\theta_\Lambda\xi\|_2\le m_\Lambda^{-1/4}\|\xi\|_2$ to see that the right-hand expression in (A-18) is at most $c_0(\lambda+c_0^{-10}\mathrm{z})\|\chi_\Lambda\xi\|_2^2+c_0^k$ with $k\le c_0$.

Meanwhile, the left-hand side of (A-18) is no less than $c_0m_\Lambda^{-2}\mathrm{z}\|\chi_\Lambda\xi\|_2$. Indeed, this follows from a standard Dirichlet eigenvalue inequality given that $|\chi_\Lambda\xi|$ has compact support in the radius $(m_\Lambda+c_0^4)\mathrm{z}^{-1/2}$ tubular neighborhood of $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+\cup\hat{\gamma}_\mathfrak{p}^-)$. These upper and lower bounds find $\|\xi\|_2^2\le c_0c_0^k\mathrm{z}^{-1}$ if $c_0\ge c_0$ and $\lambda\le c_0^{-5}\mathrm{z}^{1/2}$. This gives the first assertion of the first bullet of Lemma A.3.

Step 4 Fix $m_\Lambda=100c_0^4$ so as to invoke the conclusions of Step 2. With m_Λ fixed, use χ to construct a smooth, nonnegative function on Y which is equal to 1 at distances greater than $m_\Lambda \mathrm{z}^{-1/2}$ from $Y - Y_{\diamond \mathrm{z}}$ and equal to zero on $Y - Y_{\diamond \mathrm{z}}$. This function is denoted by $\chi_{\diamond \mathrm{z}}$. The function $\chi_{\diamond \mathrm{z}}$ can be constructed so that $|d\chi_{\diamond \mathrm{z}}| \le 32 m_\Lambda^{-1} \mathrm{z}^{1/2}$. Fix a second constant $m \in (c_0, c_0^2)$. This step makes the following two assumptions:

(A-20)
- The L^2–norm of ξ over the part of $Y_{\diamond \mathrm{z}}$ with distance greater than $m_\Lambda \mathrm{z}^{-1/2}$ from the boundary of $Y_{\diamond \mathrm{z}}$ is not less than $m_\Lambda^{-1/4} \|\xi\|_2$.
- The L^2–norm of $\chi_{\diamond \mathrm{z}} \xi$ over the part of $Y_{\diamond \mathrm{z}}$ with distance $m\mathrm{z}^{-1/2}$ or more from the zero locus of α is greater than $m^{-1/4} \|\chi_{\diamond \mathrm{z}} \xi\|_2$.

Use χ once more, now to define a smooth, nonnegative function which is 1 where the distance to $\alpha^{-1}(0)$ is greater than $m\mathrm{z}^{-1/2}$ and zero where the distance is less than $\frac{1}{2} m \mathrm{z}^{-1/2}$. Let χ_m denote this function. Given that $m \le c_0^2$, what is said by the first bullet of Property 3 and what is said by item (c) of the second bullet of Property 3 imply that the function χ_m can be constructed so that its differential obeys $|d\chi_m| \le 16 m^{-1} \mathrm{z}^{1/2}$. This bound is assumed in what follows. Introduce θ_m to denote the characteristic function for the support of $|d\chi_m|$ and $\theta_{\diamond \mathrm{z}}$ to denote the characteristic function for the support of $|d\chi_{\diamond \mathrm{z}}|$.

Take the L^2 inner product of $(\chi_m \chi_{\diamond \mathrm{z}})^2 \xi$ with the eigenvalue equation $\mathfrak{L}_{\mathfrak{c},\mathrm{z}}^2 \mathfrak{b} = \lambda \mathfrak{b}$ and use either the third bullet of (A-12) or (A-16) or (A-17) with an integration by parts and (A-11) to derive from these integrals the inequality

$$\text{(A-21)} \quad \begin{aligned} &\|\nabla(\chi_m \chi_{\diamond \mathrm{z}} \xi)\|_2^2 + \tfrac{1}{2} \mathrm{z} \|\chi_m \chi_{\diamond \mathrm{z}} \xi\|_2^2 \\ &\quad \le c_0 \lambda^2 \|\chi_m \chi_{\diamond \mathrm{z}} \xi\|_2^2 + c_0 \mathrm{z} (m^{-2} \|\theta_m \xi\|_2^2 + m_\Lambda^{-2} \|\theta_{\diamond \mathrm{z}} \xi\|_2^2) \\ &\qquad\qquad + c_0 c_0^6 \mathrm{z}^{1/2} \|\chi_m \chi_{\diamond \mathrm{z}} \xi\|_2. \end{aligned}$$

This proof of this inequality invokes Property 4, the bounds for the norms of the components of B_A and $D_A \psi$ that are asserted in Property 2 and the bound for $|\nabla_A \beta|$ that is asserted by Property 1. As noted previously, these imply that $|(\nabla \psi)_v| \le c_0 c_0^6$.

To make something of (A-21), use the first bullet in (A-20) to conclude that $\|\theta_{\diamond \mathrm{z}} \xi\|_2 \le m_\Lambda^{1/4} \|\chi_{\diamond \mathrm{z}} \xi\|_2$ and then use the second to see that $\|\theta_{\diamond \mathrm{z}} \xi\|_2 \le (m m_\Lambda)^{1/4} \|\chi_m \chi_{\diamond \mathrm{z}} \xi\|_2$. Use this bound in (A-21) to conclude that

$$\text{(A-22)} \quad \begin{aligned} &\tfrac{1}{4} \mathrm{z} \|\chi_m \chi_{\diamond \mathrm{z}} \xi\|_2^2 \\ &\quad \le c_0 \lambda^2 \|\chi_m \chi_{\diamond \mathrm{z}} \xi\|_2^2 + c_0 (m_\Lambda^{1/4} m^{-3/4} + m^{1/4} m_\Lambda^{-3/4}) \mathrm{z} \|\xi\|_2^2 + c_0 c_0^k, \end{aligned}$$

with $k < c_0$. By assumption, $m \le m_\Lambda$ and so $m^{1/4} m_\Lambda^{-3/4} \le \frac{1}{100}$. Meanwhile, $m_\Lambda^{1/4} m^{-3/4} \le \frac{1}{100}$ if $m \ge c_0 c_0^{4/3}$. Assume this to be the case. If it is also the case that $\lambda \le c_0^{-1} \mathrm{z}^{1/2}$, then (A-18) finds $\|\chi_m \chi_{\diamond \mathrm{z}} \xi\|_2 \le c_0 c_0^{k'} \mathrm{z}^{-1/2}$ with $k' \le c_0$. This last bound with (A-20) gives the bound $\|\xi\|_2^2$ by $c_0 c_0^{k''} z^{-1/2}$ with $k'' \le c_0$ if $m = c_0 c_0^{4/3}$ and if $\lambda \le c_0^{-1} z^{1/2}$.

Step 5 This step assumes that the top bullet in (A-20) is satisfied but that the lower bullet is violated. Use χ yet again, this time to construct a smooth, nonnegative function which is 0 where the distance to $\alpha^{-1}(0)$ is greater than $2mz^{-1/2}$ and 1 where the distance is less than $mz^{-1/2}$. Denote this function by χ_m^c. Given that $m \le c_0^2$, the function χ_m^c can and should be constructed so that $|d\chi_m^c| \le 32m^{-1}z^{1/2}$. Use θ_m^c to denote the characteristic function for the support of $d\chi_m^c$.

Take the L^2 inner product of $(\chi_m^c \chi_{\diamond z})^2 \xi$ with the two sides of the equation in either the third bullet of (A-12) or (A-16) or (A-17) and use an integration by parts with (A-11) to see from these integrals that

$$\text{(A-23)}\quad \|\nabla(\chi_m^c \chi_{\diamond z} \xi_m)\|_2^2 \le c_0(\lambda^2 + c_0^{-10} z)\|\chi_m^c \chi_{\diamond z}\xi\|_2^2 + m^{-2} z \|\theta_m^c \chi_{\diamond z}\xi\|_2 + m_\Lambda^{-2} z \|\theta_{\diamond z}\xi\|_2^2 + c_0^6 z^{1/2} \|\chi_m^c \chi_{\diamond z}\xi\|_2.$$

The problematic terms on the right-hand side of (A-23) are those with $\|\theta_{\diamond z}\xi\|_2$ and $\|\theta_m^c \chi_{\diamond z}\xi\|_2$. The former is dealt with as follows: The top bullet in (A-20) asserts that $\|\theta_{\diamond z}\xi\|_2 \le m_\Lambda^{1/4}\|\chi_{\diamond z}\xi\|_2$. Hold on to this for the moment. The triangle inequality finds $\|\chi_{\diamond z}\xi\|_2 \le \|\chi_m^c \chi_{\diamond z}\xi\|_2 + \|(1-\chi_m^c)\chi_{\diamond z}\xi\|_2$, and thus $\|\chi_{\diamond z}\xi\|_2 \le \|\chi_m^c \chi_{\diamond z}\xi\|_2 + m^{-1/4}\|\chi_{\diamond z}\xi\|_2$ because the lower bullet in (A-20) is violated. Thus $\|\chi_{\diamond z}\xi\|_2 \le 2\|\chi_m^c \chi_{\diamond z}\xi\|_2$ and so the bound $\|\theta_{\diamond z}\xi\|_2 \le m_\Lambda^{1/4}\|\chi_{\diamond z}\xi\|_2$ implies that $\|\theta_{\diamond z}\xi\|_2 \le 2m_\Lambda^{1/4}\|\chi_m^c \chi_{\diamond z}\xi\|_2$. Meanwhile, the problematic term $\|\theta_m^c \chi_{\diamond z}\xi\|_2$ is bounded by $c_0 m^{-1/4}\|\chi_m^c \chi_{\diamond z}\xi\|_2$ because the lower bullet in (A-20) is violated.

Insert the bounds in the preceding paragraph for $\|\theta_{\diamond z}\xi\|_2$ and $\|\theta_m^c \chi_{\diamond z}\xi\|_2$ in (A-23) and use the fact that $m_\Lambda = 100c_0^4$ and $m = c_0 c_0^{4/3}$ to see that

$$\text{(A-24)}\quad \|\nabla(\chi_m^c \chi_{\diamond z}\xi_m)\|_2^2 \le c_0(1 + (z^{-1}\lambda^2 + c_0^{-4} + m^{-5/2})z)\|\chi_m^c \chi_{\diamond z}\xi_m\| + c_0 c_0^k,$$

with $k < c_0$. Consider now the left-hand side of (A-24). To this end, let $D \subset Y_{\diamond z}$ denote a transverse disk centered at a point in $\alpha^{-1}(0)$ with radius $2mz^{-1/2}$. The points in the support of $\chi_m^c \chi_{\diamond z}$ have distance at most $2mz^{-1/2}$ from $\alpha^{-1}(0)$, and so the Dirichlet inequality implies that the L^2–norm of $|\nabla(\chi_m^c \chi_{\diamond z}\xi_m)|$ over D is no less than $c_0^{-1}m^{-1}z^{1/2}$ times that of $\chi_m^c \chi_{\diamond z}\xi_m$ over D. This being the case, (A-24) has the following consequence: Assume that $\lambda \le c_0^{-1}m^{-1}z^{1/2}$. Then $c_0^{-1}m^{-2}z\|\chi_m^c \chi_{\diamond z}\xi_m\|_2^2 \le c_0 c_0^k$. This last bound leads directly to the desired upper bound on $\|\xi\|_2$.

Step 6 Steps 2–5 established Lemma A.3's claim about the L^2–norms of ϕ, b_0 and η_1. Granted this claim, take the L^2 inner product of both sides of the eigenvalue equation $\mathfrak{L}_{c,z}\mathfrak{b} = \lambda\mathfrak{b}$ with ξ and use the third bullet of (A-12), (A-16) and (A-17) with the bounds in Properties 1 and 2 to derive the bound $\|\nabla\xi\|_2^2 \le \lambda^2\|\xi\|_2^2 + c_0 c_0^k(z\|\xi\|_2^2 + 1)$ with $k \le c_0$. This last bound implies what Lemma A.3 asserts about the L^2–norm of $\nabla\xi$. □

Ad The vortex operator

This subsection constitutes a digression to supply various observations that are used subsequently to say more about eigenvectors of $\mathfrak{L}_{c,z}$ near the zero locus of α. The discussion here is given in four parts.

Part 1 Assume in what follows that $(A, \psi = (\alpha, \beta))$ obeys the constraints given in Section Ab. Fix a point $p \in Y$ on the zero locus of α in $Y_{\diamond z}$ or on one of the curves from the set $\bigcup_{p\in\Lambda}(\hat\gamma_p^+ \cup \hat\gamma_p^-)$ in a component of $Y - Y_{\diamond z}$ that contains zeros of α. In the former case, set $c_1 = 20c_0$ and in the latter case, set $c_1 = c_0^4$. Fix an isometric, $\mathbb{C}$–linear identification between $\mathrm{Ker}(\hat a)|_p$ and $\mathbb{C}$. With this identification understood, let φ_z denote the map from $\mathbb{C}$ to Y that is obtained by composing first multiplication by $z^{-1/2}$ and then the metric's exponential map. Use (A_z, α_z) to denote the φ_z–pullback of (A, α). Use ϑ_z in what follows to denote the (A_z, α_z) version of (3-4)'s operator ϑ. Of particular interest is this operator on concentric disks about the origin in $\mathbb{C}$ with radius c_1 or less.

The analysis of ϑ_z uses the following consequence of Properties 1 and 2 in Section Ab: The pair (A_z, α_z) comes close to solving (2-8)'s vortex equations on the radius c_1 disk centered at the origin in $\mathbb{C}$ in the sense that

$$\text{(A-25)} \qquad \left|{*F_{A_z}} + i(1 - |\alpha_z|^2)\right| + |\bar\partial_{A_z}\alpha_z| \le c_0\left(c_0^{-1}\left|1 - |\alpha_z|^2\right| + c_0^{-10}\right).$$

The ramifications with regards to ϑ_z stem from the fact that the right-most term in (3-6) is bounded by $c_0 c_0^{-1}$ on a disk about the origin of radius up to $20c_1$ if $z \ge c_0^{10}$. The essential point here is that $|\bar\partial_{A_z}\alpha_z|$ is relatively small on a large radius disk about the origin. This suggests in particular that $\vartheta_z\vartheta_z^\dagger$ is uniformly positive in a suitable sense because the remaining terms in the formula for $\vartheta_z\vartheta_z^\dagger$ have the form of a covariant Laplacian plus a zero-order, nonnegative term. The constructions that follow are used to make a precise statement to this effect. These constructions assume that $z \ge \kappa_c c_0^{10}$ with κ_c larger than the versions of κ_{c_0} that appear in Lemmas A.2 and A.3.

Part 2 Suppose that $k \in \{0, 1, \dots, 7\}$ and that α lacks zeros in the open, concentric annulus in the transverse disk centered at p with respective inner and outer radii equal to $(c_1 + k c_0)\mathrm{z}^{-1/2}$ and $(c_1 + (k+3)c_0)\mathrm{z}^{-1/2}$. Zeros of α on the transverse disk through p with radius $c_1 \mathrm{z}^{-1/2}$ correspond via φ_{z} to the zeros of α_{z} on the corresponding radius c_1 disk about the origin in $\mathbb{C}$. In any event, use $\mathcal{A} \subset \mathbb{C}$ to denote the concentric annulus with inner radius $c_1 + (k+1)c_0$ and outer radius $c_1 + (k+2)c_0$.

It is a consequence of Property 4 that $|\alpha_{\mathrm{z}}| \geq 1 - c_0^{-10}$ in $\mathcal{A}$. This being the case, there is an isomorphism over $\mathcal{A}$ between $\varphi_{\mathrm{z}}^* E$ and $\mathcal{A} \times \mathbb{C}$ that maps α_{z} to $|\alpha_{\mathrm{z}}|$ with the latter viewed at any given point as a complex number with zero imaginary part. Let θ_* denote the product connection on the trivial line bundle $\mathcal{A} \times \mathbb{C}$. This isomorphism pulls back A_{z} as $\theta_* + a_{\mathrm{z}}$, where a_{z} is an $i\mathbb{R}$–valued 1–form on $\mathcal{A}$. The second bullet of Property 1 with Property 4 imply that $|a_{\mathrm{z}}| \leq c_0(c_0^{-9} + \mathrm{z}^{-1/2})$.

Fix a nonnegative, radial function on $\mathbb{C}$ which is equal to 1 where the distance to the origin is less than $c_1 + \left(k + \frac{4}{3}\right)c_0$ and equal to zero where the distance to the origin is greater than $c_1 + \left(k + \frac{5}{3}\right)c_0$. Choose a function whose derivative is bounded in absolute value by $10c_0^{-1}$. Denote the chosen function by χ_*.

Define a complex hermitian line bundle $E_{\mathrm{z}} \to \mathbb{C}$ by identifying it with E on the radius $c_1 + \left(k + \frac{5}{4}\right)c_0$ disk about the origin in $\mathbb{C}$ and with the product bundle on the complement of the radius $c_1 + (k+1)c_0$ disk about the origin in $\mathbb{C}$. Use the isomorphism between $\varphi_{\mathrm{z}}^* E|_{\mathcal{A}}$ and $\mathcal{A} \times \mathbb{C}$ to define the necessary clutching function. A unitary connection, $A_{\mathrm{z}*}$, is defined on E_{z} by setting $A_{\mathrm{z}*} = A_{\mathrm{z}}$ on the disk about the origin in $\mathbb{C}$ with radius $c_1 + \left(k + \frac{5}{4}\right)c_0$ and by setting $A_{\mathrm{z}*} = \theta_* + \chi_* a_{\mathrm{z}}$ on the complement of the disk about the origin with radius $c_1 + (k+1)c_0$. Use $\alpha_{\mathrm{z}*}$ to denote the section of E_{z} given by α over the radius $c_1 + \left(k + \frac{5}{4}\right)c_0$ disk centered at the origin and given by $(1 - \chi_*) + \chi_* |\alpha_{\mathrm{z}}|$ over the complement of the radius $c_1 + (k+1)c_0$ disk. The connection $A_{\mathrm{z}*}$ is flat and $\alpha_{\mathrm{z}*}$ has norm 1 and is also $A_{\mathrm{z}*}$–covariantly constant on the complement of the radius $c_1 + \frac{2}{3}c_0$ disk about the origin in $\mathbb{C}$. The pair $(A_{\mathrm{z}*}, \alpha_{\mathrm{z}*})$ also comes close to solving the vortex equations on the whole of $\mathbb{C}$ in the sense that (A-25) still holds.

Part 3 Use n_* to denote the integral of $\frac{i}{2\pi} F_{A_{\mathrm{z}*}}$ over $\mathbb{C}$. This is equal to 1 if the point p is in $Y_{\diamond \mathrm{z}}$, but can be greater than 1 if $p \in \bigcup_{\mathrm{p} \in \Lambda} (\hat{\gamma}_{\mathrm{p}}^+ \cup \hat{\gamma}_{\mathrm{p}}^-)$. It follows from (A-25) with Properties 1 and 2 that n_* is a positive integer.

The local Euler number of the zeros of α_{z*} sum to n_* because α_{z*} has norm 1 and is A_{z*}–covariantly constant on the complement of a compact set in $\mathbb{C}$. The following lemma says more about these zeros:

Lemma A.4 *There exists $\kappa > 100$ such that if $c_0 \geq \kappa$ and $z \geq \kappa c_0^{10}$, then $n_* \leq \kappa c_0^4$. Moreover there exists an open set in $\mathbb{C}$ with the following three properties*:

- *The set can be covered by n_* disks of radius 4.*
- *$|\alpha_z| \geq \kappa^{-1}$ on its complement.*
- *The sum of the local Euler numbers of the zeros of α_z in each component of this set is nonzero and positive.*

Proof The bound on n_* follows from Properties 3 and 4, and the other bullets follow from Property 5 using Lemma 2.9. □

Fix a set in $\mathbb{C}$ that obeys the three bullets of Lemma A.4. Denote the set of components of this set by Z_p. Given $U \in Z_p$, use $\mathfrak{m}_U \in \{1, 2, \dots, n_*\}$ to denote the sum of the local Euler numbers of α_z on U.

Part 4 Use ϑ_{z*} to denote the (A_{z*}, α_{z*}) version of (3-4)'s operator ϑ. The upcoming Lemma A.5 lists some salient features of ϑ_{z*}. This lemma uses $L^2(\mathbb{C}; \mathbb{C} \oplus E_z)$ to denote the completion of the vector space of smooth and compactly supported sections of the bundle $\mathbb{C} \times (\mathbb{C} \oplus E_z)$ using the norm whose square sends a given compactly supported section $\mathfrak{z} = (x, \iota)$ to the integral of $|\mathfrak{z}|^2$. This norm is denoted by $\|\cdot\|_2$. Use $\nabla\mathfrak{z}$ to denote $(dx, \nabla_{A_z}\iota)$. Lemma A.5 uses $L^2_1(\mathbb{C}; \mathbb{C} \oplus E_z)$ to denote the completion of this same space using the inner product whose square sends the given element $\mathfrak{z}$ to the integral over $\mathbb{C}$ of $|\nabla\mathfrak{z}|^2 + |\mathfrak{z}|^2$. This defining norm for $L^2_1(\mathbb{C}; \mathbb{C} \otimes E_z)$ is denoted by $\|\cdot\|_{2,1}$.

Lemma A.5 *There exists $\kappa > 100$ such that what follows is true if $c_0 \geq \kappa$ and $z \geq \kappa c_0^{10}$*:

- *The operator ϑ_{z*} extends as a bounded, Fredholm operator from $L^2_1(\mathbb{C}; \mathbb{C} \oplus E_z)$ to $L^2(\mathbb{C}; \mathbb{C} \oplus E_z)$ with index equal to n_* and with trivial cokernel.*
- *If $\mathfrak{z} \in L^2_1(\mathbb{C}; \mathbb{C} \oplus E_z)$, then $\|\vartheta^\dagger_{z*}\mathfrak{z}\|_2 \geq \kappa^{-1}\|\mathfrak{z}\|_{2,1}$, and if $\mathfrak{z}$ is L^2–orthogonal to the kernel of ϑ_{z*}, then $\|\vartheta_{z*}\mathfrak{z}\| \geq \kappa^{-1}\|\mathfrak{z}\|_{2,1}$.*
- *Square-integrable elements in the kernel of ϑ_{z*} are smooth and in $L^2_1(\mathbb{C}; \mathbb{C} \oplus E_z)$.*

- *If $\mathfrak{z}$ is an $L^2(\mathbb{C};\mathbb{C}\oplus E_{\mathrm{z}})$ and in the kernel of $\vartheta_{\mathrm{z}*}$, then*

$$|\mathfrak{z}| \le \kappa \sum_{U\in Z_p} \mathfrak{m}_U e^{-\mathrm{dist}(\cdot,U)/2}.$$

Moreover, if $U\in Z_p$ is a component with distance greater than κ from the others and such that $\mathfrak{m}_U = 1$, then the $L^2(\mathbb{C};\mathbb{C}\oplus E_{\mathrm{z}})$–kernel of $\vartheta_{\mathrm{z}*}$ has a nonzero element with the properties listed below. The list uses $\mathfrak{z}_U$ to denote this element.

(a) $|\mathfrak{z}_U| \le \kappa e^{-\mathrm{dist}(\cdot,U)/2}\|z_U\|_2$.

(b) *Any $L^2(\mathbb{C};\mathbb{C}\oplus E_{\mathrm{z}})$ element in the kernel of $\vartheta_{\mathrm{z}*}$ can be written as $x\mathfrak{z}_U+\mathfrak{z}'$ with $x\in\mathbb{C}$ and with $\mathfrak{z}'$ such that $|\mathfrak{z}'| \le \sum_{U'\in Z_p-\{U\}} \mathfrak{m}_{U'} e^{-\mathrm{dist}(\cdot,U')/2}$.*

Proof The fact that $\vartheta_{\mathrm{z}*}$ defines a bounded map from $L^2_1(\mathbb{C};\mathbb{C}\oplus E_{\mathrm{z}})$ to $L^2(\mathbb{C};\mathbb{C}\oplus E_{\mathrm{z}})$ follows from the appearance of the L^2–norm of the covariant derivative in the definition of $L^2_1(\mathbb{C};\mathbb{C}\oplus E_{\mathrm{z}})$. To prove it Fredholm, it is necessary to prove that the kernel in $L^2_1(\mathbb{C};\mathbb{C}\oplus E_{\mathrm{z}})$ is finite-dimensional, that the range is closed and that the cokernel is finite-dimensional. The finite-dimensional kernel and the closed range follow as a consequence of the Rellich lemma with the verification of the following:

> *There exists $\varepsilon>0$ and $R\ge 1$ such that $\|\vartheta_{\mathrm{z}*}\mathfrak{z}\|_2 \ge \varepsilon\|\mathfrak{z}\|_2$ if the support of $\mathfrak{z}$ has compact support in the complement of the radius R disk in $\mathbb{C}$ about the origin.*

This follows by virtue of the fact that $\vartheta^\dagger_{\mathrm{z}*}\vartheta_{\mathrm{z}*}(x,\iota) = \bigl((-\partial\bar\partial+\frac{1}{2})x, (-\partial_{A_{\mathrm{z}*}}\bar\partial_{A_{\mathrm{z}*}}+\frac{1}{2})\iota\bigr)$, where $A_{\mathrm{z}*}$ is flat, and $\alpha_{\mathrm{z}*}$ has norm 1 and is also $A_{\mathrm{z}*}$–covariantly constant.

The fact that the range is closed implies that the cokernel is isomorphic to the kernel of the adjoint. Standard elliptic regularity identifies the latter with the kernel of $\vartheta^\dagger_{\mathrm{z}*}$. The fact that the latter is trivial can be seen using the $(A_{\mathrm{z}*},\alpha_{\mathrm{z}*})$ version of (3-6). This is done by invoking the bounds in (A-25) after commuting covariant derivatives to equate $\bar\partial_{A_0}\partial_{A_0}$ and $\partial_{A_{\mathrm{z}*}}\bar\partial_{A_{\mathrm{z}*}}+\frac{1}{2}+\mathfrak{e}$ with $|\mathfrak{e}| \le c_0 c_0^{-1}\bigl(\bigl|1-|\alpha_{\mathrm{z}*}|^2\bigr|+1\bigr)$.

The fact that the dimension of the kernel is n_* can be seen by comparing $\vartheta_{\mathrm{z}*}$ with the version of ϑ that is defined by a pair (A_0,α_0) that obeys (2-8)'s vortex equations and is such that $1-|\alpha_0|^2$ is integrable and with integral equal to $2\pi n_*$. Such a comparison can be made by using what is said in Section 2a of [20] to construct a $[0,1]$–parametrized path of pairs in $\mathrm{Conn}(E_{\mathrm{z}})\times C^\infty(\mathbb{C};E_{\mathrm{z}})$ that starts at $(A_{\mathrm{z}*},\alpha_{\mathrm{z}*})$, ends at such a solution to (2-8) and is such that each member of the family defines a Fredholm version of ϑ. The construction of such a path amounts to little more than an exercise with cut-off functions and so no more will be said.

Granted the first bullet, the assertions of the second bullet are straightforward consequences of two facts, the first being that ϑ_{z*} is Fredholm with trivial cokernel and the second being (3-6). As for the third bullet, standard elliptic regularity arguments prove that the elements in the kernel of ϑ_{z*} are smooth. Meanwhile, the fact that the L^2–kernel of ϑ_{z*} coincides with its L^2_1–kernel follows from what was said above about $\vartheta_{z*}^\dagger\vartheta_{z*}$ where A_{z*} is flat, α_{z*} has norm 1 and α_{z*} is A_{z*}–covariantly constant.

The assertions of the fourth bullet can be proved using the same sorts of arguments as in Part 5 from Section 2a in [20]. The modifications to these arguments are straightforward given that the properties listed in Section Ab imply that (A_{z*}, α_{z*}) looks very much like a solution to the vortex equations with $1-|\alpha_0|^2$ integrable and with integral equal to $2\pi n_*$. Note in particular what is said by (A-25). Note that Part 5 of Section 2a of [20] states a stronger version of what is asserted by the fourth bullet for the version of ϑ that is defined using just such a solution to the vortex equations. As nothing fundamentally new is needed for the arguments in the case of ϑ_{z*}, the details of the proof of the fourth bullet are omitted. □

Ae The definition of Ker_ϑ and Π_ϑ

Assume here that $\mathfrak{c} = (A, \psi)$ from $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ obeys Properties 1–5 in Section Ab as defined with parameters c_0 and z, with c_0 and z chosen so as to satisfy the requirements of Lemmas A.2–A.5. Parts 1 and 2 of this subsection use versions of ϑ_{z*} to construct a complex line bundle over each component of $\alpha^{-1}(0)$ and a complex vector bundle over the sets that form an open cover of certain components of $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$. This bundle is denoted in each case by Ker_ϑ. Part 3 defines a $\mathbb{C}$–linear homomorphism from the space of sections of $\bar{K} \oplus E \to Y$ to the space of sections of each version of Ker_ϑ, this denoted by $\Pi_\vartheta(\cdot)$. Part 4 defines a norm on the direct sum of these spaces of sections. This map is used to say more about eigenvectors of the operator $\mathfrak{L}_{\mathfrak{c},z}$.

The construction of the associated complex line bundle and the associated homomorphism from $C^\infty(Y; \bar{K} \oplus E)$ for a component of $\alpha^{-1}(0)$ in $Y_{\diamond z}$ mimics constructions in Section 3 of [20]. The construction for a curve in $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$ mimics constructions in Section 5 of [20]. The definition of the norm also mimics what is done in Section 5 of [20].

Part 1 Use γ to denote a component of the zero locus of α in $Y_{\diamond z}$. Given $p \in \gamma$, define the pair (A_{z*}, α_{z*}) on $\mathbb{C}$. The L^2–kernel of the corresponding operator ϑ_{z*}

is 1–dimensional since the integral of $\frac{i}{2\pi} F_{\hat{A}}$ over the radius $c_1 \mathrm{z}^{-1/2}$ transverse disk with center p is equal to 1. (The number c_1 is defined in Part 1 of Section Ad; it is $20c_0$ for this version of γ.) The association to each point in γ of the L^2–kernel of the corresponding version of $\vartheta_{\mathrm{z}*}$ defines a complex line bundle over γ, this being Ker_ϑ.

Part 2 Use γ now to denote an element in $\bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_{\mathrm{p}}^{+} \cup \hat{\gamma}_{\mathrm{p}}^{-})$. Consider first the case when there are no zeros of α on the nearby boundary component of $Y_{\diamond\mathrm{z}}$. The associated version of Ker_ϑ is the zero-dimensional bundle if the corresponding component of $Y - Y_{\diamond\mathrm{z}}$ has no zeros of α. Suppose next that this component has zeros of α. It follows from item (c) of the second bullet of Property 3 that any given $p \in \gamma$ version of the pair $(A_{\mathrm{z}*}, \alpha_{\mathrm{z}*})$ can be defined using $k = 0$. This understood, let n_* denote the integral over $\mathbb{C}$ of $\frac{i}{2\pi}$ times the curvature 2–form of $A_{\mathrm{z}*}$. This positive integer does not depend on the chosen point in γ. Lemma A.5 asserts that any given $p \in \gamma$ version of $\vartheta_{\mathrm{z}*}$ has L^2–kernel dimension equal to n_*. As p varies in γ, these L^2–kernels define a rank n_* complex vector bundle over γ. This is the bundle Ker_ϑ.

Suppose next that $\gamma \in \bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_{\mathrm{p}}^{+} \cup \hat{\gamma}_{\mathrm{p}}^{-})$ and that the nearby boundary component of $Y_{\diamond\mathrm{z}}$ has zeros of α. Fix $p \in \gamma$ and let $D_0 \subset Y$ denote for the moment the transverse disk centered at p with radius $(c_0^4 + 10c_0)\mathrm{z}^{-1/2}$. Granted that $c_0 \geq c_0$, use the second bullet of Property 3 with the formula for v in (1-3) to find $k \in \{0, 1, \dots, 7\}$ such that the following is true: the concentric, closed annulus in D_0 with inner radius $(c_0^4 + k c_0)\mathrm{z}^{-1/2}$ and with outer radius $(c_0^4 + (k+3)c_0)\mathrm{z}^{-1/2}$ has no zeros of α. To say more about why this is so, suppose that ν is a connected, closed segment of an integral curve of v with each endpoint having distance either $c_0^4\mathrm{z}^{-1/2}$ or $(c_0^4 + 10c_0)\mathrm{z}^{-1/2}$ from γ. The formula in (1-3) implies that the ϕ angle changes monotonically on ν with total change being much less than 2π if $c_v > c_0$.

If $p \in \gamma$ and if $k \in \{0, 1, \dots, 7\}$ and there are no zeros of α in the transverse disk centered at p with distance from p between $(c_0^4 + k c_0)\mathrm{z}^{-1/2}$ and $(c_0^4 + (k+3)c_0)\mathrm{z}^{-1/2}$, then such is the case for any transverse disk centered at all points in some open neighborhood of p in γ. This being the case, γ can be written as the union of 8 open sets, $\{\gamma_k\}_{k=0,1,\dots,7}$, where the γ_k corresponds to the subset of points in γ where k has the property just described. The formula for v in (1-3) implies that γ_k will have at most two components.

Fix $k \in \{0, \dots, 7\}$ and $p \in \gamma_k$. Use the chosen value for k to construct the pair $(A_{\mathrm{z}*}, \alpha_{\mathrm{z}*})$ and the operator $\vartheta_{\mathrm{z}*}$. The association to a point $p \in \gamma_k$ of the corresponding

L^2–kernel of ϑ_{z*} defines a finite-rank, complex vector bundle over γ_k. This bundle is Ker_ϑ.

Part 3 Let γ_* denote either a component in $Y_{\diamond z}$ of the zero locus of α or else a given $\gamma \in \bigcup_{p\in\Lambda}(\hat{\gamma}_p^+ \cup \hat{\gamma}_p^-)$ and $k \in \{0, \dots, 7\}$ version of γ_k. The associated $\mathbb{C}$–linear map $\Pi_\vartheta\colon C^\infty(Y; \bar{K} \oplus E) \to C^\infty(\gamma_*; \mathrm{Ker}_\vartheta)$ is defined as follows: Fix $p \in \gamma_*$ and reintroduce the map φ_z and the function χ_* from Part 2 of Section Ad that is used to define the corresponding pair (A_{z*}, α_{z*}). If $\mathfrak{f}_0$ is a section of $\bar{K} \oplus E$, then $\chi_*\varphi_z^*(\mathfrak{f}_0)$ defines an element in $C^\infty(\mathbb{C}; \mathbb{C} \oplus E_{z*})$ with compact support. The L^2–orthogonal projection of $\chi_*\varphi_z^*(\mathfrak{f}_0)$ to the L^2–kernel of ϑ_{z*} is the value of the section $\Pi_\vartheta(\mathfrak{f}_0)$ at p.

Introduce Θ_* to denote the set whose elements are the components of α's zero locus in $Y_{\diamond z}$ and the various $\gamma \in \bigcup_{p\in\Lambda}(\hat{\gamma}_p^+ \cup \hat{\gamma}_p^-)$ and $k \in \{0, 1, \dots, 7\}$ version of γ_k with it understood in the latter case that $\gamma = \gamma_{k=0}$ and $\gamma_k = \varnothing$ for $k > 0$ if the nearby boundary component of $Y_{\diamond z}$ lacks zeros of α. The map Π_ϑ is viewed in what follows as a $\mathbb{C}$–linear map from $C^\infty(Y; \bar{K} \oplus E)$ to $\bigoplus_{\gamma_*\in\Theta_*} C^\infty(\gamma_*; \mathrm{Ker}_\vartheta)$.

Part 4 In this last part of the subsection, we define a version of the L^2–norm on $\bigoplus_{\gamma_*\in\Theta_*} C^\infty(\gamma_*; \mathrm{Ker}_\vartheta)$. To this end, let $\mathfrak{q}$ denote an element in this vector space. The corresponding norm is denoted by $\|\mathfrak{q}\|_2$. The definition that follows writes a given $\gamma_* \in \Theta_*$ component of $\mathfrak{q}$ as $\mathfrak{q}_{\gamma_*}$ and it writes the integral over γ_* of $|\mathfrak{q}_{\gamma_*}|^2$ as $\|\mathfrak{q}_{\gamma_*}\|_2^2$. Granted this notation, $\|\mathfrak{q}\|_2^2 = \sum_{\gamma_*\in\Theta_*} \|\mathfrak{q}_{\gamma_*}\|_2^2$.

Af Rewriting $\mathfrak{L}_{(\cdot)}$

What follows in this subsection is used subsequently to bring what is said by Sections Ad and Ae into the $\mathfrak{L}_{\mathfrak{c},z}$ story. It is necessary to start by introducing some new notation. The annihilator of v in T^*Y is defined to be the 2–dimensional subbundle of T^*Y that is orthogonal to the 1–form $\hat{a}$. This subbundle is dual to $\mathrm{Ker}(\hat{a})$. The almost complex structure J splits its complexification as $K \oplus \bar{K}$ with it understood that K annihilates the $-i$ eigenbundle of J's action on the complexification of the kernel of $\hat{a}$.

Introduce $I_\mathbb{C}$ to denote the product bundle $Y \times \mathbb{C}$. Write the complexification of the direct sum of the line $\mathbb{R}\hat{a} \subset T^*Y$ with $I_\mathbb{R}$ as $I_\mathbb{C} \oplus \bar{I}_\mathbb{C}$ with it understood that the projection to the $I_\mathbb{C}$ factor of a point $(b_0\hat{a}, \phi) \in \mathbb{R}\hat{a} \oplus I_\mathbb{R}$ is $-b_0 + i\phi$.

Introduce $\mathbb{V}_0$ to denote $\bar{K} \oplus E$ and $\mathbb{V}_1$ to denote $I_{\mathbb{C}} \oplus EK^{-1}$. Define an $\mathbb{R}$–linear isomorphism from $iT^*Y \oplus \mathbb{S} \oplus iI_{\mathbb{R}}$ to $\mathbb{V}_0 \oplus \mathbb{V}_1$ as follows: Let (b, η, ϕ) denote a given point in $iT^*Y \oplus \mathbb{S} \oplus iI_{\mathbb{R}}$. Write b as $b_0\hat{a} + b^{\perp}$ and use q to denote the orthogonal projection of $b^{\perp}$ to the subbundle $\bar{K}$ of the complexification of the annihilator of v. Use p to denote $(-b_0 + i\phi)$ Write η as (η_0, η_1) using the identification of $\mathbb{S}$ with $E \oplus EK^{-1}$. The desired isomorphism sends (b, η, ϕ) to $((q, \eta_0), (p, \eta_1)) \in \mathbb{V}_0 \oplus \mathbb{V}_1$. This isomorphism is used in what follows to write any given section of $iT^*Y \oplus \mathbb{R} \oplus iI_{\mathbb{R}}$ as a section of $\mathbb{V}_0 \oplus \mathbb{V}_1$ and vice versa.

To continue setting notation, suppose that q is a section of $\bar{K}$. Use $\partial^K q$ to denote the projection of the covariant derivative of q to the $\bar{K} \otimes K$ summand in $\bar{K} \otimes (T^*Y)_{\mathbb{C}}$. Meanwhile, use $(\nabla q)_v$ to denote the corresponding projection to the $\bar{K} \otimes (\mathbb{C}\hat{a})$ summand. When p denotes a section of $I_{\mathbb{C}}$, use $\bar{\partial}^K p$ and $(\nabla p)_v$ to denote the respective projections of dp to the $\bar{K}$ and $\mathbb{C}\hat{a}$ summands of $(T^*Y)_{\mathbb{C}}$. When $\eta = (\eta_0, \eta_1)$ denotes a section of $\mathbb{S} = E \oplus EK^{-1}$, write the directional covariant derivatives of η_0 and η_1 along v as $(\nabla_A \eta_0)_v$ and $(\nabla_A \eta_1)_v$, write the $\bar{K}$ part of the covariant derivative of η_0 as $\bar{\partial}^K_A \eta_0$ and write the K part of the covariant derivative of η_1 as $\partial^K_A \eta_1$.

With this notation understood, the operator $\mathfrak{L}_{\mathfrak{c},v}$ can be viewed as an operator mapping $C^\infty(Y; \mathbb{V}_0 \oplus \mathbb{V}_1)$ to itself in the manner of (3.13) and (3.14) in [20]. Viewed in this light, the operator is denoted by $\mathfrak{L}_{\mathbb{V}}$. Let $\mathfrak{f}$ denote a given section of $C^\infty(Y; \mathbb{V}_0 \oplus \mathbb{V}_1)$. To write $\mathfrak{L}_{\mathbb{V}}\mathfrak{f}$, first write the $\mathbb{V}_0 = \bar{K} \oplus E$ component of $\mathfrak{f}$ as (q, η_0) and the $\mathbb{V}_1 = I_{\mathbb{C}} \oplus EK^{-1}$ component as (p, η_1). The $\bar{K}$ and E summands of the $\mathbb{V}_0$ component of $\mathfrak{L}_{\mathbb{V}}\mathfrak{f}$ are

(A-26)
- $i(\nabla q)_v - 2i\big(-\bar{\partial}^K p + \frac{1}{\sqrt{2}}\mathrm{z}^{1/2}\bar{\alpha}\eta_1\big) - \sqrt{2}\, i\mathrm{z}^{1/2}\bar{\eta}_0\beta + \mathfrak{t}_{0q}q,$
- $i(\nabla_A \eta_0)_v - 2i\big(-\bar{\partial}^K_A \eta_1 + \frac{1}{\sqrt{2}}\mathrm{z}^{1/2}\alpha p\big) - \sqrt{2}\, i\mathrm{z}^{1/2}\bar{q}\beta + \mathfrak{t}_{0\eta}\eta_1;$

and the respective $I_{\mathbb{C}}$ and $K^{-1}E$ summands of the $\mathbb{V}_1$ component of $\mathfrak{L}_{\mathbb{V}}\mathfrak{f}$ are

(A-27)
- $-i(\nabla p)_v + 2i\big(\bar{\partial}^K q + \frac{1}{\sqrt{2}}\mathrm{z}^{1/2}\bar{\alpha}\eta_0\big) - \sqrt{2}\, i\mathrm{z}^{1/2}\bar{\eta}_1\beta + \mathfrak{t}_{1q}q + \mathfrak{t}_{1p}p,$
- $-i(\nabla_A \eta_1)_v + 2i\big(\bar{\partial}^K_A \eta_0 + \frac{1}{\sqrt{2}}\mathrm{z}^{1/2}\alpha q\big) + \sqrt{2}\, i\mathrm{z}^{1/2}\bar{p}\beta + \mathfrak{t}_{1\eta}\eta_0.$

Here, each $\mathfrak{t}_{**}$ denotes an $\mathbb{R}$–linear homomorphism between summands of $\mathbb{V}_0 \oplus \mathbb{V}_1$ that depends only on the metric and has norm bounded by c_0.

The description of the $\mathbb{V}_1$ component of $\mathfrak{L}_{\mathbb{V}}\mathfrak{f}$ given in (A-27) proves sufficient for what is to come. The description in (A-26) of the $\mathbb{V}_0$ component requires some additional rewriting. To begin this task, use γ now to denote a small length open segment of α's zero locus in $Y_{\diamond z}$ or a curve from $\bigcup_{p \in \Lambda}(\hat{\gamma}_p^+ \cup \hat{\gamma}_p^-)$ whose corresponding component of $Y - Y_{\diamond z}$ has a zero of α. Fix a point $p \in \gamma$ and use the coordinates from Part 4 of Section Aa to parametrize a neighborhood of γ in Y. This neighborhood is denoted for now by T. Nothing is lost by taking the $t = 0$ point to be the value of γ's affine parameter at p. The segment γ is assumed in what follows to be parametrized by $t \in (-\rho, \rho)$ with ρ a constant that will be specified in the applications to come. This constant ρ in any event obeys $\rho \in (c_0 z^{-1/2}, c_0^{-1})$.

Each constant t slice of $(-\rho, \rho)$ is the intersection between the transverse disk through the corresponding point in γ and the tubular neighborhood of γ. An identification of $E|_T$ with E's restriction to the $t = 0$ slice of T writes the directional covariant derivative along ∂_t of A as $(\nabla_A)_{\partial_t} = \partial_t + a_{A0}$ where a_{A0} is an $i\mathbb{R}$–valued function on T. With an identification of this sort chosen, then the terms $(\nabla q)_v$ and $(\nabla_A \eta_0)_v$ in (A-26) can be written using (A-6) as

(A-28)
$$\bullet\ \frac{\partial}{\partial t}q + 2i(\nu(z - x_\gamma) + \mu(\bar{z} - \bar{x}_\gamma))\frac{\partial}{\partial z}q - 2i(\nu(\bar{z} - \bar{x}_\gamma) + \bar{\mu}(z - x_\gamma))\frac{\partial}{\partial \bar{z}}q + \mathfrak{r}_q\, dq,$$
$$\bullet\ \frac{\partial}{\partial t}\eta_0 + a_{A0}\eta_0 + 2i(\nu(z - x_\gamma) + \mu(\bar{z} - \bar{x}_\gamma))\partial_A\eta_0 - 2i(\nu(\bar{z} - \bar{x}_\gamma) + \bar{\mu}(z - x_\gamma))\bar{\partial}_A\eta_0 + \mathfrak{r}_\eta \cdot \nabla_A q,$$

where the notation is such that ∂_A is the covariant version of $\frac{\partial}{\partial z}$ and $\bar{\partial}_A$ is the covariant version of $\frac{\partial}{\partial \bar{z}}$. What are denoted by $\mathfrak{r}_q$ and $\mathfrak{r}_\eta$ obey $|\mathfrak{r}_q| + |\mathfrak{r}_\eta| \le c_0|z|(z^{-1/2} + |z|)$. Meanwhile, the terms just to the right of $(\nabla q)_v$ and $(\nabla_A \eta_0)v$ in (A-26) can be written as

(A-29)
$$\bullet\ -2i\big(-\bar{\partial}^K p + \tfrac{1}{\sqrt{2}}z^{1/2}\bar{\alpha}\eta_1\big) = -2i\big(-\bar{\partial} p + \tfrac{1}{\sqrt{2}}z^{1/2}\bar{\alpha}\eta_1\big) + \mathfrak{e}_q \cdot \nabla p,$$
$$\bullet\ -2i\big(-\partial_A^K \eta_1 + \tfrac{1}{\sqrt{2}}z^{1/2}\alpha p\big) = -2i\big(-\partial_A \eta_1 + \tfrac{1}{\sqrt{2}}z^{1/2}\alpha p\big) + \mathfrak{e}_\eta \cdot \nabla_A \eta_1,$$

where $|\mathfrak{e}_q| + |\mathfrak{e}_\eta| \le c_0|z|^2$. The remaining terms in (A-26) can be written as

(A-30)
$$\bullet\ -\sqrt{2}\,i z^{1/2}\bar{\eta}_0\beta + 2\nu q + 2\mu\bar{q} + \mathfrak{w} \cdot q,$$
$$\bullet\ -\sqrt{2}\,i z^{1/2}\bar{q}\beta + \mathfrak{t}_{0\eta}\eta_1,$$

where $|\mathfrak{w}| \le c_0|z|$ and $\mathfrak{t}_{0\eta} \le c_0$.

Ag The operator $\mathfrak{L}_{\mathbb{V}}$ and Π_ϑ

The next lemma hints at the role played by Π_ϑ as it talks about the Π_ϑ of the $\mathbb{V}_0$ part of an eigenvector of $\mathfrak{L}_{\mathbb{V}}$. This lemma and subsequent discussions abuse notation to some extent by using Π_ϑ to denote two maps to $\bigoplus_{\gamma_* \in \Theta_*} \mathrm{Ker}_\vartheta$. The first is Section Ae's map from $C^\infty(Y; \mathbb{V}_0)$ and the second is the map from $C^\infty(Y; \mathbb{V}_0 \oplus \mathbb{V}_1)$ that is obtained from Section Ae's map by first projecting to the $\mathbb{V}_0$ summand.

Lemma A.6 *There exists $\kappa \geq 1$ and, given $c_0 \geq \kappa$, there exists $\kappa_{c_0} \geq \kappa$ with the following significance: Fix $\mathrm{z} \geq \kappa_{c_0} c_0^{10}$. Suppose that $\mathfrak{c} = (A, \psi)$ obeys the corresponding version of Properties 1–5 in Section Ab. The assumptions of Lemmas A.2–A.5 are satisfied, and, this understood, let $\mathfrak{f}$ denote an eigenvector of the operator $\mathfrak{L}_{\mathbb{V}}$ with eigenvalue bounded in absolute value by $c_0^{-\kappa}\mathrm{z}^{1/2}$. Then $\|\Pi_{\vartheta}\mathfrak{f}\|_2 \geq (1-\kappa c_0^{-1})\|\mathfrak{f}\|_2$.*

Proof Choose c_0 and z so that Lemmas A.2–A.5 can be invoked. Write $\mathfrak{f}$ as $(\mathfrak{f}_0, \mathfrak{f}_1)$. Lemma A.3 finds $\|\mathfrak{f}_1\|_2 \leq c_0 c_0^k \mathrm{z}^{-1/2}\|\mathfrak{f}\|_2$ with $k \leq c_0$, so $\mathfrak{f}_0$ accounts for most of the L^2–norm of $\mathfrak{f}$. Meanwhile, the second bullet of Lemma A.2 asserts that the L^2–norm of $\mathfrak{f}$ on the set of points with distance $2c_0\mathrm{z}^{-1/2}$ or more from $\alpha^{-1}(0)$ is bounded by $c_0 c_0^{-1}\|\mathfrak{f}\|_2$. As a consequence, the bulk of the L^2–norm of $\mathfrak{f}_0$ is accounted for by its L^2–norm on the radius $2c_0^{-1}\mathrm{z}^{-1/2}$ tubular neighborhood of α's zero locus. The contribution to the L^2–norm from this part of Y is analyzed in the four steps that follow.

Step 1 Reintroduce the set Θ_* from Part 3 of Section Ae and let γ_* denote a given element in Θ_*. This is to say that γ_* is either a component of α's zero locus in $Y_{\diamond\mathrm{z}}$ or some $\gamma \in \bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_{\mathrm{p}}^+ \cup \hat{\gamma}_{\mathrm{p}}^-)$ and $k \in \{0, \dots, 7\}$ version of γ_k. Each $p \in \gamma_*$ has an associated version of the map φ_{z} and function χ_* on $\mathbb{C}$ as described in Part 2 of Section Ad. In particular, the assignment to each point in γ_* of the L^2–norm over $\mathbb{C}$ of the corresponding version of $\chi_*\varphi_{\mathrm{z}}^*(\mathfrak{f}_0)$ defines a function on γ_*. The second bullet of Lemma A.2 implies that

$$\sum_{\gamma_*\in\Theta_*}\int_{\gamma_*}\|\chi_*\varphi_{\mathrm{z}}^*(\mathfrak{f}_0)\|_2^2 \geq (1-c_0c_0^{-1})\|\mathfrak{f}\|_2. \tag{A-31}$$

This inequality is exploited in Step 4.

Step 2 Fix $\gamma_* \in \Theta_*$. If γ is a component of α's zero locus in $Y_{\diamond\mathrm{z}}$, set $c_1 = c_0$, and if not, set $c_1 = c_0^4$, this being the definition of c_1 that is used in Part 2 of Section Ad to construct the versions of $\vartheta_{\mathrm{z}*}$ that are associated to the points in γ_*. Use T_{γ_*} in what follows to denote the union of the radius $(c_1 + 10c_0)\mathrm{z}^{-1/2}$ transverse disks with centers on γ_*.

Write $\mathfrak{f}_0 = (q, \eta_0)$ and assign to each $p \in \gamma_*$ the element $((\varphi_{\mathrm{z}}^{-1})^*(\chi_*))\mathfrak{f}_0$, this being a section of $\mathbb{V}_0$ over the transverse disk through p whose components are written as (q_*, η_{0*}). These sections define a smooth section of $\mathbb{V}_0$ over T_{γ_*} and they are viewed in this way. Use (A-27) with Lemma A.3 and Property 1 to see that

$$\Big\|\Big(\partial^K q_* + \tfrac{1}{\sqrt{2}}\mathrm{z}^{1/2}\bar{\alpha}\eta_{0*}\Big)\Big\|_2^2 + \Big\|\Big(\bar{\partial}_A^K\eta_{0*} + \tfrac{1}{\sqrt{2}}\mathrm{z}^{1/2}\alpha q_*\Big)\Big\|_2^2 \leq c_0(\lambda^2 + c_0^{-2}\mathrm{z})\|\mathfrak{f}\|_2^2 \tag{A-32}$$

with it understood that the L^2–norms on the right-hand side denote integrals over T_{γ_*}.

Step 3 Use $\mathfrak{z}_*$ in what follows to denote any given $p \in \gamma$ version of $\chi_*\varphi_{\mathrm{z}}^*(\mathfrak{f}_0)$. The operator $\vartheta_{\mathrm{z}*}$ enters the story by virtue of the fact that

$$\vartheta_{\mathrm{z}*}\mathfrak{z}_* = \mathrm{z}^{-1/2}\varphi_{\mathrm{z}}^*\big(\partial^K q_* + \tfrac{1}{\sqrt{2}}\mathrm{z}^{1/2}\bar{\alpha}\eta_{0*}, \bar{\partial}_A^K\eta_{0*} + \tfrac{1}{\sqrt{2}}\mathrm{z}^{1/2}\alpha q_*\big) + \mathfrak{e}, \tag{A-33}$$

where $\mathfrak{e}$ has compact support in the radius $c_1 + (k+2)c_0$ disk about the origin in $\mathbb{C}$ and $|\mathfrak{e}|$ is no greater than the φ_{z}–pullback of $c_0(c_0\mathrm{z}^{-1}|\nabla\mathfrak{f}_0| + c_0^{-1}|\mathfrak{f}_0|)$. The argument to prove (A-33) is identical but for notation to that used in [20] to derive the latter's (3.14) and (3.15).

Step 4 By definition, $\Pi_\vartheta\mathfrak{f}$ at points on γ_* is the L^2–orthogonal projection of $\mathfrak{z}_*$ to the L^2–kernel of $\vartheta_{\mathrm{z}*}$. This understood, write $\mathfrak{z}_* = \Pi_\vartheta\mathfrak{f} + \mathfrak{z}_*^\perp$. As the point in γ_* varies, so $\mathfrak{z}_*^\perp$ varies and, this understood, view $\|\mathfrak{z}_*^\perp\|_2$ as a function on γ_*. Lemma A.5 asserts that $\|\vartheta_{\mathrm{z}*}\mathfrak{z}_*^\perp\|_2 \ge c_0^{-1}\|\mathfrak{z}_*^\perp\|_2$. This bound with Lemma A.2's first bullet and (A-32) and (A-33) imply that

$$(1 - c_0 c_0^{-1})\int_{\gamma_*}\|\chi_*\varphi_{\mathrm{z}}^*(\mathfrak{f}_0)\|_2^2 - \int_{\gamma_*}\|\Pi_\vartheta(\mathfrak{f}_0)\|_2^2 \le c_0(\lambda^2\mathrm{z}^{-1} + c_0^{-2})\|\mathfrak{f}\|_2^2 \tag{A-34}$$

when $c_0 \ge c_0$ and $\mathrm{z} \ge c_{c_0}$ with the latter constant depending only on c_0.

The inequalities in (A-31) and (A-34) imply that $\|\Pi_\vartheta\mathfrak{f}\|_2^2 \ge (1 - c_0 c_0^{-1})\|\mathfrak{f}\|_2^2$ if it is the case that $c_0 > c_0$ and $|\lambda| \le c_0^{-1}c_0^{-1}\mathrm{z}^{1/2}$. □

Ah The equation $\Pi_\vartheta\mathfrak{L}_{\mathbb{V}}\mathfrak{f} = \lambda\Pi_\vartheta\mathfrak{f}$ on $Y - Y_{\diamond\mathrm{z}}$

What is asserted by Lemma A.6 implies that $\Pi_\vartheta\mathfrak{f}$ determines $\mathfrak{f}$ for the most part if $\mathfrak{f}$ is an eigenvector of $\mathfrak{L}_{\mathbb{V}}$ whose corresponding eigenvalue is greater than $-c_0^{-1}c_0^{-1}\mathrm{z}^{1/2}$ but less than $c_0^{-1}c_0^{-1}\mathrm{z}^{1/2}$. This fact lies behind the focus in this subsection and the next on the Π_ϑ projection of the eigenvalue equation $\mathfrak{L}_{\mathbb{V}}\mathfrak{f} = \lambda\mathfrak{f}$. By way of a look at what is to come, (A-26) with (A-28)–(A-30) are used here to rewrite a given $\gamma_* \in \Theta_*$ component of the projected equation $\Pi_\vartheta(\mathfrak{L}_{\mathbb{V}}\mathfrak{f}) = \lambda\Pi_\vartheta\mathfrak{f}$ as

$$\tfrac{i}{2}\partial_t(\Pi_\vartheta\mathfrak{f}) + \mathcal{R}\cdot\Pi_\vartheta\mathfrak{f} + \mathfrak{e}(\mathfrak{f}) = \lambda\Pi_\vartheta\mathfrak{f}, \tag{A-35}$$

where $\mathcal{R}$ is an $\mathbb{R}$–linear section of the bundle of endomorphisms of $\mathrm{Ker}_\vartheta|_{\gamma_*}$ and where $\mathfrak{e}$ is an $\mathbb{R}$–linear functional of $\mathfrak{f}$ that has small norm when $\mathfrak{f}$ has L^2–norm equal to 1. An equation of this sort appears because the Π_ϑ–image of the right-hand side of (A-29) at any given $t \in \gamma_*$ can be written schematically as $\vartheta_{\mathrm{z}_t}^\dagger\mathfrak{z} + \mathfrak{r}$, where $\mathfrak{z}$ depends on (p, η_1) and $\mathfrak{r}$ is small in a suitable sense. Meanwhile, $\vartheta_{\mathrm{z}_t}^\dagger\mathfrak{z}$ projects to 0 in $\mathrm{Ker}_\vartheta|_t$ and so the lack of an a priori small bound for the norm of $\vartheta_{\mathrm{z}_t}^\dagger\mathfrak{z}$ is of no concern.

This subsection uses (A-26) with (A-28)–(A-30) to say more about (A-35) when the given element $\gamma_* \in \Theta_*$ is some $\gamma \in \bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$ and $k \in \{0, 1, \dots, 7\}$ version of γ_k. The salient points are summarized by Lemma A.7. The first two parts of this subsection set up the background for Lemma A.7; the third part contains the lemma and its proof.

Part 1 To set the notation for what is to come, introduce $\kappa_\diamond$ and $\kappa_{c\diamond}$ to denote the larger of the respective versions of κ and κ_{c_0} that are supplied by Lemmas A.2–A.6. Fix $c_0 \geq \kappa_\diamond$ and $\mathrm{z} \geq \kappa_{c\diamond} c_0^{10}$ for use in Section Ab. Let $(A, \psi = (\alpha, \beta))$ denote a pair that satisfies Properties 1–5 in Section Ab using the given values of c_0 and z. Define the set Θ_* as in Section Ae, and focus attention on a given element in Θ_* that has the form γ_k with $\gamma \in \bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$ and integer $k \in \{0, \dots, 7\}$. If this element is the whole of γ, write γ as the union of two open sets of length $\frac{3}{4}\ell_\gamma$ with distance $\frac{1}{2}\ell_\gamma$ between their respective midpoints. These sets are denoted in what follows by γ_+ and γ_-. Introduce γ_* to denote γ_+ or γ_k if $\gamma_k = \gamma$ and to denote γ_k if $\gamma_k \neq \gamma$.

Let $T \subset Y$ denote the radius c_0^{-1} tubular neighborhood of γ with radius chosen so as to use γ's version of the coordinates from Part 4 of Section Aa for T with ν and μ constant and real, and with μ greater than $|\nu|$. To spare notation, suppose that the $t = 0$ point is in γ_*. Use $T_* \subset T$ to denote the set of points with coordinate $t \in \gamma_*$ and $|z| \leq 2c_v^4$. Fix once and for all an isomorphism between E's restriction to the transverse disk in T through the $t = 0$ point and the product bundle over this same disk. Parallel transport along the constant $z \in \mathbb{C}$ slices of T_* from the $t = 0$ transverse disk defines an isomorphism between $E|_{T_*}$ and $T_* \times \mathbb{C}$. This isomorphism writes A on T_* as

$$A = \theta_0 + \tfrac{1}{2}(\mathrm{A}\, d\bar{z} - \bar{\mathrm{A}}\, dz) \tag{A-36}$$

with A being a $\mathbb{C}$–valued function on T_*. This isomorphism makes α a $\mathbb{C}$–valued function.

Part 2 Given $t \in \gamma_*$, use ϑ_{z_t} to denote the $\mathrm{z} = \mathrm{r}$ version of Section Ad's operator $\vartheta_{\mathrm{z}*}$ that is defined by the restriction of (A, α) to the transverse disk through t. The isomorphism between $E|_{T_*}$ and $T_* \times \mathbb{C}$ writes the family $\{\vartheta_{\mathrm{z}_t}\}_{t\in\gamma_*}$ as a smooth, 1–parameter family of operators on $\mathbb{C}$ and it identifies the bundle $\mathrm{Ker}_\vartheta|_{\gamma_*}$ with a subbundle of the product bundle $\gamma_* \times C^\infty(\mathbb{C}; \mathbb{C} \oplus \mathbb{C})$. In particular, this isomorphism writes any given section of Ker_ϑ over γ_* as a map from γ_* to $C^\infty(\mathbb{C}; \mathbb{C} \oplus \mathbb{C})$. Viewed in this way, the L^2–orthogonal projection on $L^2(\mathbb{C}; \mathbb{C} \oplus \mathbb{C})$ induces a covariant derivative on sections of $\mathrm{Ker}_\vartheta|_{\gamma_*}$ as follows: Let $t \mapsto \zeta = (x_t, \iota_t)$ denote a smooth map from γ_* to $C^\infty(\mathbb{C}; \mathbb{C} \oplus \mathbb{C})$ such that ζ at each $t \in \gamma_*$ is a square-integrable element in the kernel of ϑ_{z_t}. The covariant derivative of this section is denoted by $\mathrm{D}\zeta$ and it is defined at any given $t \in \vartheta$ by the rule

$$\text{(A-37)} \qquad \mathrm{D}\zeta = \frac{d}{dt}\zeta + \vartheta^\dagger_{\mathrm{z}_t}\varpi,$$

where ϖ is the square-integrable solution to the equation

$$\text{(A-38)} \qquad \vartheta_{\mathrm{z}_t}\vartheta^\dagger_{\mathrm{z}_t}\varpi + \Big(\frac{\partial}{\partial t}\vartheta_{\mathrm{z}_t}\Big)\zeta = 0.$$

Note in this regard that $\frac{\partial}{\partial t}\vartheta_{\mathrm{z}_t}$ is an endomorphism of the product bundle $\mathbb{C} \times (\mathbb{C} \oplus \mathbb{C})$ whose coefficients are defined by the t–derivative of the function A and the corresponding covariant derivative of α on the union of the radius $(c_v^4 + 10c_v)\mathrm{z}^{-1/2}$ transverse disks in T_* with centers at the points in γ_*. This covariant derivative on $\mathrm{Ker}_\vartheta|_{\gamma_*}$ is metric compatible with it understood that the metric on this bundle is that induced by the L^2 inner product on the space of square-integrable maps from $\mathbb{C}$ to $\mathbb{C} \oplus \mathbb{C}$.

Let m denote the rank of $\mathrm{Ker}_\vartheta|_{\gamma_*}$, this being the dimension of the L^2–kernel of any $t \in \gamma_*$ version of ϑ_{z_t}. Fix an L^2–orthonormal basis for $\mathrm{Ker}_\vartheta|_{\gamma_*}$ at $t = 0$. Parallel transport this basis along γ_* using the connection defined by (A-37) and (A-38) to define an isomorphism from $\mathrm{Ker}_\vartheta|_{\gamma_*}$ to $\gamma_* \times \mathbb{C}^m$. This isomorphism is used in the upcoming Lemma A.7 to view a section of $\mathrm{Ker}_\vartheta|_{\gamma_*}$ as a map from γ_* to $\mathbb{C}^m$.

Part 3 The stage is now set for Lemma A.7:

Lemma A.7 *There exists $\kappa \geq \kappa_\diamond$ and, given $c_0 \geq \kappa$, there exists $\kappa_{c_0} \geq \kappa_{c\diamond}$ with the following significance: Fix $c_0 \geq \kappa$ and $\mathrm{z} \geq \kappa_{c_0} c_0^{10}$. Suppose that $\mathfrak{c} = (A, \psi)$ obeys the corresponding version of Properties 1–5 in Section Ab. Let $\mathfrak{f}$ denote an eigenvector of the operator $\mathfrak{L}_{\mathbb{V}}$ with eigenvalue bounded in absolute value by $c_0^{-\kappa}\mathrm{z}^{1/2}$. Define γ_* as in Part* 1 *and view both $\Pi_\vartheta \mathfrak{f}$ and $\Pi_\vartheta(\mathfrak{L}_{\mathbb{V}}\mathfrak{f})$ along γ_* as maps from γ_* to $\mathbb{C}^m$ as instructed in Part* 2*. Viewed in this way, the equation $\Pi_\vartheta(\mathfrak{L}_{\mathbb{V}}\mathfrak{f}) = \lambda\Pi_\vartheta\mathfrak{f}$ has the form*

$$\frac{i}{2}\frac{d}{dt}(\Pi_\vartheta\mathfrak{f}) + \mathfrak{r}(\mathfrak{f}) = \lambda\Pi_\vartheta\mathfrak{f}$$

with the endomorphism $\mathfrak{r}$ being an $\mathbb{R}$–linear functional of $\mathfrak{f}$ that obeys $\int_{\gamma_*}|\mathfrak{r}(\mathfrak{f})| \leq c_0^\kappa\|\mathfrak{f}\|_2$.

Proof Use $\langle\cdot,\cdot\rangle$ to denote the $\mathrm{Ker}_\vartheta|_{\gamma_*}$ inner product at a given $t \in \gamma_*$. With $\mathrm{Ker}_\vartheta|_{\gamma_*}$ viewed as $\gamma_* \times \mathbb{C}^m$, this is just the Hermitian inner product on $\mathbb{C}^m$; and with

$\mathrm{Ker}_{\vartheta}|_{\gamma_*}$ viewed as a subbundle of $\gamma_* \times C^\infty(\mathbb{C};\mathbb{C}\oplus\mathbb{C})$, this same Hermitian inner product is the L^2 inner product on the subspace of square-integrable maps from $\mathbb{C}$ to $\mathbb{C}\oplus\mathbb{C}$. Let ζ denote a covariantly constant section of $\mathrm{Ker}_{\vartheta}|_{\gamma_*}$ with unit L^2–norm. Use the two views of $\langle\cdot,\cdot\rangle$ with (A-26) and (A-28)–(A-30) to write $\langle\zeta,\Pi_\vartheta\mathfrak{L}_{\mathbb{V}}\mathfrak{f}\rangle$ as $\frac{i}{2}\frac{d}{dt}\langle\zeta,\Pi_\vartheta\mathfrak{f}\rangle+\mathfrak{r}_{\mathbb{V}}(\mathrm{F})+\mathfrak{r}_\omega(\mathfrak{f})$ where the function $t\mapsto\mathfrak{r}_{\mathbb{V}}(\mathfrak{f})|_t$ comes from the inner product of ζ with the Π_ϑ–images of all but the term $\frac{\partial}{\partial t}\mathfrak{f}_0=\bigl(\frac{\partial}{\partial t}q,\frac{\partial}{\partial t}\eta_1\bigr)$ in (A-28), and with the Π_ϑ–images of all of the terms in (A-29) and (A-30). Meanwhile, the function $t\mapsto\mathfrak{r}_\omega(\mathfrak{f})|_t$ is the right-most term in the identity

$$\frac{i}{2}\int_{\mathbb{C}}\zeta^\dagger\chi_*\varphi_{\mathrm{z}}^*\Bigl(\frac{\partial}{\partial t}\mathfrak{f}_0\Bigr)=\frac{i}{2}\frac{\partial}{\partial t}\int_{\mathbb{C}}\zeta^\dagger\chi_*\varphi_{\mathrm{z}}^*(\mathfrak{f}_0)-\frac{i}{2}\int_{\mathbb{C}}\Bigl(\frac{\partial}{\partial t}\zeta\Bigr)^\dagger\chi_*\varphi_{\mathrm{z}}^*(\mathfrak{f}_0).\tag{A-39}$$

The term $\mathfrak{r}_{\mathbb{V}}(\mathfrak{f})$ is such that

$$\int_{\gamma_*}\Bigl(\int_{\mathbb{C}}|\mathfrak{r}_{\mathbb{V}}(\mathfrak{f})|^2\Bigr)\le c_0c_0^k\|\mathfrak{f}\|_2^2\tag{A-40}$$

with $k\le c_0$. This can be seen by using the first bullet of Lemma A.2 to bound the contributions from (A-28), by using (A-33) with Lemma A.3 to bound those from (A-29), and by using Property 1 in Section Ab to bound the contribution to the terms with β in (A-30). Note with regards to (A-28) that the function a_{A0} is zero because there is no dt component on the right-hand side of (A-36).

To obtain the desired bound on the term $\mathfrak{r}_\omega(\mathfrak{f})$, use (A-37) to rewrite this term as

$$\frac{i}{2}\int_{\mathbb{C}}(\vartheta_{\mathrm{z}_t}^\dagger\varpi)\chi_*\varphi_{\mathrm{z}}^*(\mathfrak{f}_0).\tag{A-41}$$

Use the Minkowski inequality to bound (A-41) by the product of the L^2–norm of $\vartheta_{\mathrm{z}_t}^\dagger\varpi$ on $\mathbb{C}$ and that of $\chi_*\varphi_{\mathrm{r}}^*(\mathfrak{f}_0)$. The latter norm is bounded by a uniform multiple of the L^2–norm of $\mathfrak{f}$ over the transverse disk centered at the given point on γ_*. Use (A-38) with Lemma A.6 to bound the L^2–norm on $\mathbb{C}$ of $\vartheta_{\mathrm{z}_t}^\dagger\varpi$ by a multiple of the L^2–norm on $\mathbb{C}$ of $\bigl|\frac{\partial}{\partial t}\vartheta_{\mathrm{z}_t}\bigr||\zeta|$. This in turn is bounded by

$$c_0\sup_{\{(t,z)\in\gamma_*\times\mathbb{C}:|z|<c_0^4+10c_0\}}\Bigl(\mathrm{z}^{-1/2}\Bigl|\frac{\partial}{\partial t}\mathrm{A}\Bigr|+\Bigl|\frac{\partial}{\partial t}\alpha\Bigr|\Bigr)\tag{A-42}$$

because ζ has unit L^2–norm.

To say something about the size of (A-42), note first that $\frac{\partial}{\partial t}\alpha$ is the covariant derivative of α along the coordinate vector field $\frac{\partial}{\partial t}$ because Part 1's isomorphism between E over T_* writes A as depicted in (A-36). Meanwhile this covariant derivative differs from $(\nabla_A \alpha)_v$ by at most $c_0 c_0^4 \mathrm{z}^{-1/2} |\nabla_A \alpha|$ because the vector fields $\frac{\partial}{\partial t}$ and v differ on T_* by at most $c_0 c_0^4 \mathrm{z}^{-1/2}$. This being the case, use Properties 1 and 2 in Section Ab to see that the derivative of α in (A-42) is no greater than $c_0 c_0^6$. Meanwhile, $\frac{1}{2}\frac{\partial}{\partial t}\mathrm{A}$ is the $dt\, d\bar{z}$ component of the curvature 2–form of A because of the lack of a term in (A-36) that is proportional to dt. Use this fact with the aforementioned bound on $\left|\frac{\partial}{\partial t} - v\right|$ and Property 2 in Section Ab to see that the term $\mathrm{z}^{-1/2}\left|\frac{\partial}{\partial t}\mathrm{A}\right|$ in (A-42) is likewise no greater than $c_0 c_0^6$. □

Ai Lemma A.1 and the equation $\Pi_\vartheta \mathfrak{L}_{\mathbb{V}} \mathfrak{f} = \lambda \Pi_\vartheta \mathfrak{f}$

The lemma that follows talks about (A-35) in the case when (A, ψ) is described by Lemma A.1. This lemma refers to the functions ν and μ that appear in a given version of (A-6) for the case when the relevant curve from Θ lies in $Y_{*\Lambda}$. Choose a coordinate system of the sort described in Part 4 of Section Aa for each such curve from Θ with a bound by c_0 on the corresponding versions of $|\nu|$ and $|\mu|$. Such a choice is assumed implicitly in the lemma.

Lemma A.8 *There exists $\kappa \ge 100$ and, given $c_v \ge \kappa$, there exists $\kappa_{c_v} \ge \kappa$, both greater than their incarnations in Lemmas A.1–A.6 and with the following additional property: Fix $\mathrm{z} \ge \kappa_{c_v} c_v^{10}$ and a pair $(A, \psi) \in \mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ that is described by Lemma A.1 using the chosen value of c_v. Let γ denote a curve from Θ in $Y_{*\Lambda}$. Suppose that λ is an eigenvalue of the corresponding version of $\mathfrak{L}_{\mathbb{V}}$ with $|\lambda| \le c_v^{-\kappa} \mathrm{z}^{1/2}$ and let $\mathfrak{f}$ denote the corresponding eigenvector. Use ζ to denote the section of the line bundle $\mathrm{Ker}_\vartheta|_\gamma \to \gamma$ given by $(\Pi_\vartheta \mathfrak{f})|_\gamma$. There is an isomorphism of $\mathrm{Ker}_\vartheta|_\gamma$ with $\gamma \times \mathbb{C}$ that writes $\Pi_\vartheta \mathfrak{f}$ as a map $\zeta\colon \gamma \to \mathbb{C}$ and the Π_ϑ–image along γ of the eigenvalue equation $\mathfrak{L}_{\mathbb{V}} \mathfrak{f} = \lambda \mathfrak{f}$ as*

$$\frac{i}{2}\frac{d}{dt}\zeta + \nu\zeta + \mu\bar{\zeta} = \lambda\zeta + \mathfrak{e}(\mathfrak{f}), \qquad \text{(A-43)}$$

where $\mathfrak{e}(\mathfrak{f})$ is an $\mathbb{R}$–linear functional of $\mathfrak{f}$ that has L^2–norm bounded by $\kappa c_v^{-1} \|\mathfrak{f}\|_2$.

Proof Use (A-28)–(A-30) with the conclusions of Lemmas A.2–A.6 as input for the arguments that are used in Steps 9 and 10 from Section 2a in [21]. These arguments with one addition give a proof. Steps 9 and 10 in Section 2a of [21] prove the latter's Lemma 2.1, which is the analog of Lemma A.8 for the case where $\hat{a}$ is replaced by a contact 1–form. The one addition concerns the terms in (A-28) that involve x_γ. To say more about these terms, note first that they appear only when $\gamma \in Y_{\diamond \mathrm{z}}$. The relevant vortex solution defines the centered solution in $\mathfrak{C}_1$ and if (A_0, α_0) denotes such a solution, then the L^2–kernel of the corresponding version of ϑ is 1–dimensional and spanned by $\frac{1}{\sqrt{\pi}}\big(\frac{1}{\sqrt{2}}(1-|\alpha_0|^2), \bar\partial_{A_0}\alpha_0\big)$. This fact with an integration by parts shows that the x_γ terms contribute only to the $\mathfrak{e}(\mathfrak{f})$ term in the statement of the lemma. □

The next lemma states a stronger version of what is asserted by Lemma A.7 for cases when (A,ψ) on a given component of $Y-(Y_{*\Lambda}\cup T_{*\Lambda})$ is also given by the constructions in Section Aa using $\rho_* = c_v^{-4}$. To set the stage for the lemma, introduce γ to denote the curve from $\bigcup_{\mathrm{p}\in\Lambda}(\hat\gamma_\mathrm{p}^+\cup\hat\gamma_\mathrm{p}^-)$ in the given component of $Y-(Y_{*\Lambda}\cup T_{*\Lambda})$. Assume that the curves in $Y_{*\Lambda}$ from Θ have distance at least $(c_v^4+3c_v^3)\mathrm{z}^{-1/2}$ from γ. Use T here to denote the set of points with distance less than $(c_v^4+c_v^3)\mathrm{z}^{-1/2}$ from γ. Coordinates for T are given by γ's version of the coordinates from Part 4 of Section Aa with ν and μ constant and real with $\mu>|\nu|$.

The definition of (A,ψ) on T refers to a function, $\chi_{\diamond\diamond}$, of the radial coordinate $|z|$ on T. This function is nonnegative, it is equal to 1 where $|z|$ is less than $\big(c_v^4-\frac{7}{4}c_v^2\big)\mathrm{z}^{-1/2}$, it is equal to zero where $|z|$ is greater than $(c_v^4-c_v^2)\mathrm{z}^{-1/2}$, and the norm of its derivative has absolute value bounded by $32c_v^{-3}\mathrm{z}^{1/2}$. Note in particular that the function $\chi_{\diamond\diamond}$ is equal to zero on $T\cap Y_{*\Lambda}$. Such a function can be readily constructed using the function χ.

Let m denote a given positive integer. There is a unique solution to (2-8) with (3-1) equal to m and having the following properties: Write this solution as (A_{m0},α_{m0}). Then $\alpha_{m0}=|\alpha_{m0}|(z/|z|)^m$. Meanwhile, A_{m0} can be written in terms of the product connection θ_0 as $A_{m0}=\theta_0-\mathfrak{a}_{m0}\frac{m}{2}(z^{-1}\,dz-\bar z^{-1}\,d\bar z)$. Note that both $|\mathfrak{a}_{m0}|$ and $|\alpha_{m0}|$ are functions only of the radial distance to the origin in $\mathbb{C}$. The $m=1$ version of $\mathfrak{a}_{m0}$ is denoted by $\mathfrak{a}_0$ in (A-3). Any given $m\geq 1$ version of $\mathfrak{a}_{m0}$ obeys the analog of the $m=1$ bound in (A-4), this being $|1-\mathfrak{a}_{m0}|\leq c_0(1-|\alpha_{m0}|)$. The pair (A_{m0},α_{m0}) defines the point in the space $\mathfrak{C}_m$ from Part 1 of Section 3.1 that maps via the coordinates in (3-2) to the origin in $\mathbb{C}^m$. Let y_m and ς_m denote the (A_{m0},α_{m0}) versions of the functions y and ς that are described in Section Aa.

Fix an isomorphism between $E|_T$ and $T\times\mathbb{C}$ and use this isomorphism to view A as a connection on $T\times\mathbb{C}$ and the component α of ψ as a complex-valued function on T. Use this isomorphism with the coordinates from Part 4 of Section Aa to view β as a complex-valued function also. With this view understood, the connection A is written as $\theta+a_U$, where a_U is an $i\mathbb{R}$–valued 1–form on T. The 1–form a_U, α and β are defined as follows:

$$(\text{A-44})\quad \begin{aligned} a_U &= \nu\chi_{\diamond\diamond} i 2^{1/2} r_z^* y_m\, dt - \tfrac{1}{2} m(1-\chi_{\diamond\diamond} + \chi_{\diamond\diamond} r_z^* a_{m0})(z^{-1}\,dz - \overline{z}^{-1}\,d\overline{z}),\\ \alpha_U &= (1-\chi_{\diamond\diamond}(1-r_z^*|\alpha_{m0}|))\Big(\frac{z}{|z|}\Big)^m,\\ \beta_U &= i\mu \mathrm{z}^{-1/2}\chi_{\diamond\diamond} r_z^* \varsigma_m. \end{aligned}$$

Let $\mathfrak{c}_m\colon S^1 \to \mathfrak{C}_m$ denote the constant map to the point given by (A_{m0}, α_{m0}). The upcoming lemma also refers to the (A_{m0}, α_{m0}) version of the linear operator that is depicted in (3-10).

Lemma A.9 *Fix $m \geq 1$; there exists $\kappa \geq 100$ and, given $c_0 \geq \kappa$, there exists $\kappa_{c_0} \geq \kappa$, both greater than their incarnations in Lemmas A.1–A.6 and with the following significance: Fix $\mathrm{z} \geq \kappa_{c_0} c_0^{10}$. Set $c_v = c_0$ and then set $\rho_* = c_0^2 \mathrm{z}^{-1/2}$. Fix a set $T_{*\Lambda}$ and then a set Θ as described by* (A-5) *which obeys the first and second bullets of the* (z, c_0) *version of Property 3.*

- *Suppose $(A, \psi) \in \mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ is given by the $(\mathrm{z}, c_0, \rho_* = c_0^2 \mathrm{z}^{-1/2})$ version of* (A-7)–(A-10) *on $Y_{*\Lambda} \cup T_{*\Lambda}$. Fix a component of $Y - (Y_{*\Lambda} \cup T_{*\Lambda})$ and suppose that (A, ψ) is given by* (A-44) *on this component. Assume that Properties 1, 2, 4 and 5 hold on the rest of $Y - (Y_{*\Lambda} - T_{*\Lambda})$. Then (A, ψ) obeys Properties 1–5 on the whole of Y.*
- *Suppose that λ is an eigenvalue of the corresponding version of $\mathfrak{L}_{\mathbb{V}}$ with $|\lambda| \leq c_0^{-\kappa} \mathrm{z}^{1/2}$ and let $\mathfrak{f}$ denote the corresponding eigenvector. Let γ denote the curve from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$ in the given component of $Y - (Y_{*\Lambda} \cup T_{*\Lambda})$. Use ζ to denote the section of the line bundle $\mathrm{Ker}_\vartheta|_\gamma \to \gamma$ given by $(\Pi_\vartheta \mathfrak{f})|_\gamma$. There is an isomorphism of $\mathrm{Ker}_\vartheta|_\gamma$ with $\gamma \times \mathbb{C}$ that writes $\Pi_\vartheta \mathfrak{f}$ as a map $\zeta\colon \gamma_* \to \mathbb{C}$ and the Π_ϑ–image along γ of the eigenvalue equation $\mathfrak{L}_{\mathbb{V}}\mathfrak{f} = \lambda\mathfrak{f}$ as $\zeta \mapsto \frac{i}{2}\nabla_t \zeta + (\nabla_{\zeta_{\mathbb{R}}} \nabla^{1,0} \hbar)|_{c_m} + \mathfrak{e}(\zeta)$, where $\mathfrak{e}(\mathfrak{f})$ is an $\mathbb{R}$–linear functional of $\mathfrak{f}$ that has L^2–norm bounded by $\kappa c_0^{-1}\|\mathfrak{f}\|_2$.*

Proof The proof of the first bullet is a version of what is done in Sections 2e and 2f of [15]. The proof of the second bullet is a version of what is done in Steps 9 and 10 in Section 2a of [21]. □

B Vortex equation solutions and (A, ψ)

This section of the appendix supplies additional material for the proof of Proposition 2.6. To give a look ahead, suppose that $(A, \psi = (\alpha, \beta))$ is a solution to a given (r, μ)

version of (1-13) with μ being a 1–form from Ω whose $\mathcal{P}$–norm is less than 1. The replacement of (A, ψ) with a pair made from vortex solutions facilitates the upcoming analysis of the r–dependence of the spectrum of $\mathfrak{L}_{\mathbb{V}}$. By way of a reminder, the value of $\mathfrak{f}_s$ at (A, ψ) requires comparing the spectrum of the $(\mathrm{z} = \mathrm{r}, (A, \psi))$ version of $\mathfrak{L}_{\mathbb{V}}$ with that of a version defined using $\mathrm{z} = 1$. If the $\mathrm{z} = \mathrm{r}$ version of the operator $\mathfrak{L}_{\mathbb{V}}$ is defined not by (A, ψ) but by a pair made from vortex solutions, then Lemmas A.8 and A.9 can used to analyze the spectrum of $\mathfrak{L}_{\mathbb{V}}$. The inputs from these lemmas are used in Appendix C to study the spectral flow for the versions of $\mathfrak{L}_{\mathbb{V}}$ along 1–parameter family that is defined by the value of z and a corresponding z–dependent pair in $\mathrm{Conn}(E) \times C^{\infty}(Y; \mathbb{S})$ that is built from vortex solutions.

The constructions that follow in this appendix use (A, ψ) to construct a new pair in $\mathrm{Conn}(E) \times C^{\infty}(Y; \mathbb{S})$ that is defined on all of Y using solutions to the vortex equations in (2-8). This new pair is denoted by $(A_{\diamond}, \psi_{\diamond})$. The norm of the difference between the values of $\mathfrak{f}_s$ as defined using the $(\mathrm{z} = \mathrm{r}, (A, \psi))$ version of $\mathfrak{L}_{\mathbb{V}}$ and using the $(\mathrm{z} = \mathrm{r}, (A_{\diamond}, \psi_{\diamond}))$ version of $\mathfrak{L}_{\mathbb{V}}$ is shown to be bounded by an (A, ψ)– and r–independent constant. It proves convenient to construct the desired pair $(A_{\diamond}, \psi_{\diamond})$ in two stages. The first stage constructs a pair that is denoted by (A_*, ψ_*). This pair is defined on most, but not all of Y using solutions to the vortex equations in (2-8). In particular, the definition does not use vortex solutions near certain curves from the set $\bigcup_{\mathfrak{p} \in \Lambda} (\hat{\gamma}_{\mathfrak{p}}^{+} \cup \hat{\gamma}_{\mathfrak{p}}^{-})$. The second stage modifies (A_*, ψ_*) near these curves to obtain the desired pair $(A_{\diamond}, \psi_{\diamond})$.

Ba The construction of (A_*, ψ_*)

This subsection constructs the desired pair (A_*, ψ_*) from data supplied by the given solution to (1-13). The first four parts of this subsection construct (A_*, ψ_*). The fifth part of the subsection explains why (A_*, ψ_*) does not depend on the coordinates from Part 4 of Section Aa that are chosen in Part 2. The sixth and final part of the subsection constructs a path in $\mathrm{Conn}(E) \times C^{\infty}(Y; \mathbb{S})$ between (A_*, ψ_*) and the given solution to (1-13).

Part 1 The constructions in Section Aa are used to define (A_*, ψ_*) over most of Y. These constructions require as input the specification of parameters c_v, z and ρ_*. The parameter c_v is chosen in a two-step process as follows: A preliminary step chooses a parameter c_{v1} so as to be larger than the various incarnations of the constant κ that are given by Proposition 2.4 and Lemmas A.1–A.9. With c_{v1} chosen, let $\kappa_{\diamond}$

denote the largest of the various $c_0 \in [c_{v1}, 2c_{v1}]$ versions of the constant κ_{c_0} that are given by these same Lemmas A.1–A.9. Assume that $\mathrm{r} > 2^{20}\kappa_\diamond c_{v1}^{10}$ and suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. The second step to choosing c_v depends specifically on the form of the zero locus of α and thus on the chosen solution (A, ψ) of (1-13). The constant c_v should be chosen from the interval $[c_{v1}, 2c_{v1}]$ so as to satisfy certain conditions that are stated in a moment. These conditions refer to the subset $Y_{\diamond\diamond}$ of Y that consists of those points with distance at least $(c_v^4 - 3c_v^3)\mathrm{r}^{-1/2}$ from each curve in the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$.

(B-1)
- If a component of α's zero locus in $Y_{\diamond\diamond}$ is disjoint from a given boundary torus of $Y_{\diamond\diamond}$, then all of its points have distance greater than $6c_v^3\mathrm{r}^{-1/2}$ from this torus.
- If a component of α's zero locus in $Y_{\diamond\diamond}$ intersects a boundary torus of $Y_{\diamond\diamond}$, then this intersection point is an endpoint of the component and it is a transversal intersection. One endpoint of such a component lies where $u < 0$ on some boundary component of $Y_{\diamond\diamond}$ and the other where $u > 0$ on some boundary component of $Y_{\diamond\diamond}$. If a given boundary component of $Y_{\diamond\diamond}$ intersects the zero locus of α, then it does so at two points. The distance between these points is at least $100c_v^2\mathrm{r}^{-1/2}$, and one lies where $u < 0$ and the other where $u > 0$.

Use Proposition 2.4 with the formula for v in (1-3) to see that (B-1) will hold if c_v is chosen from the complement of at most G intervals of length c_0 in $[c_{v1}, 2c_{v1}]$. These intervals are determined by α and thus by the chosen (A, ψ).

Take $\mathrm{z} = \mathrm{r}$ and $\rho_* = c_v^2\mathrm{r}^{-1/2}$ to complete the specification of Section Aa's required parameters.

Part 2 With the choices just made, use $Y_{*\Lambda} \subset Y_{\diamond\diamond}$ to denote the set of points with distance at least $c_v^4\mathrm{r}^{-1/2}$ from each curve in the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$. The constructions in Section Aa require as additional input the choice of a union of components of $Y - Y_{*\Lambda}$, this denoted by $T_{*\Lambda}$. Define $T_{*\Lambda}$ as follows: a component of $Y - Y_{*\Lambda}$ is in $T_{*\Lambda}$ if and only if the component lacks zeros of α.

Having specified $T_{*\Lambda}$, the next order of business is to specify a set Θ that consists of embedded 1–manifolds in $Y_{*\Lambda} \cup T_{*\Lambda}$. These are the components of $\alpha^{-1}(0) \cap Y_{*\Lambda}$. In particular, Θ has no curves from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$. The constraint in (B-1) guarantees that the requirements of bullets two and three of (A-5) and bullets one and two of

Property 3 of Section Ab are met by the curves in Θ. That such is the case can be seen using Proposition 2.4 with the formula for v in (1-3).

Having specified the $T_{*\Lambda}$ version of Θ, Part 3 of Section Aa introduces a set denoted by U_0 and sets $\{U_\gamma\}_{\gamma\in\Theta}$. The collection of $\{U_0\}\cup\{U_\gamma\}_{\gamma\in\Theta}$ is denoted by $\mathfrak{U}$. Keep in mind that the union of the sets from $\mathfrak{U}$ contains $Y_{*\Lambda}\cup T_{*\Lambda}$. The constructions in Section Aa require choosing coordinates of the sort described in Part 4 of Section Aa for each U_γ. Make such a choice once and for all.

Section Aa also requires isomorphisms between the various $U\in\mathfrak{U}$ versions of $E|_U$ and $U\times\mathbb{C}$. Consider first the case of U_0. The chosen lower bound for r implies that $|\alpha|$ is nearly 1 on U_0 and in particular $|\alpha|>\frac{3}{4}$. This being the case, there is an isomorphism between $E|_U$ and $U\times\mathbb{C}$ that sends α to the map from U_0 to $\mathbb{C}$ given by $|\alpha|$. This is the isomorphism to use for U_0. Consider next the case for U_γ with γ a given curve from Θ. The chosen coordinates for U_γ supply an isomorphism from $E|_{U_\gamma}$ to $U_\gamma\times\mathbb{C}$ that makes α appear as the map from U_γ to $\mathbb{C}$ given by $|\alpha|z/|z|$. Use this isomorphism for U_γ.

Part 3 Section Aa uses the data supplied by Parts 1 and 2 to construct a pair of a connection on E and section of E over $U_0\cup\big(\bigcup_{\gamma\in\Theta}U_\gamma\big)$. The desired (A_*,ψ_*) is defined so as to equal this pair from Section Aa over $Y_{*\Lambda}\cup T_{*\Lambda}$. This understood, this part of the subsection and Part 4 define (A_*,ψ_*) over the components of $Y-(Y_{*\Lambda}\cup T_{*\Lambda})$.

Reintroduce the set $Y_{\diamond\diamond}$ from Part 1, this being the subset of Y whose points have distance at least $(c_v^4-3c_v^3)\mathrm{r}^{-1/2}$ from the curves in $\bigcup_{\mathrm{p}\in\Lambda}(\hat\gamma_{\mathrm{p}}^+\cup\hat\gamma_{\mathrm{p}}^-)$. Fix a component of $Y-(Y_{*\Lambda}\cup T_{*\Lambda})$ and use T to denote the radius $(c_v^4+c_v^3)\mathrm{r}^{-1/2}$ tubular neighborhood of the corresponding curve from the set $\bigcup_{\mathrm{p}\in\Lambda}(\hat\gamma_{\mathrm{p}}^+\cup\hat\gamma_{\mathrm{p}}^-)$. This set T is open and the given component is an open subset of T with compact closure.

The definition to come of (A_*,ψ_*) on T uses the coordinates from Part 4 of Section Aa that are defined by T's curve from $\bigcup_{\mathrm{p}\in\Lambda}(\hat\gamma_{\mathrm{p}}^+\cup\hat\gamma_{\mathrm{p}}^-)$. The definition also refers to the function, $\chi_{\diamond\diamond}$, that was introduced in the discussions just prior to Lemma A.9. By way of a reminder, this is a nonnegative function of the radial coordinate $|z|$ on T that is equal to 1 where $|z|$ is less than $\big(c_v^4-\frac{7}{4}c_v^2\big)\mathrm{r}^{-1/2}$, and equal to zero where $|z|$ is greater than $\big(c_v^4-\frac{5}{4}c_v^2\big)\mathrm{r}^{-1/2}$. The norm of its derivative has absolute value bounded by $32c_v^{-3}\mathrm{r}^{1/2}$. This function $\chi_{\diamond\diamond}$ is equal to 1 on $T-(T\cap Y_{\diamond\diamond})$ and it is equal to 0 on $T\cap Y_{*\Lambda}$.

The definition of (A_*,ψ_*) over T when α lacks zeros on $T\cap Y_{\diamond\diamond}$ occupies the remainder of Part 3. To start the definition in this case, define (A_*,ψ_*) over $T-(T\cap Y_{\diamond\diamond})$

to be (A,ψ)'s restriction to this same subset of T. To define (A_*,ψ_*) on the rest of T, use the first bullet of (B-1) and Proposition 2.4 to conclude that all points in $T\cap Y_{\diamond\diamond}$ have distance at least $2c_v^3\mathrm{r}^{-1/2}$ from a zero of α if $c_v\ge c_0$. This last observation has two immediate and not unrelated consequences, the first being that $T\cap Y_{*\Lambda}$ is contained in Section Aa's open set U_0. The second consequence comes via Lemma 2.3, which guarantees that $|\alpha|\ge 1-e^{-c_v^2}$ on $T\cap Y_{\diamond\diamond}$ if $c_v\ge c_0$. Granted these facts, Part 2's isomorphism from $E|_{U_0}$ to $U_0\times\mathbb{C}$ sending α to $|\alpha|$ extends over $T\cap Y_{\diamond\diamond}$ using this same rule to identify $E|_{T\cap Y_{\diamond\diamond}}$ with $(T\cap Y_{\diamond\diamond})\times\mathbb{C}$. This isomorphism depicts A on $T\cap Y_{\diamond\diamond}$ as a connection on $(T\cap Y_{\diamond\diamond})\times\mathbb{C}$, and, viewed as such, A can be written as $A=\theta_0+a_{A,U_0}$, where θ_0 denotes the product connection and where a_{A,U_0} is an $i\mathbb{R}$–valued 1–form on $T\cap Y_{\diamond\diamond}$. Use the isomorphism to write (α,β) as $(|\alpha|,\beta_{U_0})$ with β_{U_0} being a section over $T\cap Y_{\diamond\diamond}$ of the bundle K^{-1}.

Write ψ_* as (α_*,β_*) with respect to the $E\oplus EK^{-1}$ splitting of $\mathbb{S}$. Granted this notation, use the isomorphism from the preceding paragraph to define $(A_*,\psi_*=(\alpha_*,\beta_*))$ over $T\cap Y_{\diamond\diamond}$ by declaring

(B-2) $$A_*=\theta_0+\chi_{\diamond\diamond}a_{A,U_0},\quad \alpha_*=(1-\chi_{\diamond\diamond})+\chi_{\diamond\diamond}|\alpha|\quad\text{and}\quad \beta_*=\chi_{\diamond\diamond}\beta_{U_0}.$$

The definition given in (B-2) smoothly extends (A_*,ψ_*) from U_0 to $U_0\cup T$ because the pair (A_*,ψ_*) on U_0 is defined in Section Aa using Part 2's isomorphism between $E|_{U_0\cap T}$ and $(U_0\cap T)\times\mathbb{C}$ as $(A_*=\theta_0,\psi_*=(1,0))$.

Part 4 This part assumes that α has zeros in $T\cap Y_{\diamond\diamond}$. The definition in this case also sets (A_*,ψ_*) equal to (A,ψ) on $T-(T\cap Y_{\diamond\diamond})$. Four steps are used to define (A_*,ψ_*) on $T\cap Y_{\diamond\diamond}$.

Step 1 To set the stage for the definition on $T\cap Y_{\diamond\diamond}$ use (B-1), Proposition 2.4 and the depiction of v in (1-3) to see that α's zero locus in the $Y_{\diamond\diamond}$ closure of $T\cap Y_{\diamond\diamond}$ consists of two embedded, closed arcs, each with one endpoint on the boundary torus of the closure of T and the other on $Y_{\diamond\diamond}$'s boundary torus in T. Moreover:

(B-3)
- The oriented unit tangent vector to each arc differs from v by at most $c_0\mathrm{r}^{-1/2}$.
- Each arc has transversal intersections with the level sets of $|z|$.
- One arc sits where $u<0$ and the other where $u>0$ and both where $1-3\cos^2\theta>0$.
- The distance between any given point in one arc from any given point in the other is at least $100c_v^2\mathrm{r}^{-1/2}$.

Each arc from (B-3) extends a curve from the set Θ into $Y_{*\Lambda} \cup (T \cap Y_{\diamond\diamond})$ so as to move a boundary point on T's boundary component of $Y_{*\Lambda}$ to T's boundary component of $Y_{\diamond\diamond}$. Let γ denote such an extended curve. The open set U_γ from Section Aa likewise extends into T with the same definition as the union of the radius $4c_v^2 \mathrm{r}^{-1/2}$ transverse disks centered on the extension of γ. This is an open solid torus with core circle γ. The open set U_0 also extends into $T \cap Y_{\diamond\diamond}$ as the complement of the union of the radius $c_v^2 \mathrm{r}^{-1/2}$ disks centered on the relevant two arcs from (B-3).

Step 2 Granted what is said in Step 1, then Section Aa's definitions can be used to extend (A_*, ψ_*) into $T \cap Y_{\diamond\diamond}$. The extended pair over $T \cap Y_{\diamond\diamond}$ is denoted by (A_{*T}, ψ_{*T}). By way of a reminder, the extension over the complement of the radius $3c_v^2 \mathrm{r}^{-1/2}$ tubular neighborhoods of (B-3)'s arcs is written using the isomorphism of E with the product $\mathbb{C}$–bundle that sends α to $|\alpha|$. Meanwhile, (A_{*T}, ψ_{*T}) is written over the radius $4c_v^2 \mathrm{r}^{-1/2}$ tubular neighborhood of either of (B-3)'s arcs using the coordinates from Part 4 of Section Aa and the isomorphism of E with the product $\mathbb{C}$–bundle that sends α to $|\alpha|$. The respective formula on these sets are given below. These formulas write ψ_{*T} in two-component form with respect to the splitting of $\mathbb{S}$ as $E \oplus EK^{-1}$. The formulas use θ_0 to denote the product connection on the product $\mathbb{C}$–bundle:

(B-4)
- $A_{*T} = \theta_0$ and $\psi_{*T} = (1, 0)$.
- $A_{*T} = \theta_0 + i 2^{1/2} \nu \mathfrak{r}_{\mathrm{r}}^* y\, dt - \dfrac{1}{2} \mathfrak{r}_{\mathrm{r}}^* \mathfrak{a}_0 (z^{-1} dz - \bar{z}^{-1} d\bar{z})$,
  $\psi_{*T} = (\mathfrak{r}_{\mathrm{r}}^* \alpha_0, i\mu \mathrm{r}^{-1/2} \mathfrak{r}_{\mathrm{r}}^* \zeta)$.

By way of comparison, the isomorphism used in (B-4) writes (A, ψ) over the complement of the radius $3c_v^2 \mathrm{r}^{-1/2}$ tubular neighborhoods of (B-3)'s arcs and over the radius $4c_v^2 \mathrm{r}^{-1/2}$ tubular neighborhood of either arc as

(B-5)
- $A = \theta_0 + a_{A,U_0}$ and $\psi = (|\alpha|, \beta_{U_0})$,
- $A = \theta_0 + a_{A,U_\gamma}$ and $\psi = (|\alpha| z/|z|, \beta_{U_\gamma})$,

where a_{A,U_0} and a_{A,U_γ} are $i\mathbb{R}$–valued 1–forms. Keep in mind that

(B-6) $$a_{A,U_0} = a_{A,U_\gamma} + \tfrac{1}{2}(z^{-1} dz - \bar{z}^{-1} d\bar{z}) \quad \text{and} \quad \beta_{U_0} = \frac{\bar{z}}{|z|} \beta_{U_\gamma}$$

on the intersection of the respective domains of definition.

The pair (A_{*T}, ψ_{*T}) is not the desired extension of (A_*, ψ_*) because it is observedly not the same as (A, ψ) near the boundary torus in T of $Y_{\diamond\diamond}$.

Step 3 Let $\gamma \in \Theta$ denote a component with it understood that γ extends into $Y_{\diamond\diamond}$. Let $U'_\gamma \in U_\gamma$ denote the radius $c_v^2 \mathrm{r}^{-1/2}$ tubular neighborhood of γ. This step defines a smooth map $u_\gamma\colon U_\gamma \to S^1$ which is used to define

$$a_{A,\gamma} = a_{A,U_\gamma} - u_\gamma^{-1} du_\gamma, \quad \alpha_\gamma = |\alpha| u_\gamma \frac{z}{|z|} \quad \text{and} \quad \beta_\gamma = u_\gamma \beta_{U_\gamma}. \tag{B-7}$$

The map u_γ is constructed so as to obey $u_\gamma = 1$ on $U_\gamma - U'_\gamma$. This being the case, then the pair $(\theta_0 + a_{A,\gamma}, (\alpha_\gamma, \beta_\gamma))$ is gauge equivalent to (A, ψ) on U_γ and it extends as (A, ψ) to the whole of U_0.

The definition of u_γ requires the introduction of a function χ_γ which is given on the $Y_{*\Lambda}$ part of U_γ by the rule $z \mapsto \chi(2c_v^{-2}\mathrm{r}^{1/2}|z| - 1)$. The function χ_γ on the $Y_{\diamond\diamond} - Y_{*\Lambda}$ part of U_γ is the product of the function $z \mapsto \chi(2c_v^{-2}\mathrm{r}^{1/2}|z| - 1)$ with a second nonnegative function. The latter is also constructed using χ and it has the following properties: It is a function of the distance to the nearby component of $\bigcup_{\mathrm{p}\in\Lambda}(\hat\gamma_{\mathrm{p}}^+ \cup \hat\gamma_{\mathrm{p}}^-)$. This second function equals 1 where the distance to these curves is greater than $c_v^4 - \frac{9}{4}c_v^2$ and it equals 0 where the distance is less than $c_v^4 - \frac{11}{4}c_v^2$. The derivative of this second function should have absolute value no greater than $100c_v^{-2}\mathrm{r}^{1/2}$. Note in particular that this definition of χ_γ makes it zero on U_γ's intersection with a neighborhood of the boundary of $Y_{\diamond\diamond}$.

With χ_γ in hand write a_{A,U_γ} as $a_{A,U_\gamma} = a_{A0}\,dt + \frac{1}{2}(\mathrm{A}\,d\bar z - \bar{\mathrm{A}}\,dz)$ with A being a $\mathbb{C}$–valued function on $\mathbb{C}$ and a_{A0} being an $i\mathbb{R}$–valued function on $\mathbb{C}$. The map u_γ is defined by the rule

$$u_\gamma = e^{\hat o_\gamma}, \quad \text{where } \hat o_\gamma(t,z) = \chi_\gamma \frac{1}{2}\int_0^1 (\bar z \mathrm{A} - z\bar{\mathrm{A}})|_{(t,sz)}\,ds. \tag{B-8}$$

By way of explanation, the map u_γ is designed in part so that the 1–form a_γ annihilates the radial vector field $z\frac{\partial}{\partial z} + \bar z \frac{\partial}{\partial \bar z}$ where $\chi_\gamma = 1$. Note in addition that u_γ extends to the whole of Y as a smooth map to S^1 that is equal to 1 on the complement of a compact set in U_γ.

Step 4 The desired pair (A_*, ψ_*) is written below using the isomorphisms of E with the product bundle that are used in (B-4) and (B-5). The formula over the complement of the radius $3c_v^2\mathrm{r}^{-1/2}$ tubular neighborhoods of (B-3)'s arcs is

$$A_* = \theta_0 + \chi_{\diamond\diamond} a_{A,U_0} \quad \text{and} \quad \psi_* = ((1-\chi_{\diamond\diamond}) + \chi_{\diamond\diamond}|\alpha|, \chi_{\diamond\diamond}\beta_{U_0}). \tag{B-9}$$

Meanwhile, A_* and ϕ_* are written over the radius $4c_v^2\mathrm{r}^{-1/2}$ tubular neighborhood of the extension to $Y_{\diamond\diamond} \cap T$ of an arc $\gamma \in \Theta$ as $A_* = \theta_* + a_*$ and $\psi_* = (\alpha_*, \beta_*)$, where:

(B-10)
- $a_* = \chi_{\diamond\diamond} a_{A,\gamma} + (1-\chi_{\diamond\diamond})\big[\chi_{\hat{U}} i 2^{1/2} \nu r_{\mathrm{r}}^* y\, dt - \frac{1}{2}(1-\chi_{\hat{U}} + \chi_{\hat{U}} r_{\mathrm{r}}^* a_0)(z^{-1}\, dz - \bar{z}^{-1}\, d\bar{z})\big].$
- $\alpha_* = (1-\chi_{\diamond\diamond})(1-\chi_{\hat{U}}(1-r_{\mathrm{r}}^*|\alpha_0|))z/|z| + \chi_{\diamond\diamond}\alpha_\gamma.$
- $\beta_* = (1-\chi_{\diamond\diamond})(i\mu \mathrm{r}^{-1/2}\chi_{\hat{U}} r_{\mathrm{r}}^* \varsigma) + \chi_{\diamond\diamond}\beta_\gamma.$

(The function $\chi_{\hat{U}}$ was defined just prior to (A-10) in Section Aa using χ and the transverse coordinate z; it is $\chi(\rho_*^{-1}|z|-1)$.) The formulas in (B-6)–(B-8) guarantee that (B-9) and (B-10) define a smooth connection on E and section of $\mathbb{S}$ over $Y \cap T$ because $z/|z|$ is the transition function between the relevant product $\mathbb{C}$–bundles.

Part 5 This part of the subsection explains why (A_*, ψ_*) does not depend on the choices made in Part 2 of coordinate charts from Part 4 of Section Aa. What follows is the short explanation: A change in the coordinate chart for any given $\gamma \in \Theta$ also changes the product structure for the bundle E over the corresponding set U_γ. The change in the product structure must be taken into account when comparing versions of (A_*, ψ_*) that are defined by two different choices from Part 4 of Section Aa. The changed product structure compensates for the apparent coordinate dependence in the formula for (A_*, ψ_*). The next two paragraphs say somewhat more about how this comes about.

Recall that a change in the coordinate chart writes the coordinate z on U_γ as $u(t)z'$ with u being a smooth map from γ to S^1. To see the effect, consider first the formula in the second bullet of (B-4) that depicts (A_*, ψ_*) on $U'_\gamma \cap Y_{*\Lambda}$. Write the pullback of the expressions on the right-hand side of the equations in the lower bullet of (B-4) via the map $(t, z') \mapsto (t, z = u(t)z')$ in terms of $\nu' = \nu + \frac{i}{2}u^{-1}\frac{d}{dt}u$ and $\mu' = u^{-2}\mu$. Use (A-2) to write $y = -2^{-1/2}(1-a_0)$ and use the formula for ς in (A-2) to see that the $(t, z') \mapsto (t, u(t)z')$ pullback of the expression for A_{*T} in the lower bullet of (B-4) is obtained from the (z', ν', μ') version of the expression by subtracting $\big(u^{-1}\frac{d}{dt}u\big)\, dt$. Meanwhile, the pullback of the formula for ψ_{*T} in the lower bullet of (B-4) is obtained from the (z', ν', μ') version by multiplying the latter by u. These changes are precisely offset by the change in the product structure.

The invariance of (B-10) with respect to coordinate change can be seen by writing a_{A,U_γ} as $\frac{1}{2}(\alpha^{-1}\nabla_A\alpha - \bar{\alpha}^{-1}\nabla_A\bar{\alpha}) - \frac{1}{2}(z^{-1}\, dz - \bar{z}^{-1}\, d\bar{z})$ so as to compare a_{A,U_γ} with its pullback via the map $(t, z') \mapsto (t, u(t)z')$.

Part 6 Part 3 defined various $\gamma \in \Theta$ versions of a map u_γ to S^1 from the corresponding $Y_{\diamond\diamond}$ extension of U_γ. As noted at the end of Part 3, such a map extends to the whole of Y as a smooth map that sends the complement of U_γ to 1. Let u denote the product of these extended maps; this a smooth map from Y to S^1. This part of the subsection describes a path in $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ between $(A - u^{-1}du, u\psi)$ and (A_*, ψ_*). This path is parametrized by $\tau \in [0,1]$ with the $\tau = 0$ member being $(A - u^{-1}du, u\psi)$ and the $\tau = 1$ member being (A_*, ψ_*). The $\tau \in [0,1]$ member of this path is denoted in what follows by $(A_{*\tau}, \psi_{*\tau})$ and $\psi_{*\tau}$ is written as $(\alpha_{*\tau}, \beta_{*\tau})$ with respect to the splitting of $\mathbb{S}$ as $E \oplus E^{-1}$. The pair $(A_{*\tau}, \psi_{*\tau})$ on $Y - Y_{\diamond\diamond}$ is defined to be (A, ψ). The pair $(A_{*\tau}, \psi_{*\tau})$ on the $Y_{*\Lambda}$ part of $U_0 - \big(\bigcup_{\gamma\in\Theta} U_\gamma\big)$ is defined using the $\alpha \mapsto |\alpha|$ isomorphism from $E|_{U_0}$ to $U_0 \times \mathbb{C}$ by the rules

(B-11) $$A_{*\tau} = \theta_0 + (1-\tau)a_{A,U_0}, \quad \alpha_{*\tau} = \tau + (1-\tau)|\alpha| \quad \text{and} \quad \beta_{*\tau} = (1-\tau)\beta_{U_0}.$$

Meanwhile, the definition on any given $Y_{\diamond\diamond} \cap T$ part of $U_0 - \big(\bigcup_{\gamma\in\Theta} U_\gamma\big)$ is obtained from the formula in (B-9) by replacing $\chi_{\diamond\diamond}$ with $(1-\tau) + \tau\chi_{\diamond\diamond}$. The pair $(A_{*\tau}, \psi_{*\tau})$ on the $Y_{*\Lambda}$ part of any given $\gamma \in \Theta$ version of U_γ is defined using the $\alpha \mapsto |\alpha| z/|z|$ isomorphism from $E|_{U_\gamma}$ to $U_\gamma \times \mathbb{C}$ by the rules

(B-12)
- $$A_{*\tau} = \theta_0 + \tau\big(i2^{-1/2}\nu r_{\mathrm{r}}^* y\,dt - \tfrac{1}{2} r_{\mathrm{r}}^* a_0 (z^{-1}dz - \bar{z}^{-1}d\bar{z})\big) + (1-\tau)a_{A_\gamma},$$
- $$\alpha_{*\tau} = \tau r_{\mathrm{r}}^* \alpha_0 + (1-\tau)\alpha_\gamma \text{ and } \beta_{*\tau} = (1-\tau)\beta_\gamma + \tau i \mu \mathrm{r}^{-1/2} r_{\mathrm{r}}^* \varsigma.$$

The definition over any given $Y_{\diamond\diamond} \cap T$ part of U_γ is obtained from the formula in (B-10) by replacing $\chi_{\diamond\diamond}$ with $(1-\tau) + \tau\chi_{\diamond\diamond}$ and replacing $\chi_{\hat{U}}$ by $\tau\chi_{\hat{U}}$.

By way of a parenthetical remark, this path in $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ does not depend on the chosen coordinates from Part 4 of Section Aa.

Bb (A_*, ψ_*) and Properties 1–5

The upcoming Lemma B.1 asserts that (A_*, ψ_*) and each $\tau \in [0,1]$ member of the path $\tau \mapsto (A_{*\tau}, \psi_{*\tau})$ have all five of the properties that are listed in Section Ab. This lemma is proved using the a priori bounds on the various components of (A, ψ) and (A_*, ψ_*) that are supplied by Lemma B.2.

Lemma B.1 *There exists $\kappa > 1$ and, given $c_v \geq \kappa$, there exists $\kappa_{c_v} \geq \kappa$ with the following significance: Suppose that $r \geq \kappa_{c_v} c_v^{10}$ and suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of* (1-13) *with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Then the corresponding (A_*, ψ_*) satisfies the $c_0 = c_v$ and $\mathrm{z} = \mathrm{r}$ version of Properties 1–5 in Section Ab as do all $\tau \in [0,1]$ members of the path $\tau \mapsto (A_{*\tau}, \psi_{*\tau})$.*

The proof of this lemma is given in a moment.

Lemma B.2 talks about various components of (A,ψ) on the $Y_{\diamond\diamond}$ extensions of U_0 and the various $\gamma\in\Theta$ versions of U_γ. To set the stage for this lemma, use the $\alpha\mapsto|\alpha|$ isomorphism between $E|_{U_0}$ and $U_0\times\mathbb{C}$ to write $(A,(\alpha,\beta))$ on U_0 as $(\theta+a_{A,U_0},(|\alpha|,\beta_{U_0})$. The lemma also uses $(a_{A,U_0})_v$ to denote the pairing of the 1–form with v.

With $\gamma\in\Theta$ fixed, Lemma B.2 uses the coordinates from Part 4 of Section Aa for U_γ. Lemma B.2 uses the coordinates from Part 4 of Section Aa, the map u_γ from (B-8) and the $\alpha\mapsto|\alpha|z/|z|$ isomorphism between $E|_{U_\gamma}$ and $U_\gamma\times\mathbb{C}$ to write $(A-u_\gamma^{-1}\,du_\gamma,(u_\gamma\alpha,u_\gamma\beta))$ as $(\theta+a_{A,\gamma},(\alpha_\gamma,\beta_\gamma))$, and it writes $a_{A,\gamma}$ as $a_{A0,\gamma}\,dt+\frac{1}{2}(\mathrm{A}_\gamma\,d\bar z-\bar{\mathrm{A}}_\gamma\,dz)$. Lemma B.2 also borrows the functions a_0 and α_0 from (A-3).

Lemma B.2 *Fix $m\ge 1$. There exists an m–dependent $\kappa>1$ and, given $c_v\ge\kappa$, there exists $\kappa_{c_v}\ge\kappa$ with the following significance: Take $\mathrm{r}\ge\kappa_{c_v}c_v^{10}$ and suppose that $(A,\psi=(\alpha,\beta))$ is a solution to the (r,μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Define (A_*,ψ_*) as instructed in Section Ba. Then:*

- $\mathrm{r}^{-1/2}|a_{A,U_0}|+|(a_{A,U_0})_v|+\bigl|1-|\alpha|\bigr|+\mathrm{r}^{1/2}|\beta_{U_0}|\le c_v^{-m}$ *on the $Y_{\diamond\diamond}$ extension of U_0.*
- $\mathrm{r}^{-1/2}|\mathrm{A}_\gamma-r_{\mathrm{r}}^*\mathrm{A}_0|+|\alpha_\gamma-r_{\mathrm{r}}^*\alpha_0|\le c_v^{-m}$ *and $|a_{A0,\gamma}|\le\kappa c_v^2$ on the part of the $Y_{\diamond\diamond}$ extension of any given U_γ where the distance to $\bigcup_{\mathrm{p}\in\Lambda}(\hat\gamma_{\mathrm{p}}^+\cup\hat\gamma_{\mathrm{p}}^-)$ is greater than $(c_v^4-2c_v^2)\mathrm{r}^{-1/2}$.*

The proof of Lemma B.1 assumes that Lemma B.2 is true.

Proof of Lemma B.1 The two steps that follow verify the five properties. These steps use κ_c to denote a constant whose value is greater than 1 and depends only on an upper bound for Lemma B.2's constant m and c_v, but not on the particulars of (A,ψ) nor on r. This constant can be assumed to increase between subsequent appearances.

Step 1 Given the definition of Θ, what is said in Proposition 2.4 implies Property 3. The other properties hold where the distance to $\bigcup_{\mathrm{p}\in\Lambda}(\hat\gamma_{\mathrm{p}}^+\cup\hat\gamma_{\mathrm{p}}^-)$ is less than $c_v^4-2c_v^2$ if they hold for (A,ψ), which is the case when $c_v\ge c_0$ and $\mathrm{r}\ge\kappa_c$.

The remainder of this step verifies Properties 1, 2, 4 and 5 on $U_0-\bigl(\bigcup_{\gamma\in\Theta}U_\gamma\bigr)$. Consider first the $Y_{*\Lambda}$ part of this set. The inequality asserted by the first bullet of Property 1 and by the first two bullets of Property 2 follow directly from Lemmas 2.1 and 2.3. Lemma 2.3 also leads directly to Property 4 and Lemma 2.9 to Property 5. To verify the remaining parts of Properties 1 and 2, take $m \geq 100$ and use Lemma B.2's bound $|a_{A,U_0}| \leq c_v^{-m} \mathrm{r}^{1/2}$ and the bound $|(a_{A,U_0})_v| \leq c^{-m}$ with Lemmas 2.1 and 2.3 to see that $\mathrm{r}^{-1/2}|\nabla_{A_{*\tau}}\alpha_{*\tau}|$ and both $|(\nabla_{A_{*\tau}}\alpha_{*\tau})_v|$ and $|\nabla_{A_{*\tau}}\beta_{*\tau}|$ are bounded by c_0 on the $Y_{*\Lambda}$ part of $U_0 - \left(\bigcup_{\gamma\in\Theta} U_\gamma\right)$. The latter set of bounds lead directly to the bounds on the $Y_{*\Lambda}$ part of $U_0 - \left(\bigcup_{\gamma\in\Theta} U_\gamma\right)$ that are stated by the second and third bullets of Property 1 and by the third and fourth bullets of Property 2 on $Y_{*\Lambda}$.

Given (B-9) and its $(A_{*\tau}, \psi_{*\tau})$ analog, the arguments from the preceding paragraph with but one additional comment establish Properties 1, 2, 4 and 5 on the $Y_{\diamond\diamond} - Y_{*\Lambda}$ part of $U_0 - \left(\bigcup_{\gamma\in\Theta} U_\gamma\right)$. The additional comment concerns the function $\chi_{\diamond\diamond}$ that appears in (B-9), this being the fact that $|(\chi_{\diamond\diamond})_v \leq c_0 c_v^6$ on this part of $Y_{\diamond\diamond}$. Indeed, such a bound follows because $\chi_{\diamond\diamond}$ is independent of t and because (1-3) finds $\left|v - \frac{\partial}{\partial t}\right| \leq c_0 c_v^4 \mathrm{r}^{-1/2}$ at all points with distance $c_v^4 \mathrm{r}^{-1/2}$ or less from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$.

Step 2 This step verifies Properties 1, 2, 4 and 5 on any $\gamma \in \Theta$ version of U_γ. To start, take $m \geq 100$ in the second bullet of Lemma B.2. Its assertions about A_γ and α_γ imply directly the first bullet of Property 1 and Properties 4 and 5. Introduce $\nabla_{A_{*\tau}}^\perp$ to denote the $A_{*\tau}$–covariant derivative along the constant t slices of U_γ. This same part of Lemma B.2 implies that

$$|\nabla_{A_{*\tau}}^\perp(\alpha_\gamma - r_{\mathrm{r}}^*\alpha_0)| \leq c_0 c_v^{-50} \mathrm{r}^{1/2}(1 - |\alpha_\gamma|^2)^{1/2} + c_0. \tag{B-13}$$

Lemma 2.1 and Lemma B.2's assertions about A_γ give the bound $|\nabla_{A_{*\tau}}^\perp \beta_\gamma| \leq c_0$ and they imply that (A-2)'s function ς is such that $\mathrm{r}^{-1/2}|\nabla_{A_{*\tau}}^\perp(r_{\mathrm{r}}^*\varsigma)| \leq c_0$. These bounds with (B-13) imply part of what is required by the second and third bullets of Property 1 and part of what is required by Property 2. The remaining parts of Properties 1 and 2 follow directly from Lemma B.2's bound on $|a_{A0,\gamma}|$ given that the absolute value of the directional derivative of $\chi_{\diamond\diamond}$ along v is bounded by $c_0 c_v^6$ and that of $\chi_{\hat{U}}$ is bounded by $c_0 c_v^2$, the latter being a consequence of what is asserted in the first bullet of (B-3). □

Proof of Lemma B.2 The proof has four steps. The proof also uses κ_c to denote a constant greater than 1 that depends only on a given upper bound for m and c_v.

Step 1 This step proves the first bullet of Lemma B.2. The bounds for $1 - |\alpha|$ and for $\mathrm{r}^{1/2}\beta_{U_0}$ come directly from Lemmas 2.1 and 2.3 as they bound both by e^{-c_v} if $c_v \geq c_0$ and $\mathrm{r} \geq \kappa_c$. To obtain the other bounds, write $\nabla_A\alpha$ on U_0 as $d|\alpha| + a_{A,U_0}|\alpha|$. Given that $a_{A,U_0}|\alpha|$ is $i\mathbb{R}$–valued, Lemmas 2.1 and 2.3 imply that $|a_{A,U_0}| \leq e^{-c_v}\mathrm{r}^{1/2}$ if $c_v \geq c_0$ and $\mathrm{r} \geq \kappa_c$. Meanwhile, these same lemmas together with the vanishing of the EK^{-1} component of $D_A\psi$ imply the bound $|(a_{A,U_0})_v| \le e^{-c_v}$. These bounds lead directly to what is asserted by the first bullet.

Step 2 To start the proof of the second bullet of Lemma B.2, use Proposition 2.4 and Lemma 2.9 to see that

$$\left|\frac{|\alpha|}{|r_{\mathrm{r}}^*\alpha_0|} - 1\right| \le e^{-c_v} \tag{B-14}$$

on U_γ when $c_v \ge c_0$ and $\mathrm{r} \ge \kappa_c$. Keeping this in mind, use the isomorphism between E over U_γ and the product bundle that sends α to $|\alpha|z/|z|$ to write α over U_γ as $|\alpha|z/|z|$. Having done so, (B-14) implies that $|\alpha - r_{\mathrm{r}}^*\alpha_0|$ is bounded by e^{-c_v} on U_γ. Use the same isomorphism to write A over U_γ as $A = \theta_0 + a_{A,U_\gamma}$, and use the coordinates from Part 4 of Section Aa on U_γ to again write a_{A,U_γ} as $a_{A,U_\gamma} = a_{A0}\,dt + \frac{1}{2}(\mathrm{A}\,d\bar{z} - \bar{\mathrm{A}}\,dz)$. The bound in (B-14) together with Lemma 2.9 have the following additional consequence: Write A_0 as done in (A-3) using the function a_0. For any given t, the $(r_{\mathrm{r}})^{-1}$–pullback of $(\mathrm{A}, \alpha)|_t$ to the radius $4c_v^2$ ball about the origin in $\mathbb{C}$ differs from $(a_0\bar{z}^{-1}, \alpha_0)$ in the C^6–topology by less than $c_v^4 e^{-c_v}$ if $c_v \ge c_0$ and $\mathrm{r} \ge \kappa_c$. Note in this regard that $a_0\bar{z}^{-1}$ is smooth near the origin although the notation suggests otherwise. (In any event, the left-hand inequality in (A-4) implies that a_0 is $\mathcal{O}(|z|^2)$ near $z = 0$.)

The function a_0 has to be a function of $|z|^2$ because (A_0, α_0) gives the symmetric vortex in $\mathfrak{C}_1$ and $\alpha_0 = |\alpha_0|z/|z|$. It follows from this that $\bar{z}\mathrm{A} - z\bar{\mathrm{A}}$ must be very small. (The point is that A is very close to $a_0\bar{z}^{-1}$ and $\bar{z}(a_0\bar{z}^{-1})$ is real.) This understood, the fact that $|z|^2 a_0 - (r_{\mathrm{r}}^{-1})^*\mathrm{A}$ has small C^6–norm implies that the r_{r}^{-1}–pullback of the function $\hat{\mathrm{o}}_\gamma$ in (B-8) from any given constant t slice of U_γ has C^6–norm bounded by $c_0 c_v^4$ on the ball of radius $4c_v^2$ centered at the origin in $\mathbb{C}$.

The preceding observation about $\hat{\mathrm{o}}_\gamma$ has the following consequence: Write the 1–form $a_{A,\gamma}$ now as $a_{A0,\gamma}\,dt + \frac{1}{2}(\mathrm{A}_\gamma\,d\bar{z} - \bar{\mathrm{A}}_\gamma\,dz)$. For any given t, the $(r_{\mathrm{r}})^{-1}$–pullback of the pair $(\mathrm{A}_\gamma, \alpha_\gamma)|_t$ to the radius $4c_v^2$ ball about the origin in $\mathbb{C}$ differs from $(a_0|z|^2, \alpha_0)$ in the C^6–topology by less than c_v^6 if $c_v \ge c_0$ and $\mathrm{r} \ge \kappa_c$. These bounds lead directly to Lemma B.2's assertions about A_γ and α_γ.

Step 3 The bound on $a_{A0,\gamma}$ requires first a bound on $a_{A0,\gamma}$ on the $|z| \ge \mathrm{r}^{-1/2}$ part of U_γ. By way of a parenthetical remark, $(\theta_0 + a_{A,\gamma}, (\alpha_\gamma, \beta_\gamma))$ are used in (B-10) and (B-12) in lieu of $(\theta_0 + a_{A,U_\gamma}, (|\alpha|, \beta_{U_\gamma}))$ in part because no bound of the form $|a_{A0}| \le c_0 c_v^2$ has been found for the whole of U_γ. As explained below, a bound of this sort does exist on the complement of any given radius tubular neighborhood of γ and the latter bound is needed to derive the desired bound for $|a_{A0,\gamma}|$.

To bound $|a_{A0}|$, use the depiction of A on U_γ as $A = \theta_0 + a_{A0}\,dt + (\mathrm{A}\,d\bar{z} - \bar{\mathrm{A}}\,dz)$ and that of α as $|\alpha|z/|z|$ to write the A–covariant derivative of α along $\frac{\partial}{\partial t}$ as

$$(\nabla_A\alpha)_{\partial/\partial t} = \Big(\frac{\partial}{\partial t}|\alpha| + a_{A0}|\alpha|\Big)\frac{z}{|z|}. \tag{B-15}$$

Since a_{A0} is $i\mathbb{R}$–valued, the norm of this directional covariant derivative is greater than $|a_{A0}|\,|\alpha|$. Meanwhile, $\big|v - \frac{\partial}{\partial t}\big| \le c_0 c_v \mathrm{r}^{-1/2}$ on U_γ, and as $|(\nabla_A\alpha)_v| \le c_0$ it follows from the bound in Lemma 2.1 that $|a_{A0}|\alpha \le c_0 c_v$. Thus, $|a_{A0}| \le c_0 c_v |z|^{-1}$ at any $z \neq 0$ point on U_γ.

Use $d^\perp a_{A0}$ to denote the differential of a_{A0} along the constant t slices of U_γ. The identity in (B-15) is used next to obtain a bound by $c_0 c_v(\mathrm{r}^{1/2} + |z|^{-1})|z|^{-1}$ on $|d^\perp a_{A0}|$. To get this bound, first write $(\nabla_A\alpha)_{\partial/\partial t}$ as $(\nabla_A\alpha)_v + \mathfrak{R}\cdot\nabla_A\alpha$, where $\mathfrak{R}$ is an endomorphism with norm bounded by $c_0\mathrm{r}^{-1/2}$ and with derivative bounded by c_0. Use the EK^{-1} component of the equation $D_A\psi = 0$ to write $(\nabla_A\alpha)_v$ as a linear combination of covariant derivatives of β. Meanwhile, (B-15) finds $a_{A0} = \mathrm{im}(\alpha^{-1}(\nabla_A\alpha)_{\partial/\partial t})$ and so

$$\begin{aligned}(\text{B-16})\quad & |d^\perp a_{A0}| \\ &\le c_0|\alpha|^{-1}\big(|\alpha|^{-1}|\nabla_A\alpha|\,|(\nabla_A\alpha)_{\partial/\partial t}| + |\nabla\mathfrak{R}|\,|\nabla_A\alpha| + |\mathfrak{R}|\,|\nabla_A^2\alpha| + \nabla_A^2\beta|\big).\end{aligned}$$

The desired bound for $|d^\perp a_{A0}|$ follows from (B-16) and Lemma 2.1.

Step 4 The bounds for $|a_{A0}|$ and $|d^\perp a_{A0}|$ in Step 3 are used first to bound $|a_{A0,\gamma}|$ on the $|z| \ge \mathrm{r}^{-1/2}$ part of U_γ under the henceforth implicit assumption that the distance to $\bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_\mathrm{p}^+ \cup \hat{\gamma}_\mathrm{p}^-)$ is greater than $c_v^4 - 2c_v^2$. To this end, note first that $|a_{A0,\gamma}| \le |a_{A0}| + |\partial_t \hat{\mathrm{o}}_\gamma|$ and so what is needed is a suitable bound on $|\partial_t\hat{\mathrm{o}}_\gamma|$. To obtain one, use (B-8) to see that $|\partial_t\hat{\mathrm{o}}_\gamma| \le c_0|z|\,|\partial_t \mathrm{A}|$. Meanwhile, $|\partial_t\mathrm{A}| \le c_0\big(|d^\perp a_{A0}| + \big|F_A\big(\frac{\partial}{\partial t}, \cdot\big)\big|\big)$, where F_A denotes the curvature 2–form of A. Use the top bullet in (1-13) with the fact that $\frac{\partial}{\partial t}$ is very close to v to see that $\big|F_A\big(\frac{\partial}{\partial t}, \cdot\big)\big|$ is bounded by $c_0\mathrm{r}^{1/2}c_v$ on U_γ. What with Step 3's bound for $|d^\perp a_{A0}|$, the latter bound implies that $|\partial_t\hat{\mathrm{o}}_\gamma| \le c_0 c_v^2$ on the $|z| \ge \mathrm{r}^{-1/2}$ part of U_γ. This with Step 3's bound for $|a_{A0}|$ leads directly to the desired $|a_{A0,\gamma}| \le c_0 c_v^2$ bound on the $|z| \ge \mathrm{r}^{-1/2}$ part of U_γ.

To obtain the desired bound for $|a_{A0,\gamma}|$ on the $|z| \le \mathrm{r}^{-1/2}$ part of U_γ, fix z with $|z| = \mathrm{r}^{-1/2}$ and, for any given $\rho \in [0, 4c_v^2\mathrm{r}^{-1/2}]$, write

$$a_{A0,\gamma}|_{\rho z} = a_{A0,\gamma}|_z - \int_{[\rho\mathrm{r}^{-1/2},1]} \partial_s(a_{A0,\gamma}|_{sz})\,ds. \tag{B-17}$$

Meanwhile, the function $\hat{\mathrm{o}}_\gamma$ was chosen specifically so as to guarantee that the 1–form $\frac{1}{2}(\mathrm{A}_\gamma\, d\bar{z} - \bar{\mathrm{A}}_\gamma\, dz)$ annihilates the radial vector field on $\mathbb{C}$, and this implies the identity

$$\partial_s(a_{A0,\gamma}|_{sz}) = -F_A\Big(\frac{\partial}{\partial t}, \frac{\partial}{\partial |z|}\Big)\Big|_{sz}. \tag{B-18}$$

As noted in the preceding paragraph, the norm of the right-hand side of (B-18) is bounded by $c_0 c_v \mathrm{r}^{1/2}$. Use this bound for $|\partial_s(a_{A0,\gamma}|_{sz/|z|})|$ in (B-17) to obtain the bound for $|a_{A0,\gamma}|$ by $c_0 c_v^2$ on the $|z| \le \mathrm{r}^{-1/2}$ part of U_γ when $c_v \ge c_0$ and $\mathrm{r} \ge \kappa_c$. □

Bc The difference between $\mathfrak{f}_s$ at (A, ψ) and at (A_*, ψ_*)

Both the (A, ψ) and the (A_*, ψ_*) version of $L_{(\cdot),\mathrm{r}}$ might have nontrivial kernel. What follows first defines what is meant by the norm of the spectral flow difference if this is the case. The subsequent Proposition B.3 asserts that this difference is bounded by a purely c_v–dependent constant.

Let $\mathfrak{c}_0$ and $\mathfrak{c}_1$ denote a given pair in $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$. Fix $\mathrm{z}_0, \mathrm{z}_1 \ge 1$ and introduce $\mathfrak{L}_0$ and $\mathfrak{L}_1$ to denote the respective $(\mathfrak{c}_0, \mathrm{z}_0)$ and $(\mathfrak{c}_1, \mathrm{z}_1)$ versions of $\mathfrak{L}_{(\cdot)}$. The norm of the spectral flow between $\mathfrak{L}_0$ and $\mathfrak{L}_1$ is denoted here by $|\mathfrak{f}_{s1} - \mathfrak{f}_{s0}|$ and it is defined as follows: Fix $\varepsilon > 0$ and introduce $\mathcal{C}_{0\varepsilon} \subset (\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})) \times (0, \infty)$ to denote the set of pairs $(\mathfrak{c}', \mathrm{z}')$ such that $\mathfrak{c}'$ has C^2–distance less than ε from $\mathfrak{c}_0$ and $|\mathrm{z}' - \mathrm{z}_0| < \varepsilon$. Require in addition that the $(\mathfrak{c}', \mathrm{z}')$ version of $\mathfrak{L}_{(\cdot)}$ have trivial kernel. Define $\mathcal{C}_{1\varepsilon}$ likewise using $(\mathfrak{c}_1, \mathrm{z}_1)$. Granted this notation, define

$$|\mathfrak{f}_{s1} - \mathfrak{f}_{s0}| = \lim_{\varepsilon \to 0} \sup\{|\mathfrak{f}_s(\mathfrak{c}_1', \mathrm{z}_1') - \mathfrak{f}_s(\mathfrak{c}_0', \mathrm{z}_0')| : (\mathfrak{c}_0', \mathrm{z}_0') \in \mathcal{C}_{0\varepsilon} \text{ and } (\mathfrak{c}_1', \mathrm{z}_1') \in \mathcal{C}_{1\varepsilon}\}. \tag{B-19}$$

Perturbation theory can by used to prove that the lim-sup on the right in (B-19) is finite, and that it is equal to the norm of the honest spectral flow difference when both the $(\mathfrak{c}_1, \mathrm{z}_1)$ and $(\mathfrak{c}_2, \mathrm{z}_2)$ versions of $\mathfrak{L}_{(\cdot)}$ have trivial kernel. The limit in (B-19) is said in what follows to be the norm of the difference between the values $\mathfrak{f}_s$.

Proposition B.3 *There exists $\kappa \ge 100$, and, given $c_v \ge \kappa^2$, there exists $\kappa_c \ge \kappa$ with the following significance: Suppose that $\mathrm{r} \ge \kappa_c c_v^{10}$ and that $\mu \in \Omega$ has $\mathcal{P}$–norm bounded by 1. Let (A, ψ) denote a solution to the (r, μ) version of (1-13). Use (A, ψ) as directed in Section Ba to construct the pair (A_*, ψ_*). The norm of the difference between the values of $\mathfrak{f}_s$ at (A, ψ) and at (A_*, ψ_*) is bounded by κ.*

Proof The $\tau = 0$ point on the path $\tau \mapsto (A_{*\tau}, \psi_{*\tau})$ is $(A - u^{-1} du, u\psi)$, where $u\colon Y \to S^1$ is a homotopically trivial map. This being the case, it is sufficient to exhibit an r– and (A,ψ)–independent bound for the absolute value of the difference between $\mathfrak{f}_s$ at $(A-u^{-1}du, u\psi)$ and at (A_*,ψ_*). Such a bound is derived in the subsequent four parts of the proof.

Part 1 Suppose that $\mathbb{L}$ is a given Hilbert space and that $\mathcal{L}$ is an unbounded, self-adjoint operator on $\mathbb{L}$. Assume that $\mathcal{L}$ has pure point spectrum with no accumulation points and such that each eigenvalue has finite multiplicity. Let $\mathfrak{e}$ denote a bounded, self-adjoint operator on $\mathbb{L}$ and suppose that $\{\mathfrak{e}_\tau\}_{\tau\in[0,1]}$ is a real analytic family of bounded, self-adjoint operators on $\mathbb{L}$ with $\mathfrak{e}_0 = 0$ and $\mathfrak{e}_1 = \mathfrak{e}$. Section 2 of [19] explains how to label the eigenvalues of each $\tau\in[0,1]$ version of $\mathcal{L}+\mathfrak{e}$ by the integers so that the following is true: Given an integer n, let $\{\lambda_{n\tau}\}_{\tau\in[0,1]}$ denote the corresponding 1–parameter family of eigenvalues. Then the map from $[0,1]$ to $\mathbb{R}$ given by the rule $\tau\mapsto\lambda_{n\tau}$ is continuous and piecewise real analytic. Moreover, the corresponding 1–parameter family of eigenvectors varies in a real analytic fashion where $\lambda_{n(\cdot)}$ does. Let $\{\mathfrak{f}_{(\tau)}\}_{\tau\in[0,1]}$ denote a corresponding 1–parameter family of unit-length eigenvectors. The map $\tau\mapsto\mathfrak{f}_{(\tau)}$ can be assumed real analytic on the open subsets in $[0,1]$ where $\lambda_{n(\cdot)}$ is real analytic. As noted in [19], the derivative of $\lambda_{n(\cdot)}$ where it is real analytic is given by

$$\frac{d}{d\tau}\lambda_{n\tau} = \Big\langle \mathfrak{f}_{(\tau)}, \Big(\frac{d}{d\tau}\mathfrak{e}_\tau\Big)\mathfrak{f}_{(\tau)}\Big\rangle_{\mathbb{L}}, \tag{B-20}$$

where $\langle\ ,\ \rangle_{\mathbb{L}}$ denotes here the inner product on $\mathbb{L}$.

Part 2 Let $\mathbb{L}$ denote the Hilbert space $L^2(Y;\mathbb{V}_0\oplus\mathbb{V}_1)$, let $\mathcal{L}$ denote the $z=\mathrm{r}$ and (A_{*0},ψ_{*0}) version of the operator $\mathfrak{L}_{\mathbb{V}}$ (from (A-26) and (A-27)) and let $\mathcal{L}+\mathfrak{e}$ denote the corresponding (A_*,ψ_*) version of this operator. The next lemma implies in part that what is said in Part 1 can be invoked for this version of $\mathbb{L}$, $\mathcal{L}$ and $\mathfrak{e}$. This lemma uses $\kappa_\diamond$ to denote a number that is greater than the versions of the constant κ that appear in Lemmas A.1–A.8 and B.1–B.2.

Lemma B.4 *Fix $m\geq\kappa_\diamond$. There exists an m–dependent $\kappa>1$ and, given $c_v\geq\kappa$, there exists $\kappa_{c_v}\geq\kappa$ with the following significance: Suppose that $\mathrm{r}\geq\kappa_{c_v}c_v^{10}$ and suppose that $(A,\psi=(\alpha,\beta))$ is a solution to the (r,μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Construct the family of operators $\{\mathfrak{L}_{\mathbb{V},\tau}\}_{\tau\in[0,1]}$ for the path $\{(A_{*\tau},\psi_{*\tau})\}_{\tau\in[0,1]}$* (see Lemma B.1 and Part 1 above). *Fix $n\in\mathbb{Z}$ and let $\{\lambda_{n\tau}\}_{\tau\in[0,1]}$ denote the corresponding family of eigenvalues. Then $|\lambda_{n\tau}|=0$ for some $\tau\in[0,1]$ only if $|\lambda_{n\tau'}|\leq\kappa c_v^{-m}$ for all $\tau'\in[0,1]$.*

Lemma B.4 is proved in the upcoming Section Bd of this appendix.

Let $\kappa_\diamond$ now denote a constant that is greater than the various versions of κ that appear in Lemmas A.2–A.8 and Lemmas B.1–B.2 and B.4. Fix c_v and r so that the assumptions of these lemmas are met. Let Δ denote the dimension of the span of the eigenvectors of $\mathfrak{L}_{\mathbb{V},1}$ with eigenvalue between $-\kappa_\diamond c_v^{-1}$ and $\kappa_\diamond c_v^{-1}$. It is a consequence of Lemma B.4 that the norm of the spectral flow difference between (A_{*0}, ψ_{*0}) and (A_*, ψ_*) is no greater than Δ. This being the case, Proposition B.3 follows if Δ has an r– and (A, ψ)–independent bound given a suitable r– and (A, ψ)–independent choice of m and then c_v. A choice for c_v that yields such a bound Δ is derived in Parts 4 and 5 of the proof. The subsequent Part 3 of the proof supplies two observations in the form of lemmas that are used in the derivation.

Part 3 To set the stage for the first lemma, use $\kappa_{\diamond\diamond}$ to denote the version of κ given by Proposition 2.4. Fix $\mathrm{r} \geq \kappa_{\diamond\diamond}$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1. Let (A, ψ) denote a solution to the corresponding (r, μ) version of (1-13). Let $\gamma \subset Y$ denote a closed, connected segment in α's zero locus whose points have distance at least $100\kappa_{\diamond\diamond}^2 \mathrm{r}^{-1/2}$ from all curves in $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$. Use the coordinates from Part 4 of Section Aa to define the functions ν and μ on γ and having done so, use L_γ to denote the operator on $C^\infty(\gamma; \mathbb{C})$ that is defined by the rule $\zeta \mapsto \frac{i}{2}\frac{d}{dt}\zeta + \nu\zeta + \mu\bar{\zeta}$. This operator defines a function on $C^\infty(\gamma; \mathbb{C})$ by the rule $\zeta \mapsto \|L_\gamma\zeta\|_2$, where $\|\cdot\|_2$ denotes here the L^2–norm on $C^\infty(\gamma; \mathbb{C})$. This function can be restricted to any given linear subspace in $C^\infty(\gamma; \mathbb{C})$. Given $\mathrm{T} > 0$, there is always an integer that is greater than or equal to the dimension of any linear subspace in $C^\infty(\gamma; \mathbb{C})$ on which the function $\zeta \mapsto \|L_\gamma\zeta\|_2$ obeys $\|L_\gamma\zeta\|_2 \leq \mathrm{T}\|\zeta\|_2$.

The upcoming Lemma B.5 concerns L_γ and a least upper bound of the sort just described. By way of a parenthetical remark, the versions of L_γ that appear in Lemma A.8 are of particular interest with regards to the proof of Proposition B.3.

Lemma B.5 *There exists $\kappa > \kappa_{\diamond\diamond}$ with the following significance: Fix $\mathrm{r} \geq \kappa$ and $\mu \in \Omega$ with $\mathcal{P}$–norm bounded by 1. Suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the corresponding (r, μ) version of (1-13). Let γ denote a closed, connected segment of the zero locus of α whose points have distance at least $100\kappa_{\diamond\diamond}\mathrm{r}^{-1/2}$ from all of the curves in the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$. Use Δ_γ to denote the least upper bound for the dimensions of the linear subspaces in $C^\infty(\gamma; \mathbb{C})$ on which the function $\zeta \mapsto \|L_\gamma\zeta\|_2$ obeys $\|L_\gamma\zeta\|_2 \leq \kappa^{-1}\|\zeta\|_2$. This least upper bound obeys $\Delta_\gamma \leq \kappa$.*

The argument for this lemma would be straightforward were there an r–independent upper bound on γ's length, but such a bound does not exist. In any event, the proof is given in a moment. The next lemma states an analog of Lemma B.5 with the straightforward argument for its proof. This lemma enters the proof of Proposition B.3 in conjunction with Lemma A.7. Lemma B.6 also plays a role in Lemma B.5's proof.

Lemma B.6 *Fix* $\mathrm{R} > 0$ *and* $\mathrm{E} > 0$. *The least upper bound for the dimension over* $\mathbb{C}$ *of the linear subspaces in* $C^\infty([0,\mathrm{R}];\mathbb{C})$ *on which the function* $\zeta \mapsto \|\frac{d}{dt}\zeta\|_2$ *obeys* $\|\frac{d}{dt}\zeta\|_2 \le \mathrm{E}\|\zeta\|_2$ *is bounded by* $2 + \pi^{-1}\mathrm{RE}$.

Proof The subset of elements in $C^\infty([0,\mathrm{R}];\mathbb{C})$ that vanish at both endpoints has complex codimension 2. This understood, the least upper bound in question is no greater than 2 plus the number of linearly independent eigenvectors for the operator $-\frac{d^2}{dt^2}$ on $C^\infty([0,\mathrm{R}];\mathbb{C})$ that vanish at both endpoints and have eigenvalue less than E. This number is $\pi^{-1}\mathrm{RE}$. □

Proof of Lemma B.5 The four steps that follow constitute the proof.

Step 1 The lemma is proved by cutting γ into a concatenation of c_0 closed, connected segments, and then bounding a version of $\Delta_{(\cdot)}$ on each segment. To explain why such a cutting strategy works, suppose for the moment that $\gamma_0 \subset \gamma$ is a closed, connected segment. Fix $\mathrm{T} > 0$ and introduce $\Delta_{\gamma_0,\mathrm{T}}$ to denote the least upper bound for the dimensions of the linear subspaces in $C^\infty(\gamma_0;\mathbb{C})$ on which the function $\|L_{\gamma_0}(\cdot)\|_2$ is bounded by $\mathrm{T}^{-1}\|\cdot\|_2$ with $\|\cdot\|_2$ denoting here the L^2–norm on $C^\infty(\gamma_0;\mathbb{C})$. Suppose that γ_1 and γ_2 are two such segments that share at least one endpoint. Then $\Delta_{\gamma_1\cup\gamma_2,\mathrm{T}} \le 4 + \Delta_{\gamma_1,\mathrm{T}} + \Delta_{\gamma_2,\mathrm{T}}$. This is because the subspace in $C^\infty(\gamma;\mathbb{C})$ that vanishes at the common endpoints of γ_1 and γ_2 has codimension 2 if they share one endpoint and codimension 4 if they share two endpoints.

With the preceding understood, suppose that γ is written as the concatenation of N segments $\{\gamma_k\}_{1\le k\le \mathrm{N}}$. Iterate the bound given in the previous paragraph to see that $\Delta_{\gamma,\mathrm{T}}$ is no greater than $4\mathrm{N} + \sum_{1\le k\le \mathrm{N}} \Delta_{\gamma_k,\mathrm{T}}$.

Step 2 Fix $\varepsilon > 0$ and let $\gamma^\varepsilon \subset \gamma$ denote the part of γ with distance at least ε from the curves in the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat\gamma_{\mathfrak{p}}^+ \cup \hat\gamma_{\mathfrak{p}}^-)$. As explained directly, $\Delta_{\gamma^\varepsilon,\mathrm{T}} \le c_0(1+\mathrm{T}^{-2})|\ln\varepsilon|$.

To see why this bound holds, keep in mind that L_γ is defined by the pair (ν,μ) and the latter are defined by the chosen coordinates from Part 4 of Section Aa. Granted that such is the case, any version of L_γ can be obtained from a given version by conjugating the given version with a map from γ to S^1. This implies, in particular, that $\Delta_{\gamma^\varepsilon,\mathrm{T}}$ does not depend on the choice of coordinates. This being the case, choose the coordinates from Part 4 of Section Aa so that the resulting pair ν and μ are such that $|\nu|+|\mu| \le c_0$. Use B to denote to denote an upper bound for $|\nu|$ and $|\mu|$ on γ^ε.

Let L denote the length of γ^ε. Let $\mathcal{V} \subset C^\infty(\gamma^\varepsilon;\mathbb{C})$ denote a linear subspace of the sort under consideration. If γ^ε has no endpoints, then the dimension of $\mathcal{V}$ is no greater than the dimension of the span of the eigenvectors of $-\frac{d^2}{dt^2}$ with eigenvalue no greater than $(\mathrm{B}+\mathrm{T}^{-1})^2$. As noted in the proof of Lemma B.6, if γ^ε has endpoints, then $\mathcal{V}$'s dimension is at most 4 more than the span of the Dirichlet eigenvectors of $-\frac{d^2}{dt^2}$ with eigenvalue no greater than $(\mathrm{B}+\mathrm{T}^{-1})^2$. In both cases, there are at most $c_0(1+\mathrm{B}+\mathrm{T}^{-1})^2\mathrm{L}$ linearly independent eigenvectors with this eigenvalue bound. Meanwhile, Proposition 2.4 with Proposition II.2.7 and Lemma II.2.2 imply among other things that the length of γ^ε is no greater than $c_0|\ln\varepsilon|$, and that both $|\nu|$ and $|\mu|$ are bounded by c_0.

Step 3 Let $\hat{\gamma} \in \bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$ denote a given curve. As explained in Part 4 of Section 1.1, there is a version of the coordinates from Part 4 of Section Aa for $\hat{\gamma}$ with both ν and μ constant, with μ real and such that $\mu > |\nu|$. This version is assumed in what follows. The corresponding constant values for ν and μ are denoted by ν_0 and μ_0.

Fix $\varepsilon > 0$ such that the radius ε tubular neighborhood of $\hat{\gamma}$ is well inside the coordinate chart just described. Let T denote such a tubular neighborhood, and suppose that $\upsilon \subset T$ is a closed, connected segment in T of an integral curve of v. Taylor's theorem with remainder can be used with the formulas in (A-3) to see that υ has a tubular neighborhood with coordinates from Part 4 of Section Aa with $|\nu-\nu_0|+|\mu-\mu_0| < c_0\varepsilon$.

Reintroduce from Proposition 2.4 the subset $Y_{\mathrm{r}} \subset Y$. By way of a reminder, the points in $T \cap Y_{\mathrm{r}}$ have distance no less than $c_0\mathrm{r}^{-1/2}$ from $\hat{\gamma}$. Let γ^T denote a properly embedded, connected component of α's zero locus in the closure of $T \cap Y_{\mathrm{r}}$. Thus, γ^T has two boundary points, either both on the boundary of the closure of T, or one on the latter and one on Y_{r}'s boundary torus in T.

Step 4 Define the operator L_0 on $C^\infty(\mathbb{R};\mathbb{C})$ by the rule $\zeta \mapsto \frac{i}{2}\frac{d}{dt}\zeta + \nu_0\zeta + \mu_0\bar{\zeta}$. Fix $\ell > 0$ and restrict L_0^2 to the subspace of elements in $C^\infty([0,\ell];\mathbb{C})$ that vanish at the boundary points. The corresponding Dirichlet eigenvalues of L_0^2 on this domain are of the form $\mathrm{K}^2+\nu_0^2+\mu_0^2 \pm 2\nu_0(\mathrm{K}^2+\mu_0)^2$ with $\mathrm{K} = \pi(2k4\ell+1)/(4\ell)$ for $k \in \mathbb{Z}$. Note in particular that the smallest eigenvalue is greater than $(\mu_0-\nu_0)^2$ when $\mu_0 > \nu_0$, thus greater than c_0^{-1}.

Let L_{γ^T} denote the restriction of L_γ to $C^\infty(\gamma^T;\mathbb{C})$. What was said in the preceding paragraph and what was said in the final paragraph of Step 3 have the following implication: Let $\zeta \in C^\infty(\gamma^T;\mathbb{C})$ denote an element that vanishes at both boundary points of γ^T. Then $\|L_{\gamma^T}\zeta\|_2 \ge ((\mu_0-\nu_0)-c_0\varepsilon)\|\zeta\|_2$. Thus, if $\varepsilon < c_0^{-1}$, then

$$\|L_{\gamma^T}\zeta\|_2 \ge \tfrac{1}{2}(\mu_0-\nu_0)\|\zeta\|_2, \tag{B-21}$$

and this implies that $\Delta_{\gamma^T,(\nu_0-\mu_0)/2} \le 4$. Note that (B-21) and the preceding bound on $\Delta_{\gamma^T,(\nu_0-\mu_0)/2}$ hold even if the pair (ν,μ) that define L along γ^T are not ε–close to (ν_0,μ_0). This is because any two versions of (ν,μ) that arise from different choices for coordinates from Part 4 of Section Aa define corresponding versions of L that are obtained from each other by conjugating with a map from γ to S^1.

Lemma B.5 follows from the $\Delta_{\gamma^{\mathrm{T}},(\nu_0-\mu_0)/2} \le 4$ bound and those given in Step 2. □

Part 4 Fix m and then c_v and r so as to invoke the conclusions of Lemmas A.2–A.8 and Lemmas B.4–B.6. Sum Lemma B.5's integers $\{\Delta_\gamma\}_{\gamma\in\Theta}$ and use Δ_Θ to denote the result. Let $\kappa_\bullet$ denote the largest of the versions of the constant κ that appears in Lemmas A.4, A.7 and B.5. Let N denote the number of components of $Y - Y_{*\Lambda}$ with zeros of α. Each such component has the same length, this denoted by ℓ_*. Lemma A.7 associates an integer m to each such component. As noted in Part 3 of Section Ad, no version of m is greater than $\kappa_\bullet c_v^4$.

Use N to denote the number of linearly independent eigenvectors of $\mathfrak{L}_{\mathbb{V},1}$ with eigenvalue between $-c_v^{-1}$ and c_v^{-1}. As is explained in the subsequent paragraphs, N is no greater than $10^4(\Delta_\Theta + \mathrm{N}\kappa_* c_v^4(1+\ell_*\kappa_\bullet c_v^{\kappa_\bullet}))$ if $c_v \ge c_0$ and r is larger than a constant that depends only on c_v.

To see why this bound holds, suppose in what follows that N is larger than what is claimed, so as to generate nonsense. If N is larger than the asserted bound, then there exists a section, $\mathfrak{f}$, of $\mathbb{V}_0 \oplus \mathbb{V}_1$ with four properties that are described next. Lemmas B.5 and B.6 guarantee that the third and fourth properties can be satisfied. First, $\mathfrak{f}$ is a linear combination of an orthonormal set of eigenvectors of $\mathfrak{L}_{\mathbb{V},1}$ with eigenvalue between $-c_v^{-1}$ and c_v^{-1}. Second, $\mathfrak{f}$ has unit L^2–norm. The third property concerns the curves in Θ. Let γ denote such a curve. Let ζ_γ denote γ's component of $\Pi_\vartheta\mathfrak{f}$. The function $\zeta \mapsto \|L_\gamma\zeta\|_2$ from Lemma B.5 is such that $\|L_\gamma\zeta_\gamma\|_2 \ge \kappa_\bullet^{-1}\|\zeta_\gamma\|_2$. The final property concerns the components of $Y - Y_{*\Lambda}$ that contain zeros of α. Let $\gamma \in \bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$ denote a curve from such a component. Part 1 of Section Ah describes a cover of γ by open sets $\{\gamma_k\}_{0\le k\le 7}$ or by open sets γ_+ and γ_-. Let γ_* denote a component of this cover. Denote by m the rank of $\mathrm{Ker}_\vartheta|_{\gamma_*}$. Use the isomorphism between $\mathrm{Ker}_\vartheta|_{\gamma_*}$ and $\gamma_* \times \mathbb{C}^m$ in Lemma A.7 to view the component of $\Pi_\vartheta \mathfrak{f}$ in $\mathrm{Ker}_\vartheta|_{\gamma_*}$ as a map from γ_* to $\mathbb{C}^m$. Let ζ_{γ_*} denote this map. Then $\bigl\| \frac{i}{2}\frac{d}{dt}\zeta_{\gamma_*}\bigr\|_2$ is greater than $2\kappa_\bullet c_v^{\kappa_\bullet}\|\zeta_{\gamma_*}\|$.

What with the third and fourth properties, Lemmas A.7 and A.8 imply $\|\Pi_\vartheta \mathfrak{L}_{\mathbb{V},1}\mathfrak{f}\|_2 \ge \kappa_\bullet^{-1}\|\Pi_\vartheta \mathfrak{f}\|_2$. This being the case, Lemma A.6 finds $\|\Pi_\vartheta \mathfrak{L}_{\mathbb{V},1}\mathfrak{f}\|_2 \ge \frac{1}{2}\kappa_\bullet^{-1}\|\mathfrak{f}\|_2$ if $c_v \ge c_0$ and if r is greater than a purely c_v–dependent constant. Meanwhile, it follows from the definitions that $\|\mathfrak{q}\|_2 \ge c_0^{-1}\|\Pi_\vartheta \mathfrak{q}\|_2$ for any given section $\mathfrak{q}$ of $\mathbb{V}_0 \oplus \mathbb{V}_1$. Take $\mathfrak{q}$ to be $\mathfrak{L}_{\mathbb{V},1}\mathfrak{f}$ to see that $\|\mathfrak{L}_{\mathbb{V},1}\mathfrak{f}\|_2 \ge (c_0\kappa_\bullet)^{-1}\|\mathfrak{f}\|_2$. Even so, the first property listed in the preceding paragraph requires the bound $\|\mathfrak{L}_{\mathbb{V},1}\mathfrak{f}\|_2 \le c_v^{-1}\|\mathfrak{f}\|_2$ and so $c_v^{-1} \ge (c_0\kappa_\bullet)^{-1}$ unless $\mathfrak{f}$ is identically zero, and $\mathfrak{f} \ne 0$ because of the second of the listed properties.

This (A,ψ)– and r–independent lower bound for c_v^{-1} is the required nonsense because c_v has no a priori upper bound. □

Bd Proof of Lemma B.4

The proof of Lemma B.4 has three parts.

Part 1 This part of the proof states an auxiliary lemma that augments what is said by Lemma B.2. By way of a reminder, Lemma B.2 concerns the $Y_{\diamond\diamond}$ extension of a given $\gamma \in \Theta$ version of U_γ. With coordinates from Part 4 of Section Aa chosen, Lemma B.2 uses the $\alpha \mapsto |\alpha|z/|z|$ isomorphism from $E|_{U_\gamma}$ to $U_\gamma \times \mathbb{C}$ to write $(A - u_\gamma^{-1}du_\gamma, (u_\gamma\alpha, u_\gamma\beta))$ as $(\theta + a_{A,\gamma}, (\alpha_\gamma, \beta_\gamma))$ with the map $u_\gamma\colon U_\gamma \to S^1$ defined in (B-8). It goes on to write the $i\mathbb{R}$–valued 1–form $a_{A,\gamma}$ as $a_{A0,\gamma}\,dt + \frac{1}{2}(\mathrm{A}_\gamma\,d\bar{z} - \bar{\mathrm{A}}_\gamma\,dz)$. Whereas Lemma B.2 talks about the functions A and α_γ, the upcoming Lemma B.7 talks about the functions $a_{A0,\gamma}$ and β_γ. This lemma brings in the functions ς and y on $\mathbb{C}$ from (A-2), and it uses u to denote the function of the radial coordinate $|z|$ on $\mathbb{C}$ that is given by integrating the function $1 - |\alpha_0|^2$ along any ray from the origin starting at distance $|z|$ from the origin. Thus

$$u(|z|) = \int_{|z|}^{\infty} (1 - |\alpha_0|^2)|_s\,ds. \tag{B-22}$$

The lemma also invokes the coordinates from Part 4 of Section Aa to bring in the corresponding version of the function $t \mapsto x_\gamma(t)$.

Lemma B.7 *Fix $m \ge 1$. There exists an m–dependent $\kappa > 1$ and, given $c_v \ge \kappa$, there exists $\kappa_{c_v} \ge \kappa$ with the following significance: Take $\mathrm{r} \ge \kappa_{c_v} c_v^{10}$ and suppose that*

$(A,\psi=(\alpha,\beta))$ is a solution to the (r,μ) *version of* (1-13) *with μ a given element in Ω with $\mathcal{P}$–norm smaller than* 1. *Use (A,ψ) as directed in Section Ba to construct the pair (A_*,ψ_*). Let γ denote a component of α's zero locus in $Y_{*\Lambda}$.*

- $|\mathrm{r}^{1/2}\beta_\gamma - i\mu r_{\mathrm{r}}^*\varsigma| \le c_v^{-m}$ *at all points in the $Y_{*\Lambda}$ part of U_γ.*
- $|a_{A0,\gamma} - i v 2^{1/2} r_{\mathrm{r}}^* y + i\mathrm{r}(x_\gamma \bar z/|z| + \bar x_\gamma z/|z|) r_{\mathrm{r}}^* u| \le c_v^{-m}$ *at all points in the $Y_{*\Lambda}$ part of U_γ.*

Proof The proof has six steps. The first four prove the top bullet and the last two prove the lower bullet. As in the proofs of Lemmas B.1 and B.2, what is denoted by κ_c is a constant with value greater than 1 that depends only on m and c_v; in particular, it has no r– and (A,ψ)–dependence.

Step 1 The EK^{-1} component of the equation $D_A^2\psi = 0$ (with a factor of $\frac{1}{4}$ in front) can be written in the schematic form $-\frac{1}{4}((\nabla_A)_v)^2\beta - \bar\partial_A\partial_A\beta + \frac{1}{2}\mathrm{r}|\alpha|^2\beta = -\mu\partial_A\alpha + \mathfrak{r}$ where $|\mathfrak{r}| \le c_0$. This equation is used in what follows on the extension of U_γ into $Y_{\diamond\diamond}$ with the coordinates from Part 4 of Section Aa. The section β of EK^{-1} is viewed as a $\mathbb{C}$–valued function on U_γ using these coordinates and the chosen isomorphism on U_γ between E and $U_\gamma \times \mathbb{C}$. This function is denoted by β_γ. Meanwhile, the connection $A - u_\gamma^{-1} du_\gamma$ is written as $A_\gamma = a_{A0,\gamma}\,dt + \frac{1}{2}(\mathrm{A}_\gamma\, d\bar z - \bar{\mathrm{A}}_\gamma\, dz)$. Use the derivative bounds for β given by Lemma 2.1 to replace the derivative $(\nabla_A)_v$ with ∂_t so as to obtain from (A-36) the equation

$$-\tfrac{1}{4}\partial_t^2\beta_\gamma - \bar\partial_A\partial_A\beta_\gamma + \tfrac{1}{2}\mathrm{r}|\alpha|^2\beta_\gamma = -\mu\partial_A\alpha_\gamma + \mathfrak{r} \tag{B-23}$$

with $\mathfrak{r}$ different and now such that $|\mathfrak{r}| \le c_0 c_v^2$ when $c_v \ge c_0$ and $\mathrm{r} \ge \kappa_c$. The notation here uses $\partial_A = \frac{\partial}{\partial \bar z} + \frac{1}{2}\mathrm{A}_\gamma$ and it uses $\bar\partial_A$ for the complex conjugate operator. (With regards to replacing $(\nabla_A)_v$ by ∂_t, this leads to a small error because the vector fields $\frac{\partial}{\partial t}$ and v are nearly the same on U_γ. Indeed, their difference is at most $c_0 c_v^2 \mathrm{r}^{-1/2}$ because they are equal on the central arc inside U_γ and because the radius of the transverse disks in U_γ is $4\rho_*$, which is $4c_v^2\mathrm{r}^{-1/2}$.)

Reintroduce the section β_* from Part 5 of Section Aa. Of particular interest here is β_* on U_γ, where it can be written (see (A-8)) as $i\mu\mathrm{r}^{-1/2} r_{\mathrm{r}}^*\varsigma$. It follows from (3-27) that the section β_* obeys an (A_*,α_*) analog of (B-23):

$$-\tfrac{1}{4}\partial_t^2\beta_* - \bar\partial_{A_*}\partial_{A_*}\beta_* + \tfrac{1}{2}\mathrm{r}|\alpha_*|^2\beta_* = -\mu\partial_{A_*}\alpha_* + \mathfrak{r}, \tag{B-24}$$

with $\mathfrak{r}$ here different from its (B-23) incarnation but such that $|\mathfrak{r}| \le c_0 c_v^2$. With regards to (B-24): This equation is very nearly the pullback of (3-27) via r_{r}. This is because the $r_{\mathrm{r}}^*\varsigma$ factor in β_* is independent of the t–coordinate (so its t–derivatives are zero) whereas the norms of the t–derivatives of the μ factor in β_* (which can depend on t are $\mathcal{O}(1)$. Granted these facts, then the $-\frac{1}{4}\partial_t^2\beta_*$ term on the left-hand side of (B-24) is small, and it is canceled by a part of what is denoted by $\mathfrak{r}$ on the right-hand side of (B-24).

Step 2 Use Δ_A to denote $A_\gamma - A_*$ and use Δ_α to denote $\alpha_\gamma - \alpha_*$. Their absolute values on U_γ are such that $\mathrm{r}^{-1/2}|\Delta_A| + |\Delta_\alpha| \le \mathrm{r}^{1/2}e^{-c_v}$ when $c_v \ge \kappa$ and $\mathrm{r} \ge \kappa_c$. Write the connection A_* as $A_\gamma - \Delta_A$, and write α_* as $\alpha_\gamma - \Delta_\alpha$. Write A_γ as A and α_γ as α (to stem the proliferation of subscripts), and rewrite (B-24) as

$$\text{(B-25)}\quad -\tfrac{1}{4}\partial_t^2\beta_* - \bar{\partial}_A\partial_A\beta_* + \tfrac{1}{2}\mathrm{r}|\alpha|^2\beta_* = -\mu\partial_A\alpha + \mu\partial_A\Delta_\alpha - \mu\Delta_A^{(1,0)}\Delta_\alpha - \mathfrak{R}\cdot\beta_* + \mathfrak{r},$$

where the notation uses $\Delta_A^{(1,0)}$ to denote the $(1,0)$ part of Δ_α. Meanwhile, what is written as $\mathfrak{R}\cdot\beta_*$ is linear in β_* and can be written as

$$\text{(B-26)}\quad (\bar{\partial}_A\Delta_A^{(1,0)})\beta_* + \Delta_A^{(0,1)}\partial_A\beta_* + \Delta_A^{(0,1)}\bar{\partial}_A\beta_* - \mathfrak{w}_0(\Delta_A,\Delta_A)\beta_* + \mathrm{r}\mathfrak{w}_1(\Delta_\alpha)\beta_*,$$

where $|\mathfrak{w}_{0,1}| \le c_0$ and with $\mathfrak{r}$ different but still obeying $|\mathfrak{r}| \le c_0 c_v^2$.

Let $\Delta_\beta = \beta - \beta_*$. The two equations (B-23) and (B-25) imply that Δ_β obeys

$$\text{(B-27)}\quad -\tfrac{1}{4}\partial_t^2\Delta_\beta - \bar{\partial}_A\partial_A\Delta_\beta + \tfrac{1}{2}\mathrm{r}|\alpha|^2\Delta_\beta = -\mu\partial_A\Delta_\alpha + \mu\Delta_A^{(1,0)}\Delta_\alpha + \mathfrak{R}\cdot\beta_* + \mathfrak{r},$$

where $\mathfrak{r}$ is again a term with absolute value bounded by $c_0c_v^2$. Use $\mathfrak{o}_\beta$ to denote the function $\frac{1}{2}|\Delta_\beta|^2$. Take the Hermitian inner product of both sides of (B-27) with Δ_β and commute covariant derivatives of A to obtain an equation for $\mathfrak{o}_\beta$, this being the next equation. This upcoming equation uses $\nabla_A^\perp$ to denote the covariant derivative along the constant t slices of U_γ and it uses $\mathrm{Re}[\cdot]$ to denote the real part of the indicated expression. What follows is the promised equation for $\mathfrak{o}_\beta$:

$$\text{(B-28)}\quad \begin{aligned} &-\tfrac{1}{4}\partial_t^2\mathfrak{o}_\beta - \bar{\partial}\partial\mathfrak{o}_\beta + \tfrac{1}{2}\mathrm{r}(1+|\alpha|^2)\mathfrak{o}_\beta \\ &\quad = -\tfrac{1}{4}|\partial_t\Delta_\beta|^2 - \tfrac{1}{4}|\nabla_A^\perp\Delta_\beta|^2 \\ &\qquad\qquad + \mathrm{Re}[-\mu\bar{\Delta}_\beta\partial_A\Delta_\alpha + \mu\bar{\Delta}_\beta\Delta_A^{(1,0)}\Delta_\alpha + \bar{\Delta}_\beta\mathfrak{R}\cdot\beta_*] + \mathfrak{r}. \end{aligned}$$

What is denoted by $\mathfrak{r}$ here signifies a term with absolute value bounded by $c_0c_v^2$.

Step 3 Let $p = (t, z)$ denote a given point in $\mathbb{R}\times\mathbb{C}$ and introduce $G_p(\cdot)$ to denote the Green's function for the operator $-(\partial_t^2 + 4\bar{\partial}\partial) + 2\mathrm{r}$ on $\mathbb{R}\times\mathbb{C}$ with pole at p. This Green's function is positive and is such that

$$\text{(B-29)}\quad G_p \le c_0\frac{1}{|p-(\cdot)|}e^{-\sqrt{\mathrm{r}}|p-(\cdot)|}, \quad |dG_p| \le c_0\left(\frac{1}{|p-(\cdot)|^2} + \sqrt{\mathrm{r}}\right)e^{-\sqrt{\mathrm{r}}|p-(\cdot)|},$$

where d here denotes the full exterior derivative on $\mathbb{R}\times\mathbb{C}$. Introduce $\chi_\diamond$ to denote the function on the Y_{**} extended U_γ that is defined as follows: Take $\chi_\diamond=\chi_{\hat{U}}$ on $Y_{*\Lambda}$ and take it equal to $\chi_{\hat{U}}(1-\chi_{\diamond\diamond})$ on U_γ's intersection with a given component of $Y_{\diamond\diamond}-Y_{*\Lambda}$. Thus, $\chi_\diamond$ has compact support on the interior of U_γ and it is equal to 1 at the points in $Y_{*\Lambda}$ with distance $c_v^2\mathrm{r}^{-1/2}$ or less from γ. Take p to be a point in U_γ where $\chi_{\hat{U}}$ is equal to 1. Multiply both sides of (B-29) by $(\chi_\diamond)^2G_x$ and then integrate both sides to obtain an equality between two integrals. The left-hand side integral is denoted by I_L and the right-hand side by I_R. As explained in the next step of the proof, the equality $I_L=I_R$ leads directly to the bound $|\Delta_\beta|\le c_0\mathrm{r}^{-1/2}c_v^{10}e^{-c_v/2}$ at p. This proves the assertion of the top bullet of Lemma B.7 at points in $U_\gamma\cap Y_{*\Lambda}$ with distance $c_v^2\mathrm{r}^{-1/2}$ or less from γ if $c_v\ge c_0$ and $\mathrm{r}\ge\kappa_c$. Meanwhile, Lemmas 2.1 and 2.2 with (3-3) imply what is asserted by the top bullet of Lemma B.7 on the points in $U_\gamma\cap Y_{*\Lambda}$ with distances greater than $c_v^2\mathrm{r}^{-1/2}$ from γ if $c_v\ge c_0$ and $\mathrm{r}\ge\kappa_c$.

Step 4 The asserted bound on I_L is derived by integrating by parts twice in the relevant integral. The result can be written as

$$I_L=\tfrac{1}{2}|\Delta_\beta|^2(x)+\mathfrak{e},$$

where $\mathfrak{e}$ is a function with $|\mathfrak{e}|\le c_0\mathrm{r}^{-1}e^{-c_v^2/c_0}$. By way of explanation, the function $\mathfrak{e}$ comes from an integral whose integrand has a term that is bounded in absolute value by $c_0|\Delta_\beta|^2(|d\chi_\diamond|^2+|d^\dagger d\chi_\diamond|)G_x$ and one that is bounded in absolute value by $c_0|\Delta_\beta|^2|d\chi_x||dG_x|$. Since these terms are supported at distances no less than $(1+c_0^{-1})c_v^2\mathrm{r}^{-1/2}$ from x, and since $|\Delta_\beta|\le|\beta|+|\beta_*|\le c_0\mathrm{r}^{-1/2}$ in any event, the claim about I_L is a consequence of the exponential factors in (B-29).

Meanwhile, the integral I_R is bounded by $c_0\mathrm{r}^{-1}c_v^{10}e^{-c_v}$. By way of explanation, some judiciously chosen applications of integration by parts will remove derivatives along the $\mathbb{C}$ factor of $\mathbb{R}\times\mathbb{C}$ from Δ_α and Δ_A and replace them with terms that have derivatives of either G_x or $\chi_\diamond$ or covariant derivatives of β_*. A covariant derivative of β_* is bounded by $c_0(|\nabla_A^\perp\Delta_\beta|+|\nabla_A\beta|)$. Lemma 2.1 has $|\nabla_A\beta|\le c_0$ and this with (B-29) together with the bounds from Step 2 for $|\Delta_A|$ and $|\Delta_\alpha|$ can be used in a straightforward fashion to bound the integrals that result by $c_0\mathrm{r}^{-1}c_v^{20}e^{-c_v}$.

Step 5 This step begins the proof of the lower bullet of Lemma B.7. As is explained in this step and Step 6, the second bullet's assertion is a consequence of the identity in (B-18) and what is asserted by a version of Lemma B.7's top bullet that uses a suitable $m'>m$. To exploit (B-18), first write $\partial/\partial|z|$ as $(z/|z|)\partial/\partial z+(\bar{z}/|z|)\partial/\partial\bar{z}$. Next write $\partial/\partial z$ as $\frac{1}{2}(\hat{e}_1-i\hat{e}_2)+\mathfrak{r}_t\partial/\partial t+\mathfrak{r}_2$, where $\{\hat{e}_1,\hat{e}_2\}$ is an oriented, orthonormal frame for the kernel of $\hat{a}$, where $|\mathfrak{r}_t| \le c_0|z|$, and where $|\mathfrak{r}_2| \le c_0|z|^2$. Doing so writes $\partial/\partial|z|$ in terms of $\{\hat{e}_1, \hat{e}_2\}$. Use this depiction in (A-6). Meanwhile, use (A-6) to write $\partial/\partial t$ in terms of v. The result of all of this rewriting replaces the curvature component on the right-hand side of (B-18) by

$$-|z|^{-1}(z_1 F_A(v,\hat{e}_1) + z_2 F_A(v,\hat{e}_2) + (2\nu|z|^2 + \mu\bar{z}^2 + \bar{\mu}z^2 - x_\gamma\bar{z} - \bar{x}_\gamma z)F_A(\hat{e}_1,\hat{e}_2)) + \mathfrak{r}, \tag{B-30}$$

where the notation is such that z_1 and z_2 denote the respective real and imaginary parts of z, and where $\mathfrak{r}$ denotes a term with absolute value bounded by $|z|^2|F_A|$.

Since the 2–form F_A is the Hodge dual of B_A, the equation in (1-13) can be invoked to replace (B-30) with

$$-\mathrm{r}|z|^{-1}\big(\bar{\alpha}\beta\bar{z} - \alpha\bar{\beta}z + i(2\nu|z|^2 + \mu\bar{z}^2 + \bar{\mu}z^2 - x_\gamma\bar{z} - \bar{x}_\gamma z)(1-|\alpha|^2)\big) + \mathfrak{r}', \tag{B-31}$$

where $\mathfrak{r}'$ has norm obeying $|\mathrm{r}'| \le c_0 c_v e^{-\sqrt{\mathrm{r}}|z|/c_0}$, this due to Lemmas 2.1 and 2.2. A further rewriting uses the top bullet in Lemma B.4 to replace β in (B-31) by $i\mu\mathrm{r}^{-1/2} r_\mathrm{r}^* \varsigma$ plus a term with small norm. Make this substitution and then invoke the formula for ς in (A-2) to write (B-31) as

$$-i\mathrm{r}\Big(2\nu|z| - x_\gamma\frac{\bar{z}}{|z|} - \bar{x}_\gamma\frac{z}{|z|}\Big)(1-|\alpha|^2) + \mathfrak{r}'', \tag{B-32}$$

where $|\mathfrak{r}''| \le c_0(c_v^{20-m'}\mathrm{r}^{1/2} + c_v)e^{-\sqrt{\mathrm{r}}|z|/c_0}$.

Step 6 Granted (B-32), use the formula for y in (A-2) and the formula for A_0 in (A-3) to write $y = -2^{1/2}(a_0 - 1)$. Keeping in mind that a_0 is real and a function of $|z|^2$, use the formula in (A-3) and the top bullet in (2-8) to see that $\frac{d}{d|z|}a_0 = |z|(1-|\alpha_0|^2)$. This understood, it follows from (B-18) and (B-32) that

$$\partial_s(a_{A0,\gamma} - i\nu 2^{1/2} r_\mathrm{r}^* y)|_{sz} = i\mathrm{r}\Big(x_\gamma\frac{\bar{z}}{|z|} + \bar{x}_\gamma\frac{z}{|z|}\Big)(1-|r_\mathrm{r}^*\alpha_0|^2) + \mathfrak{r}''. \tag{B-33}$$

To exploit (B-33), first look again at what is said in the first paragraph from Step 4 of the proof of Lemma B.2 to see that $|a_{A0,\gamma}| \le c_0 c_v^2 e^{-\sqrt{\mathrm{r}}|z|/c_0}$, where $|z| \ge \mathrm{r}^{-1/2}$ on $U_\gamma \cap Y_{*\Lambda}$. Given (3-3), such a bound also holds for $r_\mathrm{r}^* y$. These bounds imply what is asserted by the second bullet of Lemma B.7 at the points in $U_\gamma \cap Y_{*\Lambda}$ with distance greater than $c_v^2\mathrm{r}^{-1/2}$ from γ. Given this last observation, integrate both sides of (B-33) from a given value of s to $c_v^2\mathrm{r}^{-1/2}$ with a choice for $m' > 2m + 100$ to obtain the second bullet's assertion on the part of $U_\gamma \cap Y_{*\Lambda}$ at points with distance less than $c_v^2\mathrm{r}^{-1/2}$ from γ when $c_v \ge c_0$ and $\mathrm{r} \ge \kappa_c$. □

Part 2 The next lemma writes the various versions of $\mathfrak{L}_{\mathbb{V},\tau}$ as $\mathfrak{L}_{\mathbb{V},1} + e_\tau$ and gives a bound for the norm of the τ–derivative of e_τ.

Lemma B.8 *Fix $m \geq 1$. There exists m–dependent $\kappa > 1$ and, given $c_v \geq \kappa$, there exists $\kappa_{c_v} \geq \kappa$ with the following significance: Take $\mathrm{r} \geq \kappa_{c_v} c_v^{10}$ and suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Define $\{(A_{*\tau}, \psi_{*\tau})\}_{\tau\in[0,1]}$ as instructed in Section Ba. Given $\tau \in [0,1]$, let $\mathfrak{L}_{\mathbb{V},\tau}$ denote the $(A_{*\tau}, \psi_{*\tau})$ and $z = \mathrm{r}$ version of the operator $\mathfrak{L}_{\mathbb{V}}$. Write $\mathfrak{L}_{\mathbb{V},\tau}$ as $\mathfrak{L}_{\mathbb{V},1} + e_\tau$. Then the resulting map $\tau \mapsto e_\tau$ from $[0,1]$ to $C^\infty(Y; \mathbb{V}_0 \oplus \mathbb{V}_1)$ is real analytic with derivative bound $\left|\frac{d}{d\tau} e_\tau\right| \leq c_v^{-m} \mathrm{r}^{1/2}$ at all points in Y.*

Proof Given Lemmas B.2 and B.7, the assertion is a direct consequence of the formula for $(A_{*\tau}, \psi_{*\tau})$ in Section Ba and the formula for $\mathfrak{L}_{\mathbb{V}}$ in (A-26) and (A-27).

The lemma that follows uses the $\left|\frac{d}{d\tau} e_\tau\right| \leq c_v^{-m} \mathrm{r}^{1/2}$ bound from Lemma B.8 to give a version of Lemma B.4 with an r–dependent eigenvalue bound.

Lemma B.9 *Fix $m \geq 1$. There exists m–dependent $\kappa > 1$ and, given $c_v \geq \kappa$, there exists $\kappa_{c_v} \geq \kappa$ with the following significance: Suppose that $\mathrm{r} \geq \kappa_{c_v} c_v^{10}$ and that $(A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Construct as in Lemma B.8 the family of operators $\{\mathfrak{L}_{\mathbb{V},\tau}\}_{\tau\in[0,1]}$ and introduce $\{\lambda_{n\tau}\}_{\tau\in[0,1]}$ to denote the resulting family of eigenvalues. If $|\lambda_{n\tau}| = 0$ for some $\tau \in [0,1]$ then $|\lambda_{n\tau'}| \leq c_v^{-m} \mathrm{r}^{1/2}$ for all $\tau' \in [0,1]$.*

Proof Return for a moment to the context in Part 1. Let T denote $\sup_{\tau\in[0,1]} \left\|\frac{d}{d\tau} e_\tau\right\|$. It follows from (B-20) that any $n \in \mathbb{Z}$ version of the map $\tau \mapsto \lambda_{n\tau}$ is such that $|\lambda_{n\tau'} - \lambda_{n\tau}| \leq \mathrm{T}$ for any pair $\tau, \tau' \in [0,1]$. This implies in particular that $|\lambda_{n\tau}| > 0$ for all τ if $|\lambda_{n\tau'}| > \mathrm{T}$ for any $\tau' \in [0,1]$. Given Lemma B.8, this last observation leads directly to the assertion in Lemma B.9 when applied to the family $\{\mathfrak{L}_{\mathbb{V},\tau}\}_{\tau\in[0,1]}$ with $\mathrm{T} = c_v^{-m} \mathrm{r}^{1/2}$. □

Part 3 The three steps that follow complete the proof of Lemma B.4.

Step 1 If $m > c_0$, then Lemma B.9 can be invoked. With m so chosen, suppose that $\tau \in [0,1]$ and that $\lambda_{n\tau} = 0$. Let $\{\mathfrak{f}_{(\tau')}\}_{\tau'\in[0,1]}$ denote the corresponding family of eigenvectors. Fix $\tau' \in [0,1]$. If $c_v \geq \kappa_\diamond$ and if r is greater than a purely c_v–dependent constant, then Lemma B.9's bound on $|\lambda_{n\tau}|$ implies that the assumptions of Lemmas A.2, A.3 and A.6 are met with $\mathrm{z} = \mathrm{r}$, with $(A_{*\tau'}, \psi_{*\tau'})$ used in lieu of (A, ψ), with

$\lambda = \lambda_{n\tau'}$, and with $\mathfrak{f} = \mathfrak{f}_{(\tau')}$. In particular, what is asserted by the first bullet of Lemma A.3 holds using c_v in lieu of c_0. This is to say that when $\mathfrak{f}$ is written in terms of its $\mathbb{V}_0$ and $\mathbb{V}_1$ components as $(\mathfrak{f}_0, \mathfrak{f}_1)$, then the component $\mathfrak{f}_1$ has L^2–norm bounded by $c_0 c_v^k \mathrm{r}^{-1/2}$ with $k \le c_0$.

Step 2 Assume that c_v and r are chosen so that the assumptions of Lemmas B.1–B.2 and B.7–B.9 are met. Suppose that $\tau' \in [0,1]$ is a point where the map $\lambda_{n(\cdot)}$ is real analytic. Let a'_v denote the endomorphism of $\mathbb{S}$ given by the derivative $\left(\frac{d}{d\tau}\nabla_{A_{*(\cdot)}}\right)_v$ at the point τ'. It follows from Lemmas B.2 and B.7 that $|a'v| \le c_0 c_v^{-m}$. Write $\psi_{*(\cdot)}$ as $(\alpha_{*(\cdot)}, \beta_{*(\cdot)})$ and let β' denote the derivative $\frac{d}{d\tau}\beta_{*(\cdot)}$ at τ'. Lemma B.7 implies that $|\beta'| \le c_0 c_v^{-m} \mathrm{r}^{-1/2}$. Meanwhile, Lemma B.8 has $\left|\frac{d}{d\tau} e_{(\cdot)}\right| \le c_0 c_v^{-m} \mathrm{r}^{1/2}$.

Step 3 Look at (A-26) and (A-27) to see that the absolute value of the inner product between $\mathfrak{f}$ and $\left(\frac{d}{d\tau} e_{(\cdot)}\right)\big|_{\tau'}\mathfrak{f}$ at any given point in Y is no greater than

$$c_0 c_v^{-m} \mathrm{r}^{1/2} |\mathfrak{f}_0||\mathfrak{f}_1| + c_0(|a'_v| + \mathrm{r}^{1/2}|\beta'|)|\mathfrak{f}|^2. \tag{B-34}$$

The integral of the expression in (B-34) over Y is no less than $\left|\frac{d}{d\tau}\lambda_{n(\cdot)}\right|$ at τ', this being a consequence of (B-20). Meanwhile, what is said by Steps 1 and 2 imply that the integral of the expression in (B-34) is no greater than $c_0 c_v^{-m+k}$.

Use m' to denote $m - k$. The argument used in the proof of Lemma B.9 proves that the bound by $c_0 c_v^{-m'}$ on $\left|\frac{d}{d\tau}\lambda_{n(\cdot)}\right|$ implies that $|\lambda_{n\tau}| = 0$ for some τ only if $|\lambda_{n(\tau')}| \le c_0 c_v^{-m'}$ for all $\tau' \in [0,1]$. Since m' can be any positive number greater than $\kappa_\diamond$, this last bound implies what is asserted by Lemma B.4. □

Be The pair $(A_\diamond, \psi_\diamond)$

This subsection modifies (A_*, ψ_*) on the components of $Y - (Y_{*\Lambda} \cup T_{*\Lambda})$ so that the resulting pair is given on these components by solutions to the vortex equations. The five parts of this subsection describe this modification.

Part 1 This first part describes the modification in the simplest case. To this end, fix attention on a component of $Y - (Y_{*\Lambda} \cup T_{*\Lambda})$ whose boundary is disjoint from the zero locus of α. This is the simplifying assumption. Let γ denote the curve in this component from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$ and let $T \subset Y$ denote the subset of points with distance $(c_v^4 + c_v^3)\mathrm{r}^{-1/2}$ or less from γ. Fix coordinates for T from Part 4 of Section Aa with ν constant and μ both constant, real-valued and positive.

The next lemma supplies a particular sort of isomorphism from $E|_T$ to the product bundle $T \times \mathbb{C}$.

Lemma B.10 *There exists $\kappa > 1$ and, given $c \geq \kappa$, there exists $\kappa_c \geq \kappa$ with the following significance: Take $\mathrm{r} \geq \kappa_c c^{10}$ and let $(A, \psi = (\alpha, \beta))$ denote a solution to the (r, μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Suppose that $\gamma \in \bigcup_{\mathrm{p} \in \Lambda} (\hat{\gamma}_{\mathrm{p}}^{+} \cup \hat{\gamma}_{\mathrm{p}}^{-})$, that α has zeros at distances less than $\kappa \mathrm{r}^{-1/2}$ from γ but no zeros at distance between $\kappa \mathrm{r}^{-1/2}$ and $(c^4 + 3c^3)\mathrm{r}^{-1/2}$ from γ. Fix coordinates from Part 4 of Section Aa for the radius $(c^4 + 3c^3)\mathrm{r}^{-1/2}$ tubular neighborhood of γ. There is an isomorphism on the concentric, radius $(c^4 + c^3)\mathrm{r}^{-1/2}$ tubular neighborhood of γ between E and the product bundle with the properties listed below:*

- *The isomorphism writes $A = \theta + a_{A0}\, dt + \frac{1}{2}(\mathrm{A}\, d\bar{z} - \bar{\mathrm{A}}\, dz)$ with $|a_{A0}| \leq \kappa$ and $|\mathrm{A}| \leq \kappa \mathrm{r}^{1/2}$.*
- *Use m to denote the sum of the local Euler numbers of the zeros of α on any radius $c^4 \mathrm{r}^{-1/2}$ transverse disk centered on γ. The isomorphism writes α as $|\alpha|(z/|z|)^m$ at points with distances between $2\kappa \mathrm{r}^{-1/2}$ and $(c^4 + c^3)\mathrm{r}^{-1/2}$ from γ.*

This lemma is proved in a moment.

Take $c = c_v$ in Lemma B.10 and use the lemma's isomorphism between $E|_T$ and $T \times \mathbb{C}$ to write $A_\diamond$ as $\theta + a_\diamond$ with $a_\diamond$ being an $i\mathbb{R}$–valued 1–form on T. Write $\psi_\diamond$ as $(\alpha_\diamond, \beta_\diamond)$ and use the isomorphism to view $\alpha_\diamond$ as a $\mathbb{C}$–valued function on T. Use the isomorphism and the chosen coordinate system to view $\beta_\diamond$ as a $\mathbb{C}$–valued function also. Let m denote the rank of the complex bundle $\mathrm{Ker}_\vartheta|_\gamma$. The data $a_\diamond$, $\alpha_\diamond$ and $\beta_\diamond$ are given by what is written on the right-hand side of the respective top, middle and bottom lines in (A-44).

Proof of Lemma B.10 Let κ_* denote the version of κ that appears in Proposition 2.4. Fix $x > 100$ for the moment and then choose $c > x\kappa_*$. Now suppose that there are no zeros of α with distance to γ between $x\kappa_* \mathrm{r}^{-1/2}$ and $(c^4 + 3c^3)\mathrm{r}^{-1/2}$. The first observation is that the absolute value of the sum of the local Euler numbers of the zeros of α with distance at most $x\kappa_* \mathrm{r}^{-1/2}$ from γ is bounded by $c_0 \kappa_*$. The reason is this: According to the fifth bullet of that Proposition 2.4, the 2–form $\frac{i}{2\pi} F_{\hat{A}}$ has compact support in the radius $x\kappa_* \mathrm{r}^{-1/2}$ tubular neighborhood of γ if $x \geq c_0$. (This follows from the formula for $F_{\hat{A}}$ in (2-14) and from Lemma 2.3.) Meanwhile, the section $\alpha/|\alpha|$ is $\nabla_{\hat{A}}$–covariantly constant on the boundary of this tubular neighborhood if $x \geq c_0$ (this follows from the definition of γ). These two facts imply that the integral of $\frac{i}{2\pi} F_{\hat{A}}$ on a transverse disk to γ with its boundary at distance between $x\kappa_* \mathrm{r}^{-1/2}$

and $(c^4+3c^3)\mathrm{r}^{-1/2}$ from γ computes the sum of the local Euler numbers of the zeros of α with distance less than $x\kappa_*\mathrm{r}^{-1/2}$ from γ. Meanwhile, it follows from Lemma 2.1 that such an integral is bounded by $c_0x\kappa_*$.

Lemma 2.3 finds $|\alpha|\geq 1-\frac{1}{100}$ at distances greater than $x\kappa_*\mathrm{r}^{-1/2}$ from γ but less than $(c^4+2c^3)\mathrm{r}^{-1/2}$ from γ if $x>c_0$. Granted that α's local Euler number is m, and granted this lower bound on $|\alpha|$, there exists $c_1\leq c_0$ and an isomorphism from E on the $|z|\geq c_1x\kappa_*\mathrm{r}^{-1/2}$ part of the radius $(c^4+2c^3)\mathrm{r}^{-1/2}$ tubular neighborhood of γ that writes α as $|\alpha|(z/|z|)^m$. Use this isomorphism to write A as in the first bullet of the lemma. The isomorphism writes the dt part of $\nabla_A\alpha$ as $(\partial_t|\alpha|+a_{A0}|\alpha|)(z/|z|)^m$. Given that $D_A\psi=0$, it follows that the dt part of $\nabla_A\alpha$ is bounded by $|\nabla_A\beta|+c_0\mathrm{r}^{-1/2}|\nabla_A\alpha|$. This understood, then Lemma 2.1's bound implies that $|a_{A0}|\leq c_0$ on the $|z|\geq c_1x\kappa_*\mathrm{r}^{-1/2}$ part of the radius $(c^4+2c^3)\mathrm{r}^{-1/2}$ tubular neighborhood of γ. The bound for $|\mathrm{A}|$ on this same part of the radius $(c^4+2c^3)\mathrm{r}^{-1/2}$ tubular neighborhood of γ follows from Lemma 2.1's bound for $|\nabla_A\alpha|$.

The isomorphism just described will be modified on the $|z|\leq\frac{7}{4}c_1x\kappa_*\mathrm{r}^{-1/2}$ tubular neighborhood of γ to obtain an isomorphism between E and the product bundle on the whole $(c^4+2c^3)\mathrm{r}^{-1/2}$ tubular neighborhood of γ that obeys the first bullet of the lemma. To this end, note first that there is an isomorphism between $E|_\gamma$ and $\gamma\times\mathbb{C}$ that writes the pullback of A along γ as $\hat{a}_{A2}\,dt$ with $\hat{a}_{A2}$ constant with absolute value less than $2\pi/\ell_\gamma$. Fix such an isomorphism, and then use parallel transport by A along the rays through the origin in the constant t slices of the tubular neighborhood to extend this isomorphism to the $|z|\geq 2c_1x\kappa_*\mathrm{r}^{-1/2}$ part of the tubular neighborhood. An isomorphism of this sort writes A as $\theta+a_{A2}\,dt+\frac{1}{2}\hat{\mathrm{A}}(z\,d\bar{z}-\bar{z}dz)$, where $\hat{\mathrm{A}}$ is an $\mathbb{R}$–valued function defined on the radius $2c_1x\kappa_*\mathrm{r}^{-1/2}$ tubular neighborhood of γ. This function obeys $\rho^{-1}\frac{\partial}{\partial\rho}(\rho^2\hat{\mathrm{A}})=F_A\big(\frac{\partial}{\partial z},\frac{\partial}{\partial\bar{z}}\big)$ with ρ denoting $|z|$. Integrate this identity starting at $|z|=0$ using (1-13) to see that $\rho|\hat{\mathrm{A}}|\leq c_0x\kappa_*\mathrm{r}^{1/2}$. Meanwhile, $\frac{\partial}{\partial\rho}a_{A2}=F_A\big(\frac{\partial}{\partial\rho},\frac{\partial}{\partial t}\big)$ because $A-\theta$ has no $d\rho$ component. Integrate the latter identity using (1-13) with the fact $|\partial_t-v|\leq c_0|z|$ to see that $|a_{A2}|\leq c_0x\kappa_*$ where $|z|\leq 2c_1x\kappa_*\mathrm{r}^{-1/2}$.

The preceding paragraphs describe two isomorphisms between E and the product bundle that are defined on the $c_1x\kappa_*\mathrm{r}^{-1/2}\leq|z|\leq 2cx\kappa_*\mathrm{r}^{-1/2}$ part of the radius $(c^4+2c^3)\mathrm{r}^{-1/2}$ tubular neighborhood of γ. The corresponding transition function is a map from this part of the tubular neighborhood to S^1. Use $\hat{u}$ to denote this map. The bounds on a_{A1} and a_{A2} imply that $\big|\frac{\partial}{\partial t}\hat{u}\big|\leq c_0$ and those on A and $\rho|\hat{\mathrm{A}}|$ imply that $\big|\frac{\partial}{\partial z}\hat{u}\big|\leq c_0x\kappa_*\mathrm{r}^{1/2}$. Granted these bounds, the map $\hat{u}$ with a cut-off function defined from χ can be used in a straightforward manner to modify the outer isomorphism where

$\frac{5}{4}c_1 x\kappa_* \mathrm{r}^{-1/2} < |z| \le \frac{7}{4}c_1 x\kappa_* \mathrm{r}^{-1/2}$ so that the result agrees with the inner isomorphism where $c_1 x\kappa_* \mathrm{r}^{-1/2} < |z| < \frac{5}{4}c_1 x\kappa_* \mathrm{r}^{-1/2}$ and is such that the norms of the new versions of a_{A1} and A are still bounded by $c_0 x\kappa_*$ and $c_0 x\kappa_* \mathrm{r}^{1/2}$, respectively.

Since all of what was said works for $x \le c_0$ and since $\kappa_* \le c_0$, the statement of the lemma follows from the preceding observations and constructions with κ a priori bounded by c_0 and with a suitable choice for κ_c. □

Part 2 Fix attention on a component of $Y - (Y_{*\Lambda} \cup T_{*\Lambda})$ whose boundary intersects the zero locus of α and let γ again denote the corresponding curve from the set $\bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_{\mathrm{p}}^{+} \cup \hat{\gamma}_{\mathrm{p}}^{-})$. Introduce the coordinates from Part 4 of Section Aa for γ with ν constant and with μ constant, real and positive. It follows as a consequence of what is said in (B-1) that the zero locus of α extends as two properly embedded arcs in the part of the radius $c_v^4 \mathrm{r}^{-1/2}$ tubular neighborhood of γ where $(c_v^4 - 3c_v^3)\mathrm{r}^{-1/2} \le |z| \le c_v^4 \mathrm{r}^{-1/2}$. The following lemma describes an extension of these two arcs as the end segments of a single, properly embedded arc in the radius $c_v^4 \mathrm{r}^{-1/2}$ tubular neighborhood of γ.

Lemma B.11 *There exists $\kappa > 1$ and, given $c_v \ge \kappa$, there exists $\kappa_{c_v} \ge \kappa$ with the following significance: Suppose that $\mathrm{r} \ge \kappa_{c_v} c_v^{10}$ and suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Let γ denote a curve from $\bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_{\mathrm{p}}^{+} \cup \hat{\gamma}_{\mathrm{p}}^{-})$ with points at distance $c_v^4 \mathrm{r}^{-1/2}$ from the zero locus of α. Let T denote the set of points with distance $(c_v^4 + c_v^3)\mathrm{r}^{-1/2}$ or less from γ. There exists a smooth, properly embedded arc in T with the properties listed below:*

- *The arc is the zero locus of α at points with distance $c_v \mathrm{r}^{-1/2}$ or more from α.*
- *The arc lies in the $1 - 3\cos^2\theta > 0$ part of T.*
- *Each point in the arc has distance greater than $\kappa^{-1} c_v \mathrm{r}^{-1/2}$ from γ.*
- *A unit-length tangent vector to the arc has distance at most $\kappa c_v \mathrm{r}^{-1/2}$ from v.*
- *Fix a closed, transverse disk in T with radius $c_v^4 \mathrm{r}^{-1/2}$, center point on γ, and no zeros of α on its boundary. Let m_* denote the intersection number between this disk and the arc and let m_α denote the sum of the local Euler numbers of the zeros of α on the disk. Then $m_\alpha - m_*$ is nonnegative and independent of the chosen disk.*

Proof There are various ways to construct an arc with the desired properties. The construction that follows is perhaps more complicated than is needed for now, but the resulting arc has certain extra properties that are exploited later. There are four steps.

Step 1 Fix $m \in (1, c_0^{-1} c_v)$. If $\alpha^{-1}(0) \cap T$ has a component whose points all have distance greater than $m^{-4} c_v \mathrm{r}^{-1/2}$ from γ, then the desired arc is this component of $\alpha^{-1}(0) \cap T$. Proposition 2.4 guarantees that the conditions of the lemma are met if κ is greater than a purely m–dependent constant. The remaining steps assume that the components of $\alpha^{-1}(0)$ that intersects the boundary of T have points with distance $m^{-4} c_v \mathrm{r}^{-1/2}$ from γ. This understood, keep in mind the following consequence of Proposition 2.4: the zero locus of α in the part of T with distance at least $m^{-4} c_v \mathrm{r}^{-1/2}$ from γ consists of two arcs, one where u is everywhere positive and the other where u is everywhere negative. These are denoted respectively by υ_+ and υ_- in what follows. Note also that the unit tangent vector to either arc has distance at most $c_0 \mathrm{r}^{-1/2}$ from v.

The subsequent three steps take $\cos\theta = \frac{1}{\sqrt{3}}$ on γ. The construction for the case when $\cos\theta = -\frac{1}{\sqrt{3}}$ on γ is identical but for certain sign changes and will not be given.

Step 2 Let $\theta_* \in (0, \pi)$ denote the angle that obeys $\cos\theta_* = \frac{1}{\sqrt{3}}$. It follows from the formula for v in (1-3) that any integral curve of v in T can be parametrized as a map from an interval $I \subset \mathbb{R}$ to $(\mathbb{R}/(2\pi\mathbb{Z})) \times \mathbb{R}^2$ of the form

$$t \mapsto (\phi = -t, u = b x_\Delta(t), \theta = \theta_* + y_\Delta(t)),$$

where $b = \frac{\sqrt{3}}{2\sqrt{2}} e^R (x_0 + 4e^{-2R})^{1/2}$ and where x_Δ and y_Δ are smooth functions that obey

$$\text{(B-35)} \qquad \frac{d}{dt} x_\Delta = \lambda y_\Delta + \mathfrak{e}_x(x_\Delta, y_\Delta) \quad \text{and} \quad \frac{d}{dt} y_\Delta = \lambda x_\Delta + \mathfrak{e}_y(x_\Delta, y_\Delta)$$

with $\lambda = 4\sqrt{6} e^{-R}(x_0 + 4e^{-2R})^{1/2}$ and with the pair $\mathfrak{e}_x$ and $\mathfrak{e}_y$ being smooth functions of the coordinates (x, y) on $\mathbb{R}^2$ that obey $|\mathfrak{e}_x| + |\mathfrak{e}_y| \le c_0(x^2 + y^2)$.

This parametrization of integral curves of v in T suggests the introduction of coordinates (x, y) for T by writing $u = bx$ and $y = \theta_* + y$. These coordinates are such that if $m \ge c_0$ and if $p \in [0, 4)$, then the points in T with $(x^2 + y^2)^{1/2} = m^{-p} c_v \mathrm{r}^{-1/2}$ have distance less than $c_0 m^{-p} c_v \mathrm{r}^{-1/2}$ from γ and distance greater than $c_0^{-1} m^{-p} c_v \mathrm{r}^{-1/2}$ from γ.

It follows from what is said in Proposition 2.4 that υ_+ can be parametrized as a map from an interval $I_+ \subset \mathbb{R}$ to $(\mathbb{R}/(2\pi\mathbb{Z})) \times \mathbb{R}^2$ of the form

$$\text{(B-36)} \qquad t \mapsto (\phi = -t, u = b x_+(t), \theta = \theta_* + y_+(t)),$$

where x_+ and y_+ obey a modified version of (B-35) that adds respective terms $\mathfrak{r}_{x+}$ and $\mathfrak{r}_{y+}$ of t to the left- and right-hand equations. These are smooth functions of t

with absolute value bounded by $c_0\mathrm{r}^{-1/2}$. The arc υ_- can be parametrized in a similar fashion as a map with domain an interval $I_- \subset \mathbb{R}$ by a pair of functions (x_-, y_-) that obey a modified version of (B-35) that adds respective terms $\mathfrak{r}_{x-}$ and $\mathfrak{r}_{y-}$ to the right-hand sides of these equations, both being functions of t with absolute value bounded by $c_0\mathrm{r}^{-1/2}$.

Use S to denote the torus $(x^2+y^2)^{1/2} = m^{-2}c_v\mathrm{r}^{-1/2}$. This torus is intersected transversely in one point by υ_+ and likewise by υ_-. No generality is lost by parametrizing I_+ and I_- so these points $S \cap \upsilon_+$ and $S \cap \upsilon_-$ occur at respective parameter values $t = 2\pi + t_\diamond$ and $t = -2\pi - t_\diamond$ with $t_\diamond \in [0, 2\pi)$. More is said about I_+ and I_- in a moment.

Step 3 Introduce coordinates p and q on $\mathbb{R}^2$ by the rules $p = y + x$ and $q = y - x$. Writing (B-35) or its υ_+ or υ_- analogs in terms of p and q gives an equation for a pair of maps $t \mapsto p_*(t)$ and $t \mapsto q_*(t)$. Here $*$ denotes either Δ, $+$ or $-$ as the case may be. The equation in question has the form

$$\text{(B-37)} \quad \frac{d}{dt}p_* = \lambda p_* + \mathfrak{e}_p(p_*, q_*) + \mathfrak{r}_{p_*} \quad \text{and} \quad \frac{d}{dt}q_* = -\lambda q_* + \mathfrak{e}_q(p_*, q_*) + \mathfrak{r}_{q_*},$$

where $\mathfrak{e}_p$ and $\mathfrak{e}_q$ are smooth functions of p and q that obey $|\mathfrak{e}_p| + |\mathfrak{e}_q| \le c_0(p^2+q^2)$, and where $\mathfrak{r}_{p0}$ and $\mathfrak{r}_{q0}$ are zero (this the case of (B-35)), while $\mathfrak{r}_{p+}$, $\mathfrak{r}_{q+}$, $\mathfrak{r}_{p-}$ and $\mathfrak{r}_{q-}$ are functions of t with absolute value bounded by $c_0\mathrm{r}^{-1/2}$.

It follows from (B-37) that p_+ and q_+ where $(p_+^2 + q_+^2)^{1/2} \le c_v\mathrm{r}^{-1/2}$ have the form

$$\text{(B-38)} \quad p_+(t) = p_{\diamond+}e^{\lambda(t-2\pi-t_\diamond)} + \mathfrak{w}_{p+} \quad \text{and} \quad q_+(t) = q_{\diamond+}e^{-\lambda(t-2\pi-t_\diamond)} + \mathfrak{w}_{q+},$$

where $p_{\diamond+} = p(2\pi + t_\diamond)$ and $q_{\diamond+} = q(2\pi + t_\diamond)$, and where $\mathfrak{w}_{p+}$ and $\mathfrak{w}_{q+}$ are functions of t with absolute value bounded by $c_0\mathrm{r}^{-1/2}$ where $|t| \le c0$. The t–derivatives of $\mathfrak{w}_{p+}$ and $\mathfrak{w}_{q-}$ for such t are also bounded by $c_0\mathrm{r}^{-1/2}$.

If $m > c_0$, then the fact that υ_+ intersects the locus where $(p^2+q^2)^{1/2} = c_0 m^{-4}c_v\mathrm{r}^{-1/2}$ leads to the following observations:

(B-39)
- $[-2\pi - t_\diamond, 2\pi + t_\diamond] \subset I_+$.
- $|p_{\diamond+} - m^{-2}c_v\mathrm{r}^{-1/2}| + m^2|q_{\diamond+}| \le c_0 m^{-4}c_v\mathrm{r}^{-1/2}$.

To see why this is, note first that the torus S is the locus $(p^2+q^2)^{1/2} = 2^{1/2}m^{-2}c_v\mathrm{r}^{-1/2}$ and so neither $p_{\diamond+}$ nor $q_{\diamond+}$ is greater than $2^{1/2}m^{-2}c_v\mathrm{r}^{-1/2}$. However, $p_{\diamond+} \ge 2^{-1/2}m^{-2}c_v\mathrm{r}^{-1/2}$ because $p \ge |q|$ where $u > 0$. Granted that $p_{\diamond+} \ge 2^{-1/2}m^{-2}c_v\mathrm{r}^{-1/2}$ and granted that υ_+ intersects the locus where $(p^2+q^2)^{1/2} = c_0 m^{-4}c_v\mathrm{r}^{-1/2}$, then the left-hand identity in (B-37) requires that the parameter t at this intersection is less than $\lambda^{-1}\ln(m^{-2})+c_0$. This gives the top bullet in (B-39) if $m>c_0$. The fact that t on I_+ has values less than $\lambda^{-1}\ln(m^{-2})+c_0$ with the right-hand identity in (B-38) finds q_+ at such values of t greater than $q_{\diamond+}c_0m^2$ and so $m^2|q_+|$ must be less than $c_0m^{-4}c_\upsilon\mathrm{r}^{-1/2}$. This implies the lower bullet in (B-39).

What was just said about p_+ and q_+ has its p_- and q_- analog. By way of a summary, these functions can be written as

(B-40) $p_-(t)=p_{\diamond-}e^{\lambda(t+2\pi+t_\diamond)}+\mathfrak{w}_{p-}$ and $q_-(t)=q_{\diamond-}e^{-\lambda(t+2\pi+t_\diamond)}+\mathfrak{w}_{q-}$,

where $p_{\diamond-}=p(-2\pi-t_\diamond)$ and $q_{\diamond-}=q(-2\pi-t_\diamond)$, and where $\mathfrak{w}_{p-}$ and $\mathfrak{w}_{q-}$ are functions of t with absolute value bounded by $c_0\mathrm{r}^{-1/2}$ where $|t|\le 100\pi$. Their respective t–derivatives are also bounded by $c_0\mathrm{r}^{-1/2}$ for such t. The (p_-,q_-) analog of (B-39) reads:

(B-41)
- $[-2\pi-t_\diamond, 2\pi+t_\diamond]\subset I_-$.
- $m^2|p_{\diamond-}|+|q_{\diamond-}-m^{-2}c_\upsilon\mathrm{r}^{-1/2}|\le c_0m^{-4}c_\upsilon\mathrm{r}^{-1/2}$.

The proof of (B-41) differs only cosmetically from that of (B-39).

Step 4 The arc υ coincides with the $t>2\pi+t_\diamond$ part of υ_+ and the $t<-2\pi-t_\diamond$ part of υ_-. The remaining part of υ is parametrized by $[-2\pi-t_\diamond, 2\pi+t_\diamond]$. The definition that follows for this part of υ refers to a nonnegative function on $\mathbb{R}$ that is denoted by σ and defined by the rule $t\mapsto\sigma(t)=\chi\big(1-\frac{4}{\pi}t\big)$. This function is equal to 1 where $t<\frac{\pi}{4}$ and it is equal to 0 where $t>\frac{\pi}{2}$.

The $t\in[-2\pi-t_\diamond, 2\pi+t_\diamond]$ point of υ is written as $(\varphi=-t, u=bx_\upsilon(t), \theta=\theta_*+y_\upsilon(t))$ with x_υ and y_υ being functions on the interval $[-2\pi-t_\diamond, 2\pi+t_\diamond]$. The functions x_υ and y_υ are written here as $x_\upsilon=\frac{1}{2}(p_\upsilon-q_\upsilon)$ and $y_\upsilon=\frac{1}{2}(p_\upsilon+q_\upsilon)$ with p_υ and q_υ given by the following rule:

(B-42)
- $p_\upsilon(t)=\sigma(-t)(p_{\diamond+}e^{\lambda(t-2\pi-t_\diamond)}+\mathfrak{w}_{p+}(t))+\sigma(t)p_-(t)$.
- $q_\upsilon(t)=\sigma(t)(q_{\diamond-}e^{-\lambda(2\pi+t_\diamond+t)}+\mathfrak{w}_{q-}(t))+\sigma(-t)q_+(t)$.

It is a straightforward matter to check that the arc υ has all of the required properties. □

Part 3 Use $\kappa_\diamond$ to denote a constant that is greater than the versions of κ that appear in Lemmas A.2–A.9 and in Lemmas B.10 and B.11. Assume in what follows that $c_\upsilon\ge\kappa_\diamond$ and that r is greater than the $c_0=c_\upsilon$ lower bounds given in Lemmas A.2–A.9 and the lower bounds given in Lemmas B.10 and B.11. Fix a component of $Y-(Y_{*\Lambda}\cup T_{*\Lambda})$ whose boundary has a zero of α. Let γ denote the nearby curve from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat\gamma_{\mathfrak{p}}^+\cup\hat\gamma_{\mathfrak{p}}^-)$

and let T denote the set of points in Y with distance $(c_v^4 + c_v^3)\mathrm{r}^{-1/2}$ or less from γ. This part of the subsection defines $(A_\diamond, \psi_\diamond)$ on T. The definition has four steps. These steps use υ to denote the arc that is supplied by Lemma B.11. These steps also use γ's version of the coordinates (t, z) from Part 4 of Section Aa for T.

Step 1 Define $U_{T0} \subset T$ as follows: The $|z| \geq (c_v^4 - 2c_v^2)\mathrm{r}^{-1/2}$ part of U_{T0} consists of the points in T with distance greater than $c_v^2\mathrm{r}^{-1/2}$ from υ. The $|z| < (c_v^4 - 2c_v^2)\mathrm{r}^{-1/2}$ part of U_{T0} consists of the points with distance greater than $c_v^{1/4}\mathrm{r}^{-1/2}$ from υ and with distance greater than $c_v^{1/4}\mathrm{r}^{-1/2}$ from γ. Note that the $|z| \geq c_v^4\mathrm{r}^{-1/2}$ part of U_{T0} coincides with the $Y_{*\Lambda} \cap T$ part of Section Ba's set U_0. This understood, fix an isomorphism over U_{T0} between E and $U_{T0} \times \mathbb{C}$ that sends α to $|\alpha|$ on the part of U_{T0} where $|\alpha| \geq \frac{1}{2}$. Such an isomorphism extends Section Ba's isomorphism from the $|z| \geq (c_v^4 - 2c_v^2)\mathrm{r}^{-1/2}$ part of U_{T0} to the whole of U_{T0}. This isomorphism identifies $A_\diamond$ on U_{T0} with the product connection and it identifies $\alpha_\diamond$ with the constant $1 \in \mathbb{C}$. The component $\beta_\diamond$ is everywhere zero on U_{T0}.

What follows is, for now, just a parenthetical remark: Suppose that λ and λ' are two isomorphisms from $E|_{U_{T0}}$ to $U_{T0} \times \mathbb{C}$ that agree where $|\alpha| > \frac{1}{2}$. Then $\lambda' = e^{ix}\lambda$ with x being a real-valued function which is 0 where $|\alpha| > \frac{1}{2}$. That this is so is a consequence of what is said in Proposition 2.4 about the zero locus of α in T.

Step 2 Let $U_{\upsilon-} \subset T$ denote the subset of points with $|z| < \left(c_v^4 - \frac{7}{4}c_v^3\right)\mathrm{r}^{-1/2}$ and distance less than $4c_v^{1/4}\mathrm{r}^{-1/2}$ from υ. To this end, keep in mind that the $|z| \geq (c_v^4 - 2c_v^3)\mathrm{r}^{-1/2}$ part of υ coincides with this same part of $\alpha^{-1}(0) \cap T$. This understood, fix coordinates for $U_{\upsilon-}$ from Part 4 of Section Aa that coincide on the $|z| \geq (c_v^4 - 2c_v^3)\mathrm{r}^{-1/2}$ part of $U_{\upsilon-}$ with those used in Section Ba. Denote these coordinates by (t_υ, z_υ) so as to distinguish them from the coordinates t and z that are used for T. The restriction of E to the $|z| \geq (c_v^4 - 2c_v^3)\mathrm{r}^{-1/2}$ part of $U_{\upsilon-}$ has its $\alpha \mapsto |\alpha| z_\upsilon/|z_\upsilon|$ isomorphism with the product bundle. Extend this to the product bundle so as to give an isomorphism over the whole of $U_{\upsilon-}$ between E and the product bundle. This extension should be such that the corresponding transition function for Step 1's isomorphism from $E|_{U_{T0}}$ to $U_{T0} \times \mathbb{C}$ sends the constant section 1 of $U_{T0} \times \mathbb{C}$ to the section $z_\upsilon/|z_\upsilon|$ of $U_{\upsilon-} \times \mathbb{C}$ on the $|z_\upsilon| \geq c_v^{1/4}\mathrm{r}^{-1/2}$ part of $U_{\upsilon-}$. The formula in the next equation defines $A_\diamond$ and $\alpha_\diamond$ on $U_{\upsilon-}$ by viewing them via this isomorphism as a connection on the product bundle and map to $\mathbb{C}$. This isomorphism with the coordinates (t_υ, z_υ) are used to view $\beta_\diamond$ as a map to $\mathbb{C}$ also.

The upcoming equation uses $\chi_{\hat{U}\upsilon-}$ to denote the function of $|z_\upsilon|$ given by the rule $\chi(c_v^{-1/4}\mathrm{r}^{1/2}|z_\upsilon| - 1)$. The equation also uses $(\nu_\upsilon, \mu_\upsilon)$ to denote the (t_υ, z_υ) version of the functions ν and μ from υ's version of (A-6). This equation once again brings in the functions α_0 and a_0 from (A-3) and the corresponding versions of y and ς from (A-2).

(B-43)
- $A_\diamond = \theta + \nu_\upsilon \chi_{\hat{U}\upsilon-} i 2^{1/2} \mathfrak{r}_{\mathrm{r}}^* y\, dt_\upsilon$
$$- \tfrac{1}{2}(1 - \chi_{\hat{U}\upsilon-} + \chi_{\hat{U}\upsilon-} \mathfrak{r}_{\mathrm{r}}^* a_0)(z_\upsilon^{-1}\, dz_\upsilon - \bar{z}_\upsilon^{-1}\, d\bar{z}_\upsilon),$$
- $\alpha_\diamond = (1 - \chi_{\hat{U}\upsilon-}(1 - \mathfrak{r}_{\mathrm{r}}^* |\alpha_0|)) z_\upsilon / |z_\upsilon|,$
- $\beta_\diamond = i \mu_\upsilon \mathrm{r}^{-1/2} \chi_{\hat{U}\upsilon-} \mathfrak{r}_{\mathrm{r}}^* \varsigma.$

Step 3 This step defines $A_\diamond$, $\alpha_\diamond$ and $\beta_\diamond$ on the part of T with $|z| \geq (c_\upsilon^4 - 2c_\upsilon^3)\mathrm{r}^{-1/2}$ where the distance to υ is less than $4c_\upsilon^2 \mathrm{r}^{-1/2}$. To this end, use the $\alpha \mapsto \alpha z_\upsilon/|z_\upsilon|$ isomorphism between E and the product bundle over this part of T to view $A_\diamond$ as a connection on the product bundle and $\alpha_\diamond$ as a map to $\mathbb{C}$. Use this same isomorphism with the coordinates (t_υ, z_υ) to view $\beta_\diamond$ as a map to $\mathbb{C}$ also.

Reintroduce $\chi_{\diamond\diamond}$ to denote the function that appears in (B-2). This function equals 1 where $|z| < \bigl(c_\upsilon^4 - \frac{7}{4}c_\upsilon^3\bigr)\mathrm{r}^{-1/2}$ and it equals 0 where $|z| > \bigl(c_\upsilon^4 - \frac{5}{4}c_\upsilon^3\bigr)\mathrm{r}^{-1/2}$. The definition uses $\chi_{\hat{U}\upsilon+}$ to denote the function of $|z_\upsilon|$ given by

$$(1 - \chi_{\diamond\diamond})\chi(c_\upsilon^{-2}\mathrm{r}^{1/2}|z_\upsilon| - 1) + \chi_{\diamond\diamond}\chi(c_\upsilon^{-1/4}\mathrm{r}^{1/2}|z_\upsilon| - 1).$$

This function is equal to (B-35)'s function $\chi_{\hat{U}\upsilon-}$ where $|z| \leq \bigl(c_\upsilon^4 - \frac{7}{4}c_\upsilon^3\bigr)\mathrm{r}^{-1/2}$ and it is equal to (B-10)'s function $\chi_{\hat{U}}$ where $|z| \geq \bigl(c_\upsilon^4 - \frac{5}{4}c_\upsilon^3\bigr)\mathrm{r}^{-1/2}$.

Replace $\chi_{\hat{U}\upsilon-}$ in (B-43) with $\chi_{\hat{U}\upsilon+}$ to obtain the formulas for $A_\diamond$, $\alpha_\diamond$ and $\beta_\diamond$ on the part of T with $|z| \geq (c_\upsilon^4 - 2c_\upsilon^3)\mathrm{r}^{-1/2}$ and with distance less than $4c_\upsilon^2\mathrm{r}^{-1/2}$ to υ.

Step 4 This last step defines $A_\diamond$, $\alpha_\diamond$ and $\beta_\diamond$ on the $|z| < \frac{3}{4}c_\upsilon^{1/2}\mathrm{r}^{-1/2}$ part of T. The definition requires Lemma B.11's integer m. The definition also requires the choice of an isomorphism between E on this part of T and the product bundle. A choice for such an isomorphism should be made subject to the following constraint: The resulting transition function on the $\frac{1}{2}c_\upsilon^{1/2}\mathrm{r}^{-1/2} < |z| < \frac{3}{4}c_\upsilon^{1/2}\mathrm{r}^{-1/2}$ part of T between the product bundle over $|z| < \frac{3}{4}c_\upsilon^{1/2}\mathrm{r}^{-1/2}$ part of T and the product bundle $U_{T0} \times \mathbb{C}$ sends the latter's constant section 1 to the former's section $z \mapsto (z/|z|)^m$. An isomorphism of this sort exists because γ represents the class 0 in $H^1(Y;\mathbb{Z})$. Moreover, the space of isomorphisms that obey this constraint is contractible. The chosen isomorphism is used to view $A_\diamond$ as a connection on the product bundle over this part of T and $\alpha_\diamond$ here as a $\mathbb{C}$–valued function. This isomorphism with the coordinates (t, z) is used to view $\beta_\diamond$ as a $\mathbb{C}$–valued function as well.

In the case $m=0$, the transition function is the constant function. In this case, $A_\diamond$ on the $|z|<\frac{3}{4}c_v^{1/2}\mathrm{r}^{-1/2}$ part of T is set equal to the product connection, the function $\alpha_\diamond$ is the constant function 1, and $\beta_\diamond$ is zero.

Assume next that $m>0$. Introduce χ_{**} to denote the function that is defined by the rule $z\mapsto\chi(4c_v^{-1/2}\mathrm{r}^{-1/2}|z|-1)$. This function equals 1 where $|z|\le\frac{1}{4}c_v^{1/2}\mathrm{r}^{-1/2}$ and it equals 0 where $|z|\ge\frac{1}{2}c_v^{1/2}\mathrm{r}^{-1/2}$. The definition of $A_\diamond$, $\alpha_\diamond$ and $\beta_\diamond$ in the next equation uses χ_{**}, it uses ν and μ to denote the pair of functions that are given by γ's version of (A-6), and uses the functions α_{m0}, a_{m0}, y_m and ς_m that appear in (A-44). What follows defines $A_\diamond$, $\alpha_\diamond$ and $\beta_\diamond$ on the $|z|<\frac{3}{4}c_v^{1/2}\mathrm{r}^{-1/2}$ part of T:

(B-44)
- $A_\diamond=\theta+\nu\chi_{**}i2^{1/2}r_{\mathrm{r}}^{*}y_m\,dt-\frac{1}{2}m(1-\chi_{**}+\chi_{**}r_{\mathrm{r}}^{*}a_{m0})(z^{-1}\,dz-\bar{z}^{-1}\,d\bar{z})$,
- $\alpha_\diamond=(1-\chi_{**}(1-r_{\mathrm{r}}^{*}|\alpha_{m0}|))(z/|z|)^m$,
- $\beta_\diamond=i\mu\mathrm{r}^{-1/2}\chi_{**}r_{\mathrm{r}}^{*}\varsigma_m$.

Part 4 The next lemma asserts two important features of the large c_v and r versions of $(A_\diamond,\psi_\diamond)$.

Lemma B.12 *There exists $\kappa\ge 100$ and, given $c_v\ge\kappa$, there exists $\kappa_{c_v}>\kappa$ with the following significance: Suppose that $\mathrm{r}\ge\kappa_{c_v}c_v^{10}$ and suppose that $(A,\psi=(\alpha,\beta))$ is a solution to the (r,μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1.*

- *The corresponding $(A_\diamond,\psi_\diamond)$ does not depend on the coordinates from Part 4 of Section Aa that are chosen from the various $\gamma\in\Theta$ versions of U_γ.*
- *The corresponding $(A_\diamond,\psi_\diamond)$ satisfies the $c_0=c_v$ and $z=\mathrm{r}$ versions of Properties 1–5 in Section Ab.*

Proof The fact that $(A_\diamond,\psi_\diamond)$ does not depend on the chosen coordinates from Part 4 of Section Aa follows directly from the fact that (A_*,ψ_*) does not depend on these choices. The assertion in the second bullet follows from Lemma A.1 if Properties 1, 2, 4 and 5 are obeyed on each component of $Y-(Y_{*\Lambda}\cup T_{*\Lambda})$. The fact that these properties are obeyed follows from (3-3) and the fact that y_m and ς_m and their derivatives obey similar bounds. □

Bf A path from (A_*,ψ_*) to $(A_\diamond,\psi_\diamond)$

This subsection derives an (A,ψ)– and r– independent bound for the norm of the difference between the values of $\mathfrak{f}_s$ at (A_*,ψ_*) and $(A_\diamond,\psi_\diamond)$. The proposition that follows makes the formal assertion.

Proposition B.13 *There exists* $\kappa \geq 100$ *and, given* $c_v \geq \kappa$, *there exists* $\kappa_{c_v} > \kappa$ *with the following significance: Suppose that* $\mathrm{r} \geq \kappa_{c_v} c_v^{10}$ *and suppose that* $(A, \psi = (\alpha, \beta))$ *is a solution to the* (r, μ) *version of* (1-13) *with* μ *a given element in* Ω *with* $\mathcal{P}$*–norm smaller than* 1. *Then the norm of the difference between the respective values of* $\mathfrak{f}_s$ *at* (A_*, ψ_*) *and* $\mathfrak{f}_s$ *at* $(A_\diamond, \psi_\diamond)$ *is bounded by* κ.

The proof of this proposition is given in Part 8 of the subsection. The intervening parts define a certain path in $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ from (A_*, ψ_*) to $(A_\diamond, \psi_\diamond)$ that is used in the proof.

Part 1 The path is parametrized by $[0, 1]$ and a given $\tau \in [0, 1]$ member is denoted by $(A_{\diamond\tau}, \psi_{\diamond\tau})$ with $\tau = 0$ member (A_*, ψ_*) and $\tau = 1$ member $(A_\diamond, \psi_\diamond)$. As defined, the pairs (A_*, ψ_*) and $(A_\diamond, \psi_\diamond)$ agree on $Y_{*\Lambda} \cup T_{*\Lambda}$ and this will be the case for all pairs along the path between them. The definition of the path $\{(A_{\diamond\tau}, \psi_{\diamond\tau})\}_{\tau\in[0,1]}$ on a given component of $Y - (Y_{*\Lambda} \cup T_{*\Lambda})$ is supplied in a moment. The definition uses γ to denote the corresponding curve from the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$, and it uses T to denote the set of points with distance $(c_v^4 + c_v^3)\mathrm{r}^{-1/2}$ or less from γ. The definition also uses γ's version of the coordinates from Part 4 of Section Aa for T that has ν constant and μ constant, real and greater than $|\nu|$.

By way of an overview of what is to come, the path $\tau \mapsto (A_{\diamond\tau}, \psi_{\diamond\tau})$ first moves $(A_*, \psi_*)|_T$ to a pair with two salient features: it is very close to (A_*, ψ_*) in a large k version of the C^k–topology; and it is constructed from a 1–parameter family of vortex solutions to (2-8) with the parameter being the points in γ. The parametrization is such that the pullback via the scaling map $z \mapsto \mathrm{r}^{1/2}z$ of a given $t \in \gamma$ solution to the vortex equation defines the restriction of the new pair to the constant t slice of T. A homotopy of this γ–parametrized family through γ–parametrized families of solutions to (2-8) is used to define the second part of the path $\tau \mapsto (A_{\diamond\tau}, \psi_{\diamond\tau})$. The end member of this second part of the path is very close to $(A_\diamond, \psi_\diamond)$ in a large k version of the C^k–topology. The third part of the path moves this end pair to $(A_\diamond, \psi_\diamond)$.

Part 2 Suppose for the moment α has no zeros on the boundary of the closure of a given transverse disk in T with center on γ. If this is the case, then the sum of the local Euler numbers of the zeros of α can be defined, and this sum is a positive integer. If α has no zeros on T's boundary torus, then this sum is the same for all transverse disks of this sort. If α has zeros on this torus, then there are two values that occur unless both zeros of α on the boundary of T have the same value of the parameter t.

These two values differ by 1. In any event, use m_α to denote the larger of the possible values for the sum of the local Euler numbers.

Part 3 The construction requires a suitable isomorphism between E_T and $T\times\mathbb{C}$. To obtain one, fix an isomorphism between $E|_\gamma$ and $\gamma\times\mathbb{C}$ that writes the pullback of A on γ as $\theta + a_{A0}\,dt$ with a_{A0} being constant and having absolute value less than $2\pi/\ell_\gamma$. Use parallel transport by A along the rays from the origin in each constant t disk to define an isomorphism between $E|_T$ and $T\times\mathbb{C}$. View A and α using this isomorphism as a pair of a connection on the product bundle and a map to $\mathbb{C}$, and use the coordinates (t,z) with this isomorphism to view β likewise as a map to $\mathbb{C}$. Write A as $\theta + a_{A0}\,dt + a_A^\perp$ with a_{A0} being an $i\mathbb{R}$–valued function and where $a_A^\perp$ has the form $\frac{1}{2}\hat{\mathrm{A}}(z\,d\bar z - \bar z\,dz)$ with $\hat{\mathrm{A}}$ being a real-valued function. Use this isomorphism to view α as a map from T to $\mathbb{C}$. As explained in the next paragraph, the functions α, a_{A0} and the 1–form $a_A^\perp$ are such that

$$\left|\frac{\partial}{\partial t}\alpha\right| + |a_{A0}| + \mathrm{r}^{-1/2}\Big(|a_A^\perp| + \left|\frac{\partial}{\partial t}a_A^\perp\right|\Big) \le c_0 c_v^4. \tag{B-45}$$

To justify these bounds, introduce polar coordinates on $\mathbb{C}$ by writing $z=\rho e^{i\sigma}$. The pullback of $A-\theta$ to a given constant t slice of T is $a_A^\perp$. When written using $d\rho$ and $d\sigma$, this $i\mathbb{R}$–valued 1–form appears as $a_A^\perp = -i\hat{\mathrm{A}}\rho^2\,d\sigma$ with $\hat{\mathrm{A}}$ being an $\mathbb{R}$–valued function. The fact that this pullback of $A-\theta$ lacks a $d\rho$ component implies that $\frac{\partial}{\partial\rho}a_{A0} = F_A\big(\frac{\partial}{\partial\rho},\frac{\partial}{\partial t}\big)$, where F_A is the curvature 2–form of A. Keeping in mind that $\frac{\partial}{\partial t}$ and v differ on T by no more than $c_0|z|$, integrate this identity using the bounds in Lemmas 2.1, B.2 and B.7 to obtain the asserted bound for $|a_{A0}|$. Use this bound on $|a_{A0}|$, the aforementioned bound on $\big|\frac{\partial}{\partial t}-v\big|$, the fact that $|(\nabla_A\alpha)_v|\le c_0$ and $|\nabla_A\alpha|\le c_0\mathrm{r}^{1/2}$ to see that $\big|\frac{\partial}{\partial t}\alpha\big|$ is bounded by $c_0c_v^4$. The asserted bound on $a_A^\perp$ follows by integrating the curvature identity $\rho^{-1}\frac{\partial}{\partial\rho}(\rho^2\hat{\mathrm{A}}) = F_A\big(\frac{\partial}{\partial z},\frac{\partial}{\partial\bar z}\big)$ using again Lemmas 2.1, B.2 and B.7. To obtain the bound for the t–derivative of $a_A^\perp$, first differentiate the curvature identity $\frac{\partial}{\partial\rho}a_{A0} = F_A\big(\frac{\partial}{\partial\rho},\frac{\partial}{\partial t}\big)$ to obtain an equation for $\frac{\partial}{\partial\rho}\big(\frac{\partial}{\partial\sigma}a_{A0}\big)$. Meanwhile, a third curvature identity has $\rho^2\frac{\partial}{\partial t}\hat{\mathrm{A}} + i\frac{\partial}{\partial\sigma}a_{A0} = iF_A\big(\frac{\partial}{\partial t},\frac{\partial}{\partial\sigma}\big)$. Differentiate this last equation to obtain an equation for $\frac{\partial}{\partial\rho}\big(\rho^2\frac{\partial}{\partial t}\hat{\mathrm{A}}\big)$ that involves derivatives of the components of $F_A\big(\frac{\partial}{\partial t},\cdot\big)$. Given what was said previously about $\frac{\partial}{\partial t}$ and v, and, given the bounds in Lemmas 2.1, B.2 and B.7, integration of this last equation finds $\big|\frac{\partial}{\partial t}a_A^\perp\big| \le c_0c_v^4\mathrm{r}^{1/2}$.

Part 4 This part defines $(A_{\diamond\tau},\psi_{\diamond\tau})$ for $\tau\in\big[0,\frac{1}{3}\big]$. The definition requires a preliminary lemma, Lemma B.14. To set the notation, reintroduce $\varphi_{\mathrm{r}}\colon\mathbb{C}\to\mathbb{C}$, the rescaling map given by the rule $z \mapsto \mathrm{r}^{-1/2} z$, and introduce $D_* \subset \mathbb{C}$ to denote the radius c_v^4 disk with center at the origin.

Lemma B.14 *There exists $\kappa \geq 100$ and, given $m \geq 1$ and $c_v \geq \kappa$, there exists $\kappa_{c,m} > \kappa$ with the following significance: Suppose that $\mathrm{r} \geq \kappa_{c,m} c_v^{10}$ and suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Use (A, ψ) to define T as above. There exists a smooth map $t \mapsto c(t)$ from γ to the space of solutions to (2-8) on $\mathbb{C}$ with the properties listed below:*

- *The integral in (3-1) is finite, independent of t and either m_α or $m_\alpha + 1$.*
- *Any given version of $\mathfrak{c}(t)$ has the form $(\theta + \mathcal{A}_{*t}, \alpha_{*t})$ and, for each t, the pair $(\mathcal{A}_{*t}, \alpha_{*t})$ on D_* differs from $(\varphi_\mathrm{r}^* a_A^\perp, \varphi_\mathrm{r}^* \alpha)|_t$ in the C^0–topology by at most c_v^{-m}.*
- *The assignment $t \mapsto (\mathcal{A}_{*t}, \alpha_{*t})$ is such that $\left|\frac{\partial}{\partial t}\alpha_{*t}\right| + \left|\frac{\partial}{\partial t}\mathcal{A}_{*t}\right| \leq \kappa c_v^4$ on D_*.*

This lemma is proved in Part 5; assume it for now. The $\tau \in \left[0, \frac{1}{3}\right]$ version of $A_{\diamond\tau}$ on the $|z| \leq (c_v^4 - 2c_v^3)\mathrm{r}^{-1/2}$ part of T is $A_{\diamond\tau} = \theta + (1-3\tau)a_{A0}\,dt + a_A^\perp + 3\tau(r_\mathrm{r}^* \mathcal{A}_* - a_A^\perp)$. Meanwhile, the respective E and EK^{-1} components of $\psi_{\diamond\tau}$ on this same portion of T are defined by the rule $\alpha_{\diamond\tau} = \alpha + 3\tau(r_\mathrm{r}^*\alpha_* - \alpha)$ and $\beta_{\diamond\tau} = (1-3\tau)\beta$. The definition on the rest of T is given by using the connection $\theta + (1-3\tau)a_{A0}\,dt + a_A^\perp + 3\tau(r_\mathrm{r}^* A_* - a_A^\perp)$ in lieu of A and the sections $\alpha + 3\tau(r_\mathrm{r}^*\alpha_* - \alpha)$ and $(1-3\tau)\beta$ in lieu of (α, β) to define the various functions and 1–forms that appear in (B-8)–(B-10). Keep in mind when doing so that the various isomorphisms between E and the product bundle that are invoked when writing (B-8)–(B-10) are not the isomorphisms that are used here.

Part 5 This part contains the proof of Part 4's lemma.

Proof of Lemma B.14 The proof has five steps.

Step 1 Let D_T denote the centered, radius $c_v^4 + c_v^3$ disk in $\mathbb{C}$. Fix $t \in \gamma$ and use Lemma 2.9 with the pair $(A, \alpha)|_t$ to obtain a solution to (2-8)'s vortex equations on $\mathbb{C}$ that can be written as $(\theta + \mathcal{A}_{1t}, \alpha_{1t})$ and is such that $\alpha_{1t} - \varphi_\mathrm{r}^*\alpha|_t$ and $\mathcal{A}_{1t} - \varphi_\mathrm{r}^* a_A^\perp|_t$ on D_T have C^{3m+1}–norm bounded by c_v^{-3m-1}. The γ–parametrized family $t \mapsto (\mathcal{A}_{1t}, \alpha_{1t})$ need not be continuous, let alone differentiable; nor must it obey the first bullet's requirement at any given $t \in \gamma$.

Step 2 To obtain a γ–parametrized family that obeys the first bullet's requirements, consider first the case where α lacks zeros on the boundary of T. In this case, α_{1t}

has m_α zeros counting multiplicities that lie where $|z| \le c_0$ and no zeros where $|z|$ is greater than c_0 and less than $c_v^4 + c_v^3 + 1$. Let $z \mapsto \varrho_t(z)$ denote the monic, degree m_α polynomial on $\mathbb{C}$ whose zeros with their corresponding multiplicity are those of α_{1t}. Write this polynomial as $z^{m_\alpha} + \sigma_{1t} z^{m_\alpha - 1} + \cdots + \sigma_{m_\alpha t}$ and use the values of $\{\sigma_{qt}\}_{1\le q\le m_\alpha}$ as the coordinates for an element in the vortex moduli space $\mathfrak{C}_{m_\alpha}$. Let $\mathfrak{c}_t$ denote this element. It follows from what is said in Section 2a of [20] that there exists a purely m–dependent constant $c_m > 1$, and, given $c_v > c_m$, there exists a purely m– and c_v–dependent constant $c_{m,c}$ with the following significance: if $c_v > c_m$ and $\mathrm{r} > c_{m,c}$, then there exists a vortex solution on $\mathbb{C}$ that maps to $\mathfrak{c}_t$ which when written as $(\theta + \mathcal{A}_{2t}, \alpha_{2t})$ is such that the pair $(\mathcal{A}_{2t} - \varphi_{\mathrm{r}}^* a_A^{\perp}, \alpha_{2t} - \varphi_{\mathrm{r}}^* \alpha)$ on D_* has C^{3m}–norm bounded by $2c_v^{-3m}$.

Step 3 Consider next the case when α has two zeros on the boundary of T. Let t_+ and t_- denote the t–values of the points where these zeros occur. One of these zeros will lie where $u > 0$ and the other where $u < 0$. Use $(t_{\alpha+}, z_+)$ to denote the coordinates of the former and $(t_{\alpha-}, z_-)$ to denote those of the latter. Let $I_\alpha \subset \gamma$ denote the oriented segment that starts at $t_{\alpha+}$ and ends at $t_{\alpha-}$ with I_α being the single point t_+ when $t_{\alpha+} = t_{\alpha-}$. The significance of I_α is as follows: Fix a transverse disk of radius $(c_v^4 + c_v^3)\mathrm{r}^{-1/2}$ with center in the interior of I_α. Then the sum of the local Euler number of the zeros of α on such a disk is equal to $m_\alpha - 1$. Meanwhile, this sum for a transverse disk with center on $\gamma - I_\alpha$ is equal to m_α. Keeping in mind that the coordinate t is $\mathbb{R}/\ell_\gamma$–valued, let $t_+ \in [0, \ell_\gamma)$ denote the lift to $\mathbb{R}$ of $t_{\alpha+}$ and introduce t_- to denote the lift to $\mathbb{R}$ of $t_{\alpha-}$ with $\frac{1}{2}\ell_\gamma \le t_- - t_+ < \frac{3}{2}\ell_\gamma$. Introduce $\mathfrak{p}\colon [t_+, t_-] \to \gamma$ to denote the projection map. The inverse image of any given point in $\gamma - I_\alpha$ is empty if $t_- - t_+ < \ell_\gamma$ and it contains a single point if $t_- - t_+ \ge \ell_\gamma$. The inverse image of any given point in I_α has a single point if $t_- - t_+ \le \ell_\gamma$ and two points otherwise. Fix a smooth map $z_I\colon [t_+, t_-] \to \mathbb{C}$ with $\left|\frac{\partial}{\partial t} z_I\right|$ constant, with $|z_I| \le c_v^4 + \frac{9}{8} c_v^3$ for all t, and such that $z_I(t_+) = z_+$ and $z_I(t_-) = z_-$. Require in addition that the image of (t_+, t_-) lie where $|z| > c_v^4 + c_v^3$.

For each $t \in \gamma$, define monic polynomials $z \mapsto \varrho_{1t}(z)$ and $z \mapsto \varrho_{2t}(z)$ as follows:

(B-46)
- Suppose that $t \notin I_\alpha$. The zeros of ϱ_{1t} with their corresponding multiplicity are those of α_{1t} with distance 1 or less from some $|z| \le c_v^4 + c_v^3$ zero of $\varphi_{\mathrm{r}}^* \alpha$. If $\mathfrak{p}^{-1}(t) = \varnothing$, then $\varrho_{2t} = 1$; and $\varrho_{2t} = z - z_I(\mathfrak{p}^{-1}(t))$ if $\mathfrak{p}^{-1}(t) \ne \varnothing$.
- Suppose that $t \in I_\alpha$. The zeros of ϱ_{1t} with their corresponding multiplicity are those of α_{1t} with distance 1 or less from some $|z| \le c_v^4 + c_v^3 - 2$ zero of $\varphi_{\mathrm{r}}^* \alpha$. Meanwhile, ϱ_{2t} is $\prod_{t' \in \mathfrak{p}^{-1}(t)} (z - z_I(t'))$.

For each $t \in \gamma$, use ϱ_t to denote the product $\varrho_{1t}\varrho_{2t}$. This is a monic polynomial with t–independent degree, either m_α or $m_\alpha + 1$. Let m_* denote this degree. Use the coefficients of ϱ_t to specify a point in the vortex moduli space $\mathfrak{C}_{m_*}$. The observations in Section 2a of [20] can be used to derive a purely m–dependent constant $c_m > 1$, and, given $c_v > c_m$, a purely m– and c_v–dependent constant $c_{m,c}$ with the following significance: If $c_v > c_m$ and if $\mathrm{r} > c_{m,c}$, then there is a solution on $\mathbb{C}$ to (2-8) that maps to ϱ_t's point in $\mathfrak{C}_{m_*}$ and can be written as $(\theta + \mathcal{A}_{2t}, \alpha_{2t})$ with $(\mathcal{A}_{2t}, \alpha_{2t})$ such that $(\mathcal{A}_{2t} - \varphi_{\mathrm{r}}^* a_A^{\perp}, \alpha_{2t} - \varphi_{\mathrm{r}}^* \alpha)$ on D_* has C^{3m}–norm bounded by $2c_v^{-3m}$.

Step 4 The map $t \mapsto (\mathcal{A}_{2t}, \alpha_{2t})$ satisfies the requirements of the first and second bullets of Lemma B.14, but it need not be smooth and, if smooth, it need not satisfy the requirements of the third bullet. To remedy this defect, first introduce c_χ to denote the integral of the function $t \mapsto \chi(|t|-1)$ over $\mathbb{R}$. Fix for the moment $L \geq 1$ and define the map from γ into $C^\infty(D_T; iT^*\mathbb{C} \oplus \mathbb{C})$ given by the rule $t \mapsto (\mathcal{A}_t^L, \alpha_t^L)$, where

$$(\mathcal{A}_t^L, \alpha_t^L) = c_\chi^{-1} \int L\chi(L|t-s|-1)(\mathcal{A}_{1s}, \alpha_{1s})\, ds. \tag{B-47}$$

This is a smooth map. What follows are two consequences of (B-47). If $L \leq c_v^{3m/2+4}$ then the C^0–norms of $\alpha_t^L - \varphi_{\mathrm{r}}^* \alpha|_t$ and $\mathcal{A}_t^L - \varphi_{\mathrm{r}}^* a_A^{\perp}|_t$ on D_* are bounded by $c_v^{-3m/2}$. Moreover, the map $t \mapsto (\mathcal{A}_t^L, \alpha_t^L)$ is such that $\left|\frac{\partial}{\partial t}\alpha_t^L\right| + \left|\frac{\partial}{\partial t}\mathcal{A}_t^L\right| \leq c_0 c_v^4$ on D_*. However, any given t version $(\theta + \mathcal{A}_t^L, \alpha_t^L)$ need not obey the vortex equations. Even so, the pair comes very close to doing so.

Step 5 To obtain a pair that obeys the vortex equation, introduce the $(\theta + \mathcal{A}_t^L, \alpha_t^L)$ version of (3-4)'s operator ϑ. Denote this operator by ϑ_{tL}. As explained in a moment, there is a smooth map $t \mapsto \mathfrak{h}_t$ from γ to $C^\infty(\mathbb{C}; \mathbb{C} \oplus \mathbb{C}) \cap L^2(\mathbb{C}; \mathbb{C} \oplus \mathbb{C})$ with the following properties: Write $\theta_{tL}^\dagger \mathfrak{h}_t$ as $(2^{-1/2} e_{0t}, e_{1t})$. Then (e_{0t}, e_{1t}) has C^0–norm bounded by $c_0 c_v^{-3m/2}$ on D_*, its t–derivative on D_* has pointwise norm bounded by $c_0 c_v^{4-3m/2}$ and the pair of connection and map to $\mathbb{C}$ given by $(\theta + \mathcal{A}_t^L + e_{0t}\, d\bar{z} - \bar{e}_{0t}\, dz, \alpha_t^L + e_{1t})$ obeys the vortex equations on $\mathbb{C}$ and defines a point in $\mathfrak{C}_{m_*}$. To explain, note that the vortex equations are obeyed if $\mathfrak{h}_t$ obeys an equation having the schematic form

$$\vartheta_{tL} \vartheta_{tL}^\dagger \mathfrak{h}_t + (\vartheta_{tL}^\dagger \mathfrak{h}_t) \# (\vartheta_{tL}^\dagger \mathfrak{h}_t) = \mathfrak{q}_t, \tag{B-48}$$

where $\mathfrak{q}_t$ has C^0– and L^2–norm bounded by $c_0 c_v^{-3m/2}$. (The notation $\mathfrak{f}_1 \# \mathfrak{f}_2$ denotes a certain bilinear expression in the components of $\mathfrak{f}_1$ and $\mathfrak{f}_2$ with norm bounded by $c_0|\mathfrak{f}_1||\mathfrak{f}_2|$.) Given (3-6) and this small norm for $\mathfrak{q}_t$, the contraction mapping theorem on a suitable Hilbert space can be used to find a smooth solution $t \mapsto \mathfrak{h}_t$ from γ to $C^\infty(\mathbb{C}; \mathbb{C} \oplus \mathbb{C}) \cap L^2(\mathbb{C}; \mathbb{C} \oplus \mathbb{C})$ with C^1–norm bounded by $c_0 c_v^{-3m/2}$. The contraction mapping theorem construction will guarantee a pointwise norm bound by $c_0 c_v^4$ for the t–derivative of $\vartheta_{tL}^\dagger \mathfrak{h}_t$ on D.

Granted the preceding, set $\mathcal{A}_{*t} = \mathcal{A}_t^L + e_{0t}\, d\bar{z} - \bar{e}_{0t}\, dz$ and set $\alpha_{*t} = \alpha_t^L + e_{1t}$. The resulting map $t \mapsto (\mathcal{A}_{*t}, \alpha_{*t})$ obeys all of the lemma's requirements. □

Part 6 This part defines the pair $(A_{\diamond\tau}, \psi_{\diamond\tau})$ for $\tau \in \left[\frac{1}{3}, \frac{2}{3}\right]$. To this end, let D denote a given constant t slice of T, this being a transverse disk with center on γ and radius $(c_v^4 + c_v^3)\mathrm{r}^{-1/2}$. If the sum of the local Euler numbers of α on D is defined, then it is also defined for $\alpha_\diamond$ and these sums are the same. Note that all of the local Euler numbers of $\alpha_\diamond$ are positive. Let $t \in \gamma$ denote the center point of such a disk. Define a monic polynomial $z \mapsto \varrho_{\diamond t}(z)$ on $\mathbb{C}$ using the rules that follow. If α has no zeros on the boundary of T, then $\varrho_{\diamond t}(z) = z^{m_\alpha}$. If α has zeros on the boundary of T, define $\varrho_{\diamond t}$ by using $\alpha_{\diamond t}$ in lieu of α_{1t} in (B-46). Meanwhile, let ϱ_{*t} denote the monic polynomial on $\mathbb{C}$ whose roots with their corresponding multiplicity are the zeros of the function α_{*t} from Lemma B.14. Note that all such zeros have positive local Euler number. The polynomials $\varrho_{\diamond t}$ and ϱ_{*t} have the same degree. Denote this degree by m_*.

Given $\tau \in \left[\frac{1}{3}, \frac{2}{3}\right]$, set $\varrho_{\tau t}$ to be the monic polynomial $(2-3\tau)\varrho_{*t} + (3\tau - 1)\varrho_{\diamond\tau}$. The resulting 1–parameter family of polynomials interpolates between ϱ_{*t} and $\varrho_{\diamond t}$. For any given pair (τ, t), the coefficients of $\varrho_{\tau t}$ defines a point in $\mathfrak{C}_{m_*}$ that varies smoothly with variations in τ and t with the variation in τ being real analytic. With τ fixed for the moment, let $t \mapsto \mathfrak{c}_\tau(t)$ denote the corresponding map from γ to $\mathfrak{C}_m$. Lemma B.14 describes a lift of the map $t \mapsto \mathfrak{c}_{\tau=1/3}(t)$ to a smooth map $t \mapsto (\theta + \mathcal{A}_{*t}, \alpha_{*t})$ from γ into the space of solutions to (2-8) on $\mathbb{C}$. The next lemma describes a corresponding smooth lift of the two-variable map $(\tau, t) \mapsto \mathfrak{c}_\tau(t)$ from $\left[\frac{1}{3}, \frac{2}{3}\right] \times \gamma$ to $\mathfrak{C}_{m_*}$.

Lemma B.15 *There exists $\kappa \geq 100$ and, given $c_v \geq \kappa$, there exists $\kappa_{c_v} > \kappa$ with the following significance: Suppose that $\mathrm{r} \geq \kappa_{c_v} c_v^{10}$ and suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Fix a component of the corresponding version of $Y - (Y_{*\Lambda} \cup T_{*\Lambda})$ and introduce the latter's version of the integer m and the map $(\tau, t) \mapsto \mathfrak{c}_\tau(t)$ from $\left[\frac{1}{3}, \frac{2}{3}\right] \times \gamma$ to $\mathfrak{C}_m$. There is a smooth map $(\tau, t) \mapsto (\mathcal{A}_{\tau t}, \alpha_{\tau t})$ from $\left[\frac{1}{3}, \frac{2}{3}\right] \times \gamma$ to $C^\infty(\mathbb{C}; iT^*\mathbb{C} \oplus \mathbb{C})$ which is real analytic with respect to variations in τ and such that at each $(\tau, t) \in \left[\frac{1}{3}, \frac{2}{3}\right] \times \gamma$, the pair of connection on the product bundle $\mathbb{C} \times \mathbb{C}$ and map to $\mathbb{C}$ given by $(\theta + \mathcal{A}_{\tau t}, \alpha_{\tau t})$ satisfies (2-8) and projects to $\mathfrak{c}_\tau(t)$. In addition,*

- $\left|\frac{\partial}{\partial t}\alpha_{\tau t}\right| + \left|\frac{\partial}{\partial t}\mathcal{A}_{\tau t}\right| \leq \kappa c_v^4$ *on* D_*;
- $\left|\frac{\partial}{\partial \tau}\partial\alpha_{\tau t}\right| + \left|\frac{\partial}{\partial \tau}\mathcal{A}_{\tau t}\right| \leq \kappa c_v^4$ *on* D_*.

This lemma is proved in a moment. By way of notation, any given $\tau \in \left[\frac{1}{3}, \frac{2}{3}\right]$ version of the map $t \mapsto (\mathcal{A}_{\tau t}, \alpha_{\tau t})$ from γ into $C^\infty(\mathbb{C}; iT^*\mathbb{C} \oplus \mathbb{C})$ is denoted in what follows by $(\mathcal{A}_\tau, \alpha_\tau)$.

The $\tau \in \left[\frac{1}{3}, \frac{2}{3}\right]$ version of the connection $A_{\diamond\tau}$ on the $|z| \leq (c_v^4 - 2c_v^3)\mathrm{r}^{-1/2}$ part of T is $A_{\diamond\tau} = \theta + r_\mathrm{r}^* \mathcal{A}_\tau$ and the respective E and EK^{-1} components of $\psi_{\diamond\tau}$ on this same portion of T are defined by the rule $\alpha_{\diamond\tau} = r_\mathrm{r}^*\alpha_\tau$ and $\beta_{\diamond\tau} = 0$. The definition on the rest of T is given by using the connection $\theta + r_\mathrm{r}^* A_\tau$ in lieu of A and the sections $r_\mathrm{r}^*\alpha_\tau$ and 0 in lieu of (α, β) to define the various functions and 1–forms that appear in (B-8)–(B-10). Keep in mind when doing so that the various isomorphisms between E and the product bundle that are invoked when writing (B-8)–(B-10) are not the isomorphisms that are used here.

Proof of Lemma B.15 The existence of a lift of the map $(\tau, t) \mapsto \mathfrak{c}_\tau(t)$ follows from what is said in Section 2c of [20]. The existence of a lift with t– and τ–derivatives bounded by $c_0 c_v^4$ follows from what is said in this same Section 2c of [20] using (2.5), (2.11), (2.12) and (2.19) in [20]. □

Part 7 This part defines $(A_{\diamond\tau}, \psi_{\diamond\tau})$ for $\tau \in \left[\frac{2}{3}, 1\right]$. This definition is given below by (B-49). To set the notation, view $A_\diamond$ and the pair $(\alpha_\diamond, \beta_\diamond)$ as a respective connection on $T \times \mathbb{C}$ and pair of maps from T to $\mathbb{C}$ using the same isomorphism of $E|_T$ with $T \times \mathbb{C}$ that is used to define the $\tau = \frac{2}{3}$ version of $(A_{\diamond\tau}, \psi_{\diamond\tau})$. The definition writes this depiction of $A_\diamond$ as $\theta + \mathrm{A}'_\diamond$ and it writes this depiction of $(\alpha_\diamond, \beta_\diamond)$ as $(\alpha'_\diamond, \beta'_\diamond)$. The connection $A_{\diamond\tau=\frac{2}{3}}$ is written below as $\theta + \mathrm{A}_{\diamond\frac{2}{3}}$. Equation (B-49) refers to a map $\hat{u}\colon \gamma \times \mathbb{C} \to S^1$ that is described below by Lemma B.16. Fix $\tau \in \left[\frac{2}{3}, 1\right]$ and what follows defines $(A_{\diamond\tau}, \psi_{\diamond\tau})$ on T:

(B-49)
- $A_{\diamond\tau} = \theta + (3\tau - 2)(\mathrm{A}'_\diamond - \hat{u}^{-1} d\hat{u}) + (3 - 3\tau)\mathrm{A}_{\diamond\frac{2}{3}}$.
- $\alpha_{\diamond\tau} = (3\tau - 2)\hat{u}\alpha'_\diamond + (3 - 3\tau)\alpha_{\diamond\frac{2}{3}}$ and $\beta_{\diamond\tau} = (3\tau - 2)\hat{u}\beta'_\diamond$.

The map $\hat{u}$ is constructed in the proof of Lemma B.16.

Lemma B.16 *There exists $\kappa \geq 100$ and, given $m \geq 1$ and $c_v \geq \kappa$, there exists $\kappa_{c,m} > \kappa$ with the following significance: Suppose that $\mathrm{r} \geq \kappa_{c,m} c_v^{10}$ and let $(A, \psi = (\alpha, \beta))$ denote a solution to the (r, μ) version of (1-13) with μ an element in Ω with $\mathcal{P}$–norm less than 1. Use (A, ψ) to define T and the corresponding versions of $(A'_\diamond, \alpha'_\diamond)$ and $(A_{\diamond\frac{2}{3}}, \alpha_{\diamond\frac{2}{3}})$. There exists a smooth map $\hat{u}\colon T \to S^1$ such that*

$$\mathrm{r}^{-1/2}|\mathrm{A}'_\diamond - \hat{u}^{-1} d\hat{u} - \mathrm{A}_{\diamond\frac{2}{3}}| + |\hat{u}\alpha'_\diamond - \alpha_{\diamond\frac{2}{3}}| \leq c_v^{-m} \quad \textit{and} \quad \left|\mathrm{A}'_\diamond\left(\frac{\partial}{\partial t}\right) - \hat{u}^{-1}\frac{\partial}{\partial t}\hat{u}\right| \leq \kappa c_v^4.$$

Proof The two steps that follow construct $\hat{u}$ on the $|z| \le \frac{1}{2} c_v^{1/2} \mathrm{r}^{-1/2}$ portion of T. But for cosmetic changes, the same construction supplies $\hat{u}$ on the rest of T.

Step 1 To define $\hat{u}$ where $|z| \le \frac{1}{2} c_v^{1/2} \mathrm{r}^{-1/2}$, recall from Part 3 that the pullback of $(A_\diamond, \psi_\diamond)$ to the $|z| \le \frac{1}{4} c_v^{1/2} \mathrm{r}^{-1/2}$ portion of a transverse disk centered on any given $t \in \gamma$ is the solution to the vortex equations in (2-8) given by

$$\big(\theta - \tfrac{1}{2} m r_\mathrm{r}^* a_{m0}(z^{-1}\,dz - \bar{z}^{-1}\,d\bar{z}), r_\mathrm{r}^* \alpha_{m0}\big).$$

Meanwhile, the pullback of $(A_{\diamond\tau=\frac{2}{3}} \alpha_{\diamond\tau=\frac{2}{3}})$ to the same part of the transverse disk centered at $t \in \gamma$ is a solution to (2-8) that was written as $(\theta + r_\mathrm{r}^* \mathcal{A}_{\frac{2}{3}t}, r_\mathrm{r}^* \alpha_{\frac{2}{3}t})$. The two $\mathbb{C}$–valued functions α_{m0} and $\alpha_{\frac{2}{3}t}$ have the same zero locus on the $|z| \le \frac{3}{4} c_v^{1/2}$, this being the origin. Moreover, they have the same local degree at 0. What follows is a consequence: there exists a smooth map, denoted here by u, from the $|z| \le \frac{9}{16} c_v^{1/2}$ part of $\gamma \times \mathbb{C}$ to S^1 such that $u\alpha_{\frac{2}{3}} = |\alpha_{\frac{2}{3}}| z^m$.

Fix a positive integer m. Granted the preceding, use what is said in Part 4 of Section 2a in [20] about solutions to (2-8) to find a purely m–dependent lower bound for c_v such that the subsequent assertion is true when c_v exceeds this bound. Introduce $d^\perp u$ to denote the exterior derivative of u along the constant t slices of $\gamma \times \mathbb{C}$. For $t \in \gamma$, the pairs $(\mathcal{A}_{\frac{2}{3}t} - u^{-1} d^\perp u, u\alpha_{\frac{2}{3}t})$ and $\big(-\frac{1}{2} m a_{m0}(z^{-1}\,dz - \bar{z}^{-1}\,d\bar{z}), \alpha_{m0}\big)$ differ by at most c_v^{-3m} in the C^{2m} topology on the $|z| \le \frac{5}{8} c_v^{1/2}$ disk in $\mathbb{C}$.

This last conclusion has the following consequence: if c_v is greater than a purely m–dependent lower bound, then the map $u_1 = u^{-1}(\varphi_\mathrm{r}^* \alpha'_\diamond)^{-1} \alpha_{m0}$, from the $|z| \le \frac{9}{16} c_v^{1/2}$ part of $\gamma \times \mathbb{C}$ to S^1 is such that for any $t \in \gamma$, the pair $(\mathcal{A}_{\frac{2}{3}t}, \alpha_{\frac{2}{3}t})$ and the pullback to $\{t\} \times \mathbb{C}$ of the pair $(\varphi_\mathrm{r}^* \mathrm{A}'_\diamond - u_1^{-1} du_1, u_1 \varphi_\mathrm{r}^* \alpha'_\diamond)$ differ by less than $c_v^{-5m/2}$ in the C^{2m}–topology on the disk $|z| \le \frac{5}{8} c_v^{1/2}$.

Step 2 The map u_1 can be replaced by a map $u_2 \colon \gamma \times \mathbb{C} \to S^1$ such that if c_v is greater than a purely m–dependent constant, then:

(B-50)
- For $t \in \gamma$, the pullback to $\{t\} \times \mathbb{C}$ of $(\varphi_\mathrm{r}^* \mathrm{A}'_\diamond - u_2^{-1} du_2, u_2 \varphi_\mathrm{r}^* \alpha'_\diamond)$ and $(\mathcal{A}_{\frac{2}{3}t}, \alpha_{\frac{2}{3}t})$ differ pointwise on the $|z| \le \frac{5}{8} c_v^{1/2}$ part of $\mathbb{C}$ by at most c_v^{-m}.
- $\big|\mathrm{A}'_\diamond\big(\frac{\partial}{\partial t}\big) - u_2^{-1} \frac{\partial}{\partial t} u_2\big| \le c_0 c_v^4$ on the $|z| \le \frac{5}{8} c_v^{1/2}$ part of $\mathbb{C}$.

The map $\hat{u}$ on the $|z| \le \frac{5}{8} c_v^{1/2} \mathrm{r}^{-1/2}$ part of T is defined to be $r_\mathrm{r}^* u_2$.

To construct u_2, write u_1 as $u^{-1}(\varphi_\mathrm{r}^* \alpha'_\diamond)^{-1} \alpha_{m0}$ and write u as $e^{i(2\pi n t/\ell_\gamma + x)}$ with n being an integer and x being a real-valued function on $\gamma \times \mathbb{C}$. The map u_2 has the form

$e^{i(2\pi n/\ell_\gamma + x_2)}$ with x_2 being a real-valued function on $\gamma \times \mathbb{C}$. The function x_2 is the smoothing of the function x that is given by the rule $x_2|_t = c_\chi^{-1}\int L\chi(L|t-s|-1)x|_s\,ds$ with c_χ being the constant that appears in (B-47) and with $L = c_v^{5m/4}$. The resulting map u_2 obeys the inequality in the first bullet of (B-50) if c_v is greater than a purely m–dependent constant. This is a direct consequence of the $c_v^{-5m/2}$ bound obtained in Step 1. Meanwhile, $\left|\frac{\partial}{\partial t}x_2\right| \le c_0 c_v^4$, this being a consequence of this same $c_v^{-5m/2}$ bound and the bound in the top bullet of Lemma B.15. Granted all of this, then Lemma B.16's right-most inequality is obeyed if the integer n is such that $|n| \le c_0 c_v^4$.

To obtain such a bound for n, fix a constant z circle in T with $|z| = \frac{1}{4}\kappa_\diamond^{-1} c_v$ and with distance at least $\frac{1}{100}\kappa_\diamond^{-1} c_v^{1/4}$ or more from α's zero locus. Proposition 2.4 guarantees the existence of such circles if $c_v \ge c_0$ and if r is greater than a purely c_v–dependent constant. The integral over the chosen circle of $-iu^{-1}\frac{\partial}{\partial t}u$ is equal to $2\pi n \ell_\gamma^{-1}$, and so upper and lower bounds on this integral give a bound for $|n|$. A suitable bound is obtained by writing $\alpha_{\frac{2}{3}}$ as $|\alpha_{\frac{2}{3}}|u^{-1}z^m$ to derive the identity

$$\text{(B-51)} \qquad i(\alpha_{\frac{2}{3}})^{-1}\frac{\partial}{\partial t}\alpha_{\frac{2}{3}} = 2\pi n \ell_\gamma^{-1} + \frac{\partial}{\partial t}x + i\frac{\partial}{\partial t}\ln(|\alpha_{\frac{2}{3}}|).$$

Integrate both sides of this identity on the given circle. The integral of the right-hand side is $2\pi n$, and the top bullet in Lemma B.15 bounds the absolute value of the integral of the left-hand side by $c_0 c_v^4$. □

Part 8 The promised proof of Proposition B.13 is given below. By way of a look ahead, the proof uses the results from Appendix A in much the same way as does the proof of Proposition B.3. Most of what is said in Appendix A requires Properties 1–5 in Section Ab; the fact that each $\tau \in [0,1]$ version of $(A_{\diamond\tau}, \psi_{\diamond\tau})$ has these properties is asserted by the next lemma.

Lemma B.17 *There exists $\kappa \ge 100$ and, given $c_v \ge \kappa$, there exists $\kappa_{c_v} > \kappa$ with the following significance: Suppose that $\mathrm{r} \ge \kappa_{c_v} c_v^{10}$ and suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Each element in the corresponding path $\{(A_\tau, \psi_\tau)\}_{\tau\in[0,1]}$ obeys Properties 1–5 in Section Ab.*

Proof The assertion follows from Lemma A.1 if it is the case that Properties 1–5 hold on $Y - (Y_{*\Lambda} \cup T_{*\Lambda})$. To verify that this is indeed the case, focus attention now on a given component of this set. Let γ denote the corresponding curve from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$ and let T denote the radius $(c_v^4 + c_v^3)\mathrm{r}^{-1/2}$ tubular neighborhood of γ.

The fact that Properties 4 and 5 hold on T when $c_v \ge c_0$ and r is larger than a purely c_v–dependent constant follows from (A-4) and Lemmas B.14–B.16. The fact that Properties 1 and 2 hold on T follows from (A-4), and Lemmas B.14–B.15 given that the vectors fields $\frac{\partial}{\partial t}$ and v differ on T by at most $c_0 c_v^4$. The details of the argument are straightforward and left to the reader but for the remark that the verification of the second and third bullets of Property 2 require the third bullet of Lemma B.14, the first bullet of Lemma B.15 and the bound for $\left|\mathrm{A}'_\diamond\big(\frac{\partial}{\partial t}\big) - \hat{u}^{-1}\frac{\partial}{\partial t}\hat{u}\right|$ in Lemma B.16. □

Proof of Proposition B.13 The assertion of the proposition follows if there is a purely c_v–dependent $\kappa_c \ge 1$ with the following property: Assume that $c_v \ge c_0$ and $\mathrm{r} \ge \kappa_c$. Fix any interval $[\tau, \tau'] \subset [0, 1]$ of length at most κ_c^{-1}. Then the norm of the difference between the values of $\mathfrak{f}_s$ at $(A_{\diamond\tau}, \psi_{\diamond\tau})$ and at $(A_{\diamond\tau'}, \psi_{\diamond\tau'})$ is bounded by c_0. The three steps of the proof exhibit a purely c–dependent κ_c with this property.

Step 1 What is said in Part 1 of the proof of Proposition B.3 applies to the family $\{\mathfrak{L}_{\mathbb{V}\tau}\}_{\tau\in[0,1]}$, where any given $\tau \in [0, 1]$ member is the $(A_{\diamond\tau}, \psi_{\diamond\tau})$ version of the operator $\mathfrak{L}_{\mathbb{V}}$ that is depicted in (A-26) and (A-27). This being the case, there is the corresponding set of eigenvalue families $\{\lambda_{n\tau}\}_{n\in\mathbb{Z},\tau\in[0,1]}$. Keep in mind that all $\tau \in [0, 1]$ versions of $\mathfrak{L}_{\mathbb{V}\tau}$ are identical on $Y_{*\Lambda} \cup T_{*\Lambda}$. This has the following consequence: Fix $n \in \mathbb{Z}$ and an interval in $[0, 1]$ where the map $\tau \mapsto \lambda_{n\tau}$ is differentiable. Let $\tau \mapsto \mathfrak{f}_{(\tau)}$ denote the corresponding family of unit L^2–norm eigenvectors. Then the relevant version of (B-20) has the form

$$\text{(B-52)} \qquad \frac{d}{d\tau}\lambda_{n\tau} = \int_{Y-(Y_{*\Lambda}\cup T_{*\Lambda})} \mathfrak{f}^{\dagger}_{(\tau)}\Big(\frac{d}{d\tau}\mathfrak{L}_{\mathbb{V}\tau}\Big)\mathfrak{f}_{(\tau)}.$$

Fix $m > c_0$ and take c_v and r so as to invoke Lemmas B.14 and B.16. If I is either of the intervals $\big[0, \frac{1}{3}\big]$ or $\big[\frac{2}{3}, 1\big]$, then these lemmas imply that the $\tau \in I$ versions of $\frac{d}{d\tau}\mathfrak{L}_{\mathbb{V}(\cdot)}$ is an endomorphism of $\mathbb{V}$ with pointwise norm bounded by $c_0 c_v^{-m}\mathrm{r}^{1/2}$. This being the case, integrate (B-52) to draw the following conclusion: Fix $n \in \mathbb{Z}$. If $\lambda_{n(\cdot)}$ has a zero in I, then $|\lambda_{n\tau}| \le c_0 c_v^{-m}\mathrm{r}^{1/2}$ for all $\tau \in I$.

Suppose in addition that c_v and r are such that Lemma B.15 can also be invoked. Lemma B.15 implies that the $\tau \in \big[\frac{1}{3}, \frac{2}{3}\big]$ version of $\frac{d}{d\tau}\mathfrak{L}_{\mathbb{V}(\cdot)}$ has pointwise norm bounded by $c_0\mathrm{r}^{1/2}$. This understood, fix an interval $I \subset \big[\frac{1}{3}, \frac{2}{3}\big]$ with length at most c_v^{-m}. Integrate (B-52) on the interval I to deduce the analog of what is said at the end of the preceding paragraph: Fix $n \in \mathbb{Z}$. If $\lambda_{n(\cdot)}$ has a zero in I, then $|\lambda_{n\tau}| \le c_0 c_v^{-m}\mathrm{r}^{1/2}$ for all $\tau \in I$.

Step 2 With $m \geq 1$ fixed, take c_v and r large enough to invoke the preceding lemmas in this Appendix B and the $c_0 = c_v$ versions of the lemmas in Appendix A. Let I now denote any given interval in [0, 1] of length at most c_v^{-m}. If $m > c_0$, then Lemma A.6 can be invoked to draw the following conclusion: Let $n \in \mathbb{Z}$ be such that $\lambda_{n(\cdot)}$ has a zero in I. Fix $\tau \in I$ and use $\mathfrak{f}_{(\tau)}$ to denote an eigenvector of $\mathfrak{L}_{\mathbb{V}\tau}$ with eigenvalue $\lambda_{n\tau}$. Then $\|\Pi_\theta \mathfrak{f}_{(\tau)}\| \geq (1 - c_0 c_v^{-1})\|\mathfrak{f}_{(\tau)}\|_2$.

Supposing that $m \geq c_0$, that c_v is greater than a purely m–dependent constant, and that r is greater than a purely m– and c_v–dependent constant, then Lemmas A.7 and A.8 can be invoked to conclude the following: Let $I \subset [0, 1]$ denote an interval of length $c_0^{-1} c_v^{-m}$. If $n \in \mathbb{Z}$ and $\lambda_{n(\cdot)}$ has a zero in I, then $|\lambda_{n\tau}| \leq c_v^{-m}$ for all $\tau \in I$.

Step 3 Let $I \subset [0, 1]$ denote an interval of length at most $c_0^{-1} c_v^{-m}$. Write I as $[\tau, \tau']$. Granted that the conclusion of the preceding step holds for I, then the argument used in Part 4 of the proof of Proposition B.3 can be repeated with only notational changes to see that the norm of the difference between the respective values of $\mathfrak{f}_s$ at $(A_{\diamond\tau}, \psi_{\diamond\tau})$ and at $(A_{\diamond\tau'}, \psi_{\diamond\tau'})$ is at most c_0. □

C Paths in $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ from vortex solutions

This last section of the appendix first constructs a deformation of $(A_\diamond, \psi_\diamond)$ through a family of pairs in $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$, all made from vortex solutions as in Section Aa and (A-44) using $\mathrm{z} = \mathrm{r}$. The end result is then deformed through a family that is defined using vortex solutions as done in Section Aa and (A-44) using ever-*increasing* values of z. The end result of this deformation is a pair whose resulting version of $\mathfrak{L}_{\mathbb{V}}$ as defined using $\mathrm{z} \gg \mathrm{r}$ can be compared with that of a $z = \mathcal{O}(1)$ version using a strategy from [21]. These comparisons are used in Section Ce of this appendix to complete the proof of Proposition 2.6.

Ca Deforming the zero locus of $\alpha_\diamond$

The zero locus of $\alpha_\diamond$ is a disjoint union of two sorts of embedded circles. The first are curves from the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$. The remainder consist of a finite set of at most G embedded circles that look very much like the subset of curves from a generator of the embedded contact homology chain complex that intersect the $f^{-1}(1, 2)$ part of M_δ. With this in mind, this subsection constructs a path in $\mathrm{Conn}(E) \times C^\infty(Y; \mathbb{S})$ from $(A_\diamond, \psi_\diamond)$ that ends at a pair of connection on E and section of $\mathbb{S}$ with the following property: the section of $\mathbb{S}$ when written with respect to the decomposition

$\mathbb{S} = E \oplus EK^{-1}$ has E component whose zero locus consists entirely of closed integral curves of v.

The construction of the desired path occupies the first three parts of the subsection. The fourth part of the subsection states and then proves a proposition that supplies an r–independent bound for the absolute value of the difference between $\mathfrak{f}_s$ at $(A_\diamond, \psi_\diamond)$ and at the end member of the path.

Part 1 Given that $\mathrm{r} \geq c_0$, it follows from Proposition 2.4 and Proposition II.2.7 that there exists a set of closed integral curves of v whose intersection with M_δ is everywhere very close to $\alpha^{-1}(0) \cap M_\delta$. This set of curves is denoted here by Θ^α; it is parametrized as in Proposition II.2.7 as $\Theta^\alpha = (\hat{v}^\alpha, (\mathfrak{k}_\mathfrak{p}^\alpha)_{\mathfrak{p}\in\Lambda})$. The component $\hat{v}^\alpha$ from Θ^α describes how the curves from Θ^α intersect M_δ, and each $\mathfrak{p} \in \Lambda$ version of $\mathfrak{k}_\mathfrak{p}^\alpha$ is an integer that describes how the curves from Θ^α intersect $\mathcal{H}_\mathfrak{p}$. The paragraphs that follow say more about the significance of the parametrization that is used by [9].

What is denoted by $\hat{v}^\alpha$ signifies a certain set of G segments of integral curves of v in the $f^{-1}(1,2)$ part of M_δ, these being integral curves that extend into M as integral curves of the pseudogradient vector field for f that was used in Section II.1 to define the geometry of Y. The segments that form $\hat{v}^\alpha$ define a pairing between the index 1 critical points of the incarnation of f as a function on M and the latter's index 2 critical points in the following sense: Each arc from this set starts on the boundary of the radius δ coordinate ball in M_δ corresponding to an index 1 critical point of f, and each ends on the boundary of the radius δ coordinate ball in M_δ of an index 2 critical point of f. Moreover, distinct arcs start on distinct radius δ coordinate balls and end on distinct radius δ coordinate balls. The section α determines $\hat{v}^\alpha$ in the following way: The pairing of index 1 critical points of $f|_M$ with index 2 critical points that is determined via α as described in the third bullet of Proposition 2.4 is the same pairing given by $\hat{v}^\alpha$. Moreover, the respective components of $\alpha^{-1}(0) \cap M_\delta$ and $\hat{v}^\alpha$ that pair the same index 1 and index 2 critical points of $f|_M$ are in each other's radius $c_0^{-1}\delta$ tubular neighborhoods.

As noted above, the component $(\mathfrak{k}_\mathfrak{p}^\alpha)_{\mathfrak{p}\in\Lambda}$ of Θ^α consist of a set of integers that are labeled by the pairs in Λ. The remainder of Part 1 explains how α determines this set. To this end, let υ denote a component of the zero locus of $\alpha_\diamond$ that intersects M_δ and let $\hat{v}^{\alpha,\upsilon} \subset \hat{v}^\alpha$ denote the subset which corresponds to $\upsilon \cap M_\delta$ in the sense that corresponding arcs label the same index 1 and index 2 critical points of $f|_M$. Introduce Λ_υ to denote the subset of $\mathfrak{p} \in \Lambda$ with $\upsilon \cap \mathcal{H}_\mathfrak{p} \neq \varnothing$ and suppose for the moment that

$\mathfrak{k} = (\mathfrak{k}_{\mathfrak{p}})_{\mathfrak{p}\in\Lambda_\upsilon}$ is a given set of integers parametrized by Λ_υ. Proposition II.2.7 uses the sets $\hat{\upsilon}^{\alpha,\upsilon}$ and $\mathfrak{k}$ to define a closed integral curve of v. Let $\upsilon^{\mathfrak{k}}$ denote this integral curve of v. The next paragraph summarizes some facts about $\upsilon^{\mathfrak{k}}$ that follow from Proposition II.2.7.

The label $\mathfrak{k}$ makes a significant difference with regards to the behavior of $\upsilon^{\mathfrak{k}}$ on the various $\mathfrak{p} \in \Lambda_\upsilon$ versions of $\mathcal{H}_{\mathfrak{p}}$. To say more, fix an element $\mathfrak{p} \in \Lambda_\upsilon$. Then $\upsilon^{\mathfrak{k}} \cap \mathcal{H}_{\mathfrak{p}}$ is an arc that crosses $\mathcal{H}_{\mathfrak{p}}$ where $1 - 3\cos^2\theta > 0$ starting from the $u < 0$ boundary of $\mathcal{H}_{\mathfrak{p}}$ and ending on the $u > 0$ boundary of $\mathcal{H}_{\mathfrak{p}}$. These endpoints have distance at most $c_0^{-1}\delta$ from the corresponding endpoints of $\upsilon \cap \mathcal{H}_{\mathfrak{p}}$. This understood, define a continuous and piecewise smooth loop in $\mathcal{H}_{\mathfrak{p}}$ as follows: Start on the $u < 0$ boundary point of $\upsilon^{\mathfrak{k}} \cap \mathcal{H}_{\mathfrak{p}}$ and travel along $\upsilon^{\mathfrak{k}} \cap \mathcal{H}_{\mathfrak{p}}$ to its $u > 0$ boundary. Take the short geodesic arc from this boundary point of $\upsilon^{\mathfrak{k}} \cap \mathcal{H}_{\mathfrak{p}}$ to the nearby boundary point of $\upsilon \cap \mathcal{H}_{\mathfrak{p}}$. Having done so, travel in the reverse direction along $\upsilon \cap \mathcal{H}_{\mathfrak{p}}$ to its boundary point on the $u < 0$ boundary of $\mathcal{H}_{\mathfrak{p}}$. Then take the short geodesic arc to the starting point on $\upsilon^{\mathfrak{k}} \cap \mathcal{H}_{\mathfrak{p}}$. The result is an oriented, piecewise smooth loop in the $1 - 3\cos^2\theta > 0$ part of $\mathcal{H}_{\mathfrak{p}}$ and thus a class in the first homology of the $1 - 3\cos^2\theta > 0$ part of $\mathcal{H}_{\mathfrak{p}}$. Meanwhile, the first homology of this part of $\mathcal{H}_{\mathfrak{p}}$ is isomorphic to $\mathbb{Z}$ with generator being the $u = 0$, $\cos\theta = 0$ circle. The loop just constructed from $\upsilon \cap \mathcal{H}_{\mathfrak{p}}$ and $\upsilon^{\mathfrak{k}} \cap \mathcal{H}_{\mathfrak{p}}$ defines an element in this homology class, thus an integer multiple of the generator. This integer can be written as $\mathfrak{m}_{\upsilon,\mathfrak{p}} + \mathfrak{k}_{\mathfrak{p}}$ with $\mathfrak{m}_{\upsilon,\mathfrak{p}}$ depending on $\upsilon \cap \mathcal{H}_{\mathfrak{p}}$ but not on $\mathfrak{k}$.

Granted the preceding, any given $\mathfrak{p} \in \Lambda_\upsilon$ version of the integer $\mathfrak{k}_{\mathfrak{p}}^{\alpha}$ coming from Θ^α is $-\mathfrak{m}_{\upsilon,\mathfrak{p}}$. This is to say that the $\mathfrak{k}_{\mathfrak{p}} = \mathfrak{k}_{\mathfrak{p}}^{\alpha}$ version of the loop in $\mathcal{H}_{\mathfrak{p}}$ described in the preceding paragraph is null-homotopic.

The subsequent parts of this subsection use $\upsilon^\alpha \subset \Theta^\alpha$ to denote the loop that is defined by the subsets $\hat{\upsilon}^{\alpha,\upsilon} \subset \hat{\upsilon}^{\alpha}$ and components $(\mathfrak{k}_{\mathfrak{p}}^{\alpha})_{\mathfrak{p}\in\Lambda_\upsilon} \subset (\mathfrak{k}_{\mathfrak{p}})_{\mathfrak{p}\in\Lambda}$.

Part 2 The introduction promises a path in $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ from $(A_\diamond, \psi_\diamond)$ that ends at a pair whose section of $\mathbb{S} = E \oplus EK^{-1}$ has E component with zero locus consisting entirely of closed integral curves of v, these being the curves from Θ^α and the curves from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^{+} \cup \hat{\gamma}_{\mathfrak{p}}^{-})$ that lie in $\alpha_\diamond^{-1}(0)$. The path in $\mathrm{Conn}(E)\times C^\infty(Y;\mathbb{S})$ is parametrized by $[0,1]$ and a given $\tau \in [0,1]$ member of this path is denoted in what follows by $(A_{\diamond 1\tau}, \psi_{\diamond 1\tau})$. The definition of this element in $\mathrm{Conn}(E) \times C^\infty(Y;\mathbb{S})$ is given in a moment. The lemma that follows directly supplies input for the definition.

Lemma C.1 *Fix $m \geq 1$. There an m–dependent constant $\kappa \geq 100$ and, given $c_v \geq \kappa$, there exists $\kappa_{c_v} > \kappa$ with the following significance: Suppose that $\mathrm{r} \geq \kappa_{c_v} c_v^{10}$ and* *suppose that* $(A, \psi = (\alpha, \beta))$ *is a solution to the* (r, μ) *version of* (1-13) *with* μ *a given element in* Ω *with* $\mathcal{P}$*–norm smaller than* 1. *The parameters* κ, c_υ *and* r *are suitable for use in Lemma B.11 and in particular for constructing* $(A_\diamond, \psi_\diamond = (\alpha_\diamond, \beta_\diamond))$ *and the corresponding set* Θ^α. *Let* υ *denote a component of* $\alpha_\diamond^{-1}(0)$ *that intersects* M_δ *and let* υ^α *denote the corresponding element in* Θ^α. *There exists an isotopy from* $[0,1] \times \upsilon$ *into* Y *starting from* υ, *ending at* υ^α *and with the properties listed below. The list uses* υ^α_τ *to denote the* $\tau \in [0,1]$ *curve of the isotopy.*

- *Each point in* υ^α_τ *has distance at most* m^{-1} *from the corresponding point in* υ.
- *Each point in* υ^α_τ *has distance at least* $\kappa^{-1} c_\upsilon \mathrm{r}^{-1/2}$ *from each curve in the set* $\bigcup_{\mathrm{p}\in\Lambda}(\hat\gamma_\mathrm{p}^+ \cup \hat\gamma_\mathrm{p}^-)$.
- *The unit tangent vector to* υ^α_τ *has distance at most* $c_\upsilon \mathrm{r}^{-1/2}$ *from* v, *and it has distance at most* $\kappa \mathrm{r}^{-1/2}$ *from* v *at the points where the distance is at least* $c_\upsilon \mathrm{r}^{-1/2}$ *from each curve in* $\bigcup_{\mathrm{p}\in\Lambda}(\hat\gamma_\mathrm{p}^+ \cup \hat\gamma_\mathrm{p}^-)$.
- *The pushforward via this isotopy of* $\frac{\partial}{\partial\tau}$ *is bounded by* κ_{c_υ}.

This lemma is proved in Section Cb.

Part 3 Granted Lemma C.1, fix $\tau \in [1,2]$ so as to define $(A_{\diamond 1\tau}, \psi_{\diamond 1\tau})$. The definition of this pair is identical to that of $(A_\diamond, \psi_\diamond)$ given in Section Be but for one change and one added remark. What follows directly is the one change to Section Be's definition. Let υ denote a given component of the zero locus of $\alpha_\diamond$ that intersects M_δ. By way of a reminder, υ's intersection with $Y_{*\Lambda} \cup T_{*\Lambda}$ is a union of components of α's zero locus in $Y_{*\Lambda} \cup T_{*\Lambda}$, and υ's intersection with any given component of $Y - (Y_{*\Lambda} \cup T_{*\Lambda})$ is described by Lemma B.11. This understood, replace υ in the formula that appear in Section Be with the corresponding curve υ^α_τ that is supplied by Lemma C.1.

The added remark addresses the need to specify an isomorphism between E and the product bundle over a certain neighborhood of each curve υ^α_τ and over the complement of the union of a certain smaller neighborhood about $\bigcup_{\upsilon^\alpha \in \Theta^\alpha} \upsilon^\alpha_\tau$ and a neighborhood of the components of the zero locus of $\alpha_\diamond$ from $\bigcup_{\mathrm{p}\in\Lambda}(\hat\gamma_\mathrm{p}^+ \cup \hat\gamma_\mathrm{p}^-)$. The required isomorphisms are already specified for the $\tau = 0$ case, these being the ones needed to define (A_*, ψ_*) and $(A_\diamond, \psi_\diamond)$. The three steps that follow describe the $\tau > 0$ versions of these isomorphisms.

Step 1 Let υ^α denote a given component of Θ^α. The definition of $(A_\diamond, \psi_\diamond)$ referred to larger and smaller neighborhoods of the corresponding curve υ. Each $\tau \in [0,1]$ version of υ^α_τ has its analogous neighborhoods, these denoted by $U_{\upsilon,\tau}$ and $U'_{\upsilon,\tau}$.

The set $U_{\upsilon,\tau}$ is a neighborhood of υ^α_τ that is defined as follows: Its intersection with $Y_{*\Lambda} \cup T_{*\Lambda}$ is the radius $4c_v^2 \mathrm{r}^{-1/2}$ tubular neighborhood of υ^α_τ. To describe the remainder of $U_{\upsilon,\tau}$, fix a component of $Y - (Y_{*\Lambda} \cup T_{*\Lambda})$ and let γ denote for the moment the corresponding curve from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}^+_{\mathfrak{p}} \cup \hat{\gamma}^-_{\mathfrak{p}})$. Reintroduce γ's version of the coordinates (t,z) that are used on this component to define $(A_\diamond, \psi_\diamond)$. Let T denote the $|z| \le (c_v^4 + c_v^3)\mathrm{r}^{-1/2}$ part of the coordinate chart. The set $U_{\upsilon,\tau}$ intersects the $|z| \ge (c_v^4 - 2c_v^3)\mathrm{r}^{-1/2}$ part of T as the radius $4c_v^2\mathrm{r}^{-1/2}$ tubular neighborhood of this part of υ^α_τ. The intersection of $U_{\upsilon,\tau}$ with the rest of T is the radius $4c_v^{1/2}\mathrm{r}^{-1/2}$ tubular neighborhood of this part of υ^α_τ. The set $U'_{\upsilon,\tau}$ is defined analogously, but with the factor of 4 missing.

Introduce for each $\tau \in [0,1]$ the neighborhood of υ^α_τ that is defined as follows: This neighborhood intersects the complement in Y of the radius $(c_v^4 - 3c_v^3)\mathrm{r}^{-1/2}$ tubular neighborhoods of the curves from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}^+_{\mathfrak{p}} \cup \hat{\gamma}^-_{\mathfrak{p}})$ as the tubular neighborhood of υ^α_τ with radius $8c_v^2\mathrm{r}^{-1/2}$, and it intersects the remaining part of Y as the concentric tubular neighborhood of υ^α_τ with radius $8c_v^{1/2}\mathrm{r}^{-1/2}$. This neighborhood is denoted by $U_{\upsilon,\tau*}$. The set $U_{\upsilon,\tau}$ is a proper subset of $U_{\upsilon,\tau*}$.

Step 2 Fix an isomorphism between $K^{-1}|_\upsilon$ and $\upsilon \times \mathbb{C}$ that gives a version of the coordinates from Part 4 of Section Aa for υ with $|\nu| + |\mu| \le c_0$. The pushforward of $\frac{d}{d\tau}$ by the map that defines Lemma C.1's isotopy gives a vector field along the image of the isotopy. Parallel transport along the integral curves of this vector field defines an isomorphism over any given $\tau \in [0,1]$ version of υ^α_τ between K^{-1} and the product bundle. Use this isomorphism to define a υ^α_τ version of the coordinates from Part 4 of Section Aa. The associated pair (ν,μ) is such that $|\nu| + |\mu| \le c_0$, this being a consequence of the fourth bullet in Lemma C.1.

Fix $\tau \in [0,1]$. The pushforward of $\frac{d}{d\tau}\big|_\tau$ appears with respect to the υ^α_τ version of the (t,z) coordinate chart as a vector that is defined at $z = 0$. View this vector as a vector field on $U_{\upsilon,\tau*}$ whose coefficients have no z–dependence. Use the function χ to extend the latter vector field from $U_{\upsilon,\tau}$ to the rest of Y so as to be equal to 0 on the complement of $U_{\upsilon,\tau*}$ and so that its commutator with $\frac{\partial}{\partial z}$ on $U_{\upsilon,\tau*}$ is bounded by $\kappa_{c_v}\mathrm{r}^{1/2}$ with κ_{c_v} denoting the constant from Lemma C.1. The existence of an extension with this property follows from what is said by the fourth bullet of Lemma C.1. This extension is denoted in what follows by $v_{\upsilon,\tau}$. Use v^α to denote the vector field on $[0,1] \times Y$ that is defined by the rule $v_\alpha|_\tau = \frac{d}{d\tau} + \sum_{\upsilon\in\Theta_\alpha} v_{\upsilon,\tau}$.

Define $\pi_\alpha\colon [0,1] \times Y \to Y$ to be the map that sends any given point (τ, p) to the point in $\{0\} \times Y$ that lies on the integral curve through p of the vector field v^α. The map π_α is a surjection that restricts to any $\tau \in [0,1]$ version of υ^α_τ as a diffeomorphism onto υ.

Step 3 Let $\pi\colon [0,1]\times Y \to Y$ denote the standard projection. The respective pullbacks of E by π and π_α are isomorphic. Use $\varphi_\alpha\colon \pi^*E \to \pi^*_\alpha E$ to denote the isomorphism that is defined by parallel transport along the fibers of π_α by the connection $\pi^*_\alpha A_{\diamond 1}$. The pullback $\pi^*_\alpha \alpha_{\diamond 1}$ defines a section of $\pi^*_\alpha E$ and so $\varphi^{-1}_\alpha(\pi^*_\alpha \alpha_{\diamond 1})$ defines a section of $\pi^* E$. This section is denoted by $\hat\alpha_\diamond$ and its restriction, $\hat\alpha_\diamond|_\tau$, to any given constant τ slice is a section of E. The zero locus of the latter is $\bigcup_{\upsilon^\alpha\in\Theta^\alpha} \upsilon^\alpha_\tau$; it vanishes transversely with degree 1 on each component curve.

Step 4 Fix $\tau \in [0,1]$ and introduce $U_{0,\tau}$ to denote the $\{\upsilon^\alpha_\tau\}_{\upsilon^\alpha\in\Theta^\alpha}$ version of the set U_0. This is the complement of $\bigcup_{\upsilon^\alpha\in\Theta^\alpha} U'_{\upsilon,\tau}$ and the union of the radius $c_0^{-1} c_\upsilon^{1/2} \mathrm{r}^{-1/2}$ tubular neighborhoods of the curves from $\bigcup_{\mathrm{p}\in\Lambda}(\hat\gamma^+_{\mathrm{p}} \cup \hat\gamma^-_{\mathrm{p}})$ in the zero locus of $\alpha_\diamond$. The constructions that define $(A_{\diamond 1\tau}, \psi_{\diamond 1\tau})$ require an isomorphism over $U_{0,\tau}$ between E and the product bundle. Use the isomorphism that sends $\hat\alpha_\diamond|_\tau$ to $\bigl|\hat\alpha_\diamond|_\tau\bigr| \cdot 1$.

The constructions that define $(A_{\diamond 1\tau}, \psi_{\diamond 1\tau})$ also require an isomorphism between the bundle E and the product bundle over each set from the collection $\{U_{\upsilon,\tau}\}_{\upsilon^\alpha\in\Theta^\alpha}$. This isomorphism is defined using the υ^α_τ version of the coordinates (t,z). The desired isomorphism sends the section $\hat\alpha_\diamond|_\tau$ to $\bigl|\hat\alpha_\diamond|_\tau\bigr| z/|z|$.

Part 4 The next proposition compares $\mathfrak{f}_s$ at $(A_\diamond, \psi_\diamond)$ with $\mathfrak{f}_s$ at $(A_{\diamond 1\tau=1}, \psi_{\diamond 1\tau=1})$.

Proposition C.2 *There exists $\kappa \geq 100$ and, given $c_\upsilon \geq \kappa$, there exists $\kappa_{c_\upsilon} > \kappa$ with the following significance: Suppose that $\mathrm{r} \geq \kappa_{c_\upsilon} c_\upsilon^{10}$ and suppose that $(A, \psi = (\alpha,\beta))$ is a solution to the (r,μ) version of* (1-13) *with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Assume that the parameters κ, c_υ and r are suitable for use in Lemma B.17 and in particular for constructing the path $\{(A_{\diamond 1\tau}, \psi_{\diamond 1\tau})\}_{\tau\in[0,1]}$. Then the norm of the difference between the values of $\mathfrak{f}_s$ at $(A_\diamond, \psi_\diamond)$ and $\mathfrak{f}_s$ at $(A_{\diamond 1\tau=1}, \psi_{\diamond 1\tau=1})$ is no greater than κ_{c_υ}.*

Proof The proof is much like that of Proposition B.13. In any event, there are four steps.

Step 1 Use the same arguments that prove Proposition B.13 to prove that Proposition B.13's assertion also holds for each $\tau \in [0,1]$ version of $(A_{\diamond\tau}, \psi_{\diamond\tau})$.

Step 2 For any given $\tau \in [0,1]$, use $\mathfrak{L}_{\mathbb{V}\tau}$ for the moment to denote the $(A_{\diamond 1\tau}, \psi_{\diamond 1\tau})$ version of the operator $\mathfrak{L}_{\mathbb{V}}$ that is depicted in (A-26) and (A-27). This family is not real analytic, but there are as small as desired perturbations that make it so, and, this being the case, what is said in Part 1 of the proof of Proposition B.3 can be assumed to apply. Let $\{\lambda_{n\tau}\}_{n\in\mathbb{Z},\tau\in[0,1]}$ denote the corresponding set of eigenvalue families. The analog of (B-52) in this case reads

$$\text{(C-1)} \qquad \frac{d}{d\tau}\lambda_{n\tau} = \sum_{\upsilon^\alpha\in\Theta^\alpha}\int_{U_{\upsilon,\tau}} \mathfrak{f}^\dagger_{(\tau)}\Big(\frac{d}{d\tau}\mathfrak{L}_{\mathbb{V}\tau}\Big)\mathfrak{f}_{(\tau)},$$

this because the τ–derivative of $(A_{\alpha 1\tau}, \psi_{\alpha 1\tau})$ has support only in $\bigcup_{\upsilon^\alpha\in\Theta^\alpha} U_{\upsilon,\tau}$.

Step 3 It follows from what is said in the fourth bullet of Lemma B.11, in Lemma C.1 and in Part 3 that the endomorphism $\frac{d}{d\tau}\mathfrak{L}_{\mathbb{V}\tau}$ of $\mathbb{V}_0\oplus\mathbb{V}_1$ has pointwise norm bounded by $\kappa_{c1}\mathrm{r}^{1/2}$, where κ_{c1} is a constant that is purely c_v–dependent. With this in mind, fix an integer $m \geq 1$ and let $I \subset [0,1]$ denote an interval of length at most m^{-1}. The formula in (C-1) implies that $\lambda_{n(\cdot)}$ has a zero on I only if $|\lambda_{n\tau}| \leq m^{-1}\kappa_{c1}\mathrm{r}^{1/2}$ for each $\tau \in I$.

This understood, it follows from the lemmas in Section Aa that if $c_v \geq c_0$ and r is greater than a purely c_v–dependent constant, then there is a second purely c_v–dependent constant $\kappa_{c2} > \kappa_{c1}$ with the following significance: Take $m > \kappa_{c2}$ and suppose that $n \in \mathbb{Z}$ is such that $\lambda_{n(\cdot)}$ has a zero in I. Fix $\tau \in I$ and use $\mathfrak{f}_{(\tau)}$ to denote an eigenvector of $\mathfrak{L}_{\mathbb{V}\tau}$ with eigenvalue $\lambda_{n\tau}$. Then $\|\Pi_\vartheta \mathfrak{f}_{(\tau)}\| \geq (1 - c_0 c_v^{-1})\|\mathfrak{f}_{(\tau)}\|_2$.

Granted the preceding, use Lemmas A.2, A.3, A.7 and A.8 with (A-28)–(A-30) to deduce the following: Suppose that $c_v \geq c_0$, r is greater than a purely c_v–dependent constant, and that m is greater than yet another purely c_v–dependent constant. Suppose that $n \in \mathbb{Z}$ and $\lambda_{n(\cdot)}$ has a zero on I. Then $|\lambda_{n\tau}| \leq c_v^{-4}$ for all $\tau \in I$.

Step 4 Take c_v, r and m so as to use what is said in Steps 1–3. Fix $\tau' > \tau \in [0,1]$ with $\tau' - \tau < m^{-1}$. Since m need only be greater than a purely c_v–dependent constant, assume that it is no greater than this constant plus 1. The argument used in Part 4 of the proof of Proposition B.3 can be repeated with only notational changes to see that the norm of the difference between the values of $\mathfrak{f}_s$ at $(A_{\diamond 1\tau}, \psi_{\diamond 1\tau})$ and at $(A_{\diamond 1\tau'}, \psi_{\diamond 1\tau'})$ is at most κ_c with κ_c being a purely c_v–dependent constant. This conclusion implies what is asserted by Proposition C.2 as $[0,1]$ can be covered by $2m$ intervals of length less than $m-1$.

Cb The proof of Lemma C.1

The proof has fourteen steps.

Step 1 Fix $\mathfrak{p} \in \Lambda$ such that υ crosses $\mathcal{H}_{\mathfrak{p}}$. The curve υ crosses the $u = R + \ln\delta$ sphere in $\mathcal{H}_{\mathfrak{p}}$ and quickly intersects the $f = 1 + \delta^2$ surface in $\mathcal{H}_{\mathfrak{p}}$ as it continues out of $\mathcal{H}_{\mathfrak{p}}$ to cross M_δ. Let $z_{\mathfrak{p}+}$ denote this intersection point. By way of a reminder, the function f where $u \geq R + \ln\delta$ in $\mathcal{H}_{\mathfrak{p}}$ is given by $f = 1 + e^{-2(R-u)}(1 - 3\cos^2\theta)$. The point $z_{\mathfrak{p}+}$ is the starting point of a component of the $f \in (1 + \delta^2, 2 - \delta^2)$ part of $\upsilon \cap M_\delta$. The ending point of this component lies on the $f = 2 - \delta^2$ surface in one of the handles $\{\mathcal{H}_{\mathfrak{p}'}\}_{\mathfrak{p}' \in \Lambda}$. Let $\mathfrak{p}' \in \Lambda$ denote the relevant pair and let $z_{\mathfrak{p}'-}$ denote this ending point on the $f = 2 - \delta^2$ surface in $\mathcal{H}_{\mathfrak{p}'}$.

By way of a reminder from Part 2 in Section II.1C, the index 1 critical point from $\mathfrak{p}$ has an ascending disk in M_δ that intersects the Heegaard surface σ as a smoothly embedded circle, this denoted by C_{p+}; and the index 2 critical point from $\mathfrak{p}'$ has a descending disk in M_δ that intersects the Heegaard surface Σ as a smoothly embedded circle, this denoted by $\mathrm{C}_{p'-}$. The segment of υ that starts at $z_{\mathfrak{p}+}$ and ends at $z_{\mathfrak{p}'-}$ intersects Σ at a point with distance c_0^{-1} or less from a point in $\mathrm{C}_{p+} \cap \mathrm{C}_{p-}$. Use z_υ for this point in $\upsilon \cap \Sigma$ and use z_* for the nearby point in $\mathrm{C}_{p+} \cap \mathrm{C}_{p'-}$. The point z_υ is well inside a certain coordinate neighborhood of z_*. This neighborhood has coordinates $(\varphi, \hbar)$ which are defined where $|\varphi|^2 + |\hbar|^2$ is bounded by a constant that depends only on the geometry of M. The pair $(\varphi, \hbar)$ is such that $z = \varphi + i\hbar$ is a holomorphic coordinate for the neighborhood.

Lie transport by v of the functions $(\varphi, \hbar)$ along v's integral curves defines coordinates $(t, \varphi, \hbar)$ for a closed cylinder in M_δ with t being the value of f along the integral curves of v. The coordinate t is restricted to the interval $[1 + \delta^2, 2 - \delta^2]$. The corresponding $t = 1 + \delta^2$ boundary disk of the cylinder is a disk in the $u > R + \ln\delta$ part of $\mathcal{H}_{\mathfrak{p}}$. The function φ on this boundary disk is such that $d\varphi = d\phi$. The function $\hbar$ on this disk is the function $e^{-2(R-u)}\cos\theta\sin^2\theta$. The $t = 2 - \delta^2$ boundary disk of this coordinate cylinder is a disk in the $u \leq -R - \ln\delta$ part of $\mathcal{H}_{\mathfrak{p}'}$. The function φ on this boundary disk is either $e^{-2(R+u)}\cos\theta\sin^2\theta$ or it is $-e^{-2(R+u)}\cos\theta\sin^2\theta$. In the former case, $d\hbar = d\phi$ on this boundary disk; and $d\hbar = -d\phi$ in the latter case.

The segment of υ between $z_{\mathfrak{p}+}$ and $z_{\mathfrak{p}-}$ is in this coordinate cylinder and as such, it appears as the graph of the form $t \mapsto (t, z = z_\upsilon(t))$. The function $z_\upsilon(\cdot)$ solves the $\tau = 0$ version of the $\tau \in [0, 1]$ family of differential equations depicted in the upcoming (C-2). The depiction of this family introduces a certain $\mathbb{C}$–valued function, x_υ, on $[1+\delta^2, 2-\delta^2]$ with norm bounded by $c_0 \mathrm{r}^{-1/2}$. A given $\tau \in [0,1]$ member of the family requires a $\mathbb{C}$–valued function of t to obey

$$\frac{i}{2}\frac{d}{dt}z + (1-\tau)x_\upsilon = 0 \tag{C-2}$$

for $t \in [1+\delta^2, 2-\delta^2]$. Given $z_0 \in \mathbb{C}$ with norm bounded by c_0^{-1}, integration finds a unique solution to (C-2) with $z(1+\delta^2) = z_0$. There is also a unique solution with $z(2-\delta^2) = z_0$. In either case, the solution obeys $|z(\cdot) - z_0| \le (1-\tau)c_0 \mathrm{r}^{-1/2}$.

Step 2 Fix $\varepsilon_1 \in (0, c_0^{-1})$. This section uses c_ε to denote a purely ε_1–dependent constant that is greater than 1 and whose value can be assumed to increase on successive appearances.

Fix $\mathrm{r} \ge c_\varepsilon$. Suppose that $\mathfrak{p} \in \Lambda$ is such that υ crosses $\mathcal{H}_\mathfrak{p}$. Assume in addition that each point of υ has distance ε_1 or greater from both $\hat{\gamma}_\mathfrak{p}^+$ and $\hat{\gamma}_\mathfrak{p}^-$. This being the case, υ coincides with a segment in $\mathcal{H}_\mathfrak{p}$ of α's zero locus and so its tangent vector here has distance at most $c_0 \mathrm{r}^{-1/2}$ from v. Reintroduce $z_{\mathfrak{p}-}$ to denote the point on $\upsilon \cap \mathcal{H}_\mathfrak{p}$ where υ intersects the $e^{-2(R+u)}(1-3\cos^2\theta) = \delta^2$ surface in $\mathcal{H}_\mathfrak{p}$ and introduce again $z_{\mathfrak{p}+}$ to denote the point where υ intersects the $e^{-2(R-u)}(1-3\cos^2\theta) = \delta^2$ surface in $\mathcal{H}_\mathfrak{p}$.

Let γ denote the segment in $\mathcal{H}_\mathfrak{p}$ of the integral curve of v that starts at $z_{\mathfrak{p}-}$ and lies in the $e^{-2(R-|u|)}(1-3\cos^2\theta) \le \delta^2$ part of $\mathcal{H}_\mathfrak{p}$. Given that υ's tangent vector differs from v by at most $c_0 \mathrm{r}^{-1/2}$, the $e^{-2(R-|u|)}(1-3\cos^2\theta) \le \delta^2$ part of υ in $\mathcal{H}_\mathfrak{p}$ lies entirely in the radius $c_\varepsilon \mathrm{r}^{-1/2}$ tubular neighborhood of γ. The function $1-3\cos^2\theta$ is positive on γ if $\mathrm{r} \ge c_\varepsilon^{-1}$ and the segment γ ends on the $e^{-2(R-u)}(1-3\cos^2\theta) = \delta^2$ surface in $\mathcal{H}_\mathfrak{p}$. In fact, the radius c_ε^{-1} tubular neighborhood of γ lies entirely in the $1-3\cos^2\theta > 0$ part of $\mathcal{H}_\mathfrak{p}$ and its boundary consists of one disk on the $e^{-2(R+u)}(1-3\cos^2\theta) = \delta^2$ surface and the other on the $e^{-2(R-u)}(1-3\cos^2\theta) = \delta^2$ surface. This neighborhood has coordinates (t, z) as described in Part 4 of Section Aa with $|\nu| + |\mu| \le c_0$ and with the $t = 0$ point being the $e^{-2(R+u)}(1-3\cos^2\theta)$ point on γ.

The segment of υ in $\mathcal{H}_\mathfrak{p}$ between $z_{\mathfrak{p}-}$ and $z_{\mathfrak{p}+}$ appears in these coordinates as the graph $t \mapsto (t, z = z_\upsilon(t))$ where $z_\upsilon(0) = 0$. The function $z_\upsilon(\cdot)$ is a solution to the $\tau = 0$ member of a $\tau \in [0,1]$ family of differential equations for a $\mathbb{C}$–valued function of t, this being an equation of the form

$$\frac{i}{2}\frac{d}{dt}z + \nu z + \mu\bar{z} + (1-\tau)x_\upsilon + \mathfrak{e}(z) = 0, \tag{C-3}$$

where $\mathfrak{e}$ is a smooth function on the radius c_0^{-1} ball in $\mathbb{C}$ centered at the origin with the property that $|\mathfrak{e}| \le c_0|z|^2$ and $|d\mathfrak{e}| \le c_0|z|$. Meanwhile, x_υ is a smooth function of t that obeys $|x_\upsilon| \le c_0\mathrm{r}^{-1/2}$.

Fix $\tau \in [0,1]$ and $z_0 \in \mathbb{C}$ in the domain of $\mathfrak{e}$ with $|z_0| \le c_0^{-1}$. Then there is a unique solution to τ's version of (C-3) that is defined on a neighborhood of 0 with $t = 0$ value z_0. Let $z(\cdot)$ denote this solution. If $|z_0| \le c_\varepsilon^{-1}$, then this $z(\cdot)$ will be defined for all values of t and it will obey $|z(\cdot)| \le c_\varepsilon|z_0|$. The solution depends smoothly on the data (τ, z_0). The solutions to the $\tau = 1$ version of (C-3) are the segments of the integral curves of v in the $e^{-2(R-|u|)}(1-3\cos^2\theta) \ge \delta^2$ part of $\mathcal{H}_\mathfrak{p}$ that start at distances less than c_ε^{-1} from $z_{\mathfrak{p}-}$.

Step 3 The observations in Steps 1 and 2 suggest the lemma that follows.

Lemma C.3 *Given $\varepsilon > 0$, there exists $\kappa_\varepsilon > 1$ with the following significance: Fix $\mathrm{r} \ge \kappa_\varepsilon$ and suppose that $(A, \psi = (\alpha,\beta))$ is a solution to the (r,μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Let υ denote a component of $\alpha^{-1}(0)$ whose points have distance ε or more from each curve in $\bigcup_{\mathfrak{p}\in\Lambda}(\hat\gamma_\mathfrak{p}^+ \cup \hat\gamma_\mathfrak{p}^-)$. Then υ is in the radius $\kappa_\varepsilon\mathrm{r}^{-1/2}$ tubular neighborhood of a closed, integral curve of v.*

Proof The proof also uses c_ε to denote a purely ε–dependent constant that is greater than 1. The value of c_ε can be assumed to increase between successive appearances. Fix a point $p \in \upsilon \cap \Sigma$ and use what is said in Steps 1 and 2 to construct a segment of an integral curve of v that starts at p, ends at a point $p' \in \Sigma$ with distance at most $c_\varepsilon\mathrm{r}^{-1/2}$ from p and is such that υ lies in its radius $c_\varepsilon\mathrm{r}^{-1/2}$ tubular neighborhood. Let γ_p denote this segment. With this in mind, the arguments used in Step 4 of the proof of Proposition II.2.7 can be used with only cosmetic modifications to prove that γ_p has distance at most $c_\varepsilon\mathrm{r}^{-1/2}$ from a closed integral curve of v. □

What follows directly is a proof of Lemma C.1 in the case when υ obeys the assumptions of Lemma C.3 for a given ε. To start, let γ_0 denote now the closed integral curve of v that is supplied by Lemma C.3. If $\mathrm{r} \ge c_\varepsilon$, then the radius $4\kappa_\varepsilon$ tubular neighborhood of γ_0 will intersect each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_\mathfrak{p}$ only where $1-3\cos^2\theta > 0$. Keeping this in mind, use Part 4 of Section Aa to define coordinates (t,z) for this tubular neighborhood of γ_0 with $|\nu|$ and $|\mu|$ bounded by c_0. The curve υ appears in these coordinates as the graph of a map $t \mapsto z(t)$ with $|z| \le \kappa_\varepsilon\mathrm{r}^{-1/2}$ and with $\left|\frac{d}{dt}z\right| \le c_0\mathrm{r}^{-1/2}$. Define the family $\{\upsilon_\tau^\alpha\}_{\tau\in[0,1]}$ by writing any given member as the graph of the map $t \mapsto (1-\tau)z(\cdot)$.

Step 4 Let γ denote $\hat{\gamma}_{\mathfrak{p}}^{+}$. Use $\theta_* \in \left(0, \frac{\pi}{2}\right)$ in what follows to denote the angle with $\cos\theta_* = \frac{1}{\sqrt{3}}$. Use b to denote $\frac{3}{2\sqrt{2}} e^{R}(x_0 + 4e^{-2R})^{1/2}$. Fix $\varepsilon \in (0, c_0^{-1})$ with the upper bound chosen so that the $\mathbb{R}/(2\pi\mathbb{Z})$–valued function ϕ and the pair $(x = b^{-1}u, y = \theta - \theta_*)$ define coordinates on the radius ε tubular neighborhood of γ. Let $p = y + x$ and $q = y - x$ and fix $\varepsilon_0 < \varepsilon$ so that the locus where $(p^2 + q^2)^{1/2} = \varepsilon_0$ lies in the radius ε tubular neighborhood of γ. The notation that follows uses c_ε to denote a constant that is greater than 1 and depends solely on ε_0. Its value can be assumed to increase between successive appearances.

Fix ε_1 and r as in Step 2 with ε_1 chosen so that $p^2 + q^2 \le 18\varepsilon_0$ on the radius ε_1 tubular neighborhood of γ. Suppose that $\mathfrak{p} \in \Lambda$ is such that $\upsilon \cap \mathcal{H}_{\mathfrak{p}} \ne \varnothing$ but assume in this case that υ has boundary points on the radius ε_1 tubular neighborhood of $\hat{\gamma}_{\mathfrak{p}}^{+}$. Much the same argument holds if υ has boundary points on $\hat{\gamma}_{\mathfrak{p}}^{-}$ and so the latter case will not be discussed.

The part of $\upsilon \cap \mathcal{H}_{\mathfrak{p}}$ where $e^{-2(R-|u|)}(1-3\cos^2\theta) \le \delta^2$ but not in the $(p^2+q^2)^{1/2} < \frac{1}{2}\varepsilon_0$ part of the radius ε tubular neighborhood of γ consists of two segments, these denoted by υ_- and υ_+ in what follows. The function u is negative on υ_- and positive on υ_+. Both segments lie in the zero locus of α and have transversal intersection with the $(p^2 + q^2)^{1/2} = \frac{1}{2}\varepsilon_0$ locus. The starting point of υ_- is $z_{\mathfrak{p}-}$. Use z_- to denote the point of υ_- on the $(p^2 + q^2)^{1/2} = \varepsilon_0$ surface in γ's radius ε tubular neighborhood.

The $u < 0$ part of the segment of the integral curve of v in $\mathcal{H}_{\mathfrak{p}}$ that contains z_- and lies where $e^{-2(R+u)}(1 - 3\cos^2\theta) \le \delta^2$ will start on the $e^{-2(R+u)}(1 - 3\cos^2\theta) = \delta^2$ surface at distance $c_\varepsilon \mathrm{r}^{-1/2}$ or less from $z_{\mathfrak{p}-}$ and it will intersect the $(p^2+q^2)^{1/2} = \frac{1}{2}\varepsilon_0$ surface in the radius ε tubular neighborhood of γ. Introduce γ_- to denote the segment of the $u < 0$ part of this integral curve that runs between its $e^{-2(R+u)}(1-3\cos^2\theta) = \delta^2$ point and its intersection with the $(p^2 + q^2)^{1/2} = \frac{1}{2}\varepsilon_0$ surface in the radius ε tubular neighborhood of γ. The part of υ_- that lies outside the locus where $p^2 + q^2 < \frac{1}{2}(1 + c_\varepsilon \mathrm{r}^{-1/2})\varepsilon_0$ is in the radius $c_\varepsilon \mathrm{r}^{-1/2}$ tubular neighborhood of γ_-.

Fix coordinates (t, z) for the radius c_0^{-1} tubular neighborhood of γ_- from Part 4 of Section Aa with the $z = 0$ locus being γ_- and the $t = 0$ point being z_-. Require in addition that $|\nu|$ and $|\mu|$ are bounded by c_0. The intersection of this tubular neighborhood with the $e^{-2(R+u)}(1 - 3\cos^2\theta) = \delta^2$ surface is a disk neighborhood of a boundary point of γ_- in this surface. If the radius of this tubular neighborhood is less than c_ε^{-1}, then its intersection with the surfaces in the radius ε tubular neighborhood of γ where $(p^2 + q^2)^{1/2}$ is constant and between $\frac{3}{4}\varepsilon_0$ and $2\varepsilon_0$ are disks that lie in the $u < 0$ part of these surfaces.

The segment υ_- appears in the coordinates (t,z) as a graph $t \mapsto (t, z(t))$ with $z(t)$ obeying the $\tau = 0$ version of a $\tau \in [0,1]$ family of equations that has the same form as that depicted in (C-3). This solution has $z(0) = 0$ and $|z(\cdot)| \le c_\varepsilon \mathrm{r}^{-1/2}$. Note that the solutions to the γ_- and $\tau = 1$ version of (C-3) are integral curves of v.

Solutions to (C-3) for all values of τ can readily be found. In particular, there exists a purely ε_0–dependent constant, $c_{\mathfrak{p}\varepsilon}$, that is greater than 1 and has the following significance: Fix $\tau \in [0,1]$ and a point z_0 in the $p^2 + q^2 = \varepsilon_0$ surface with distance less than $c_{\mathfrak{p}\varepsilon}^{-1}$ from z_-. Use Δ to denote this distance. There is a unique solution to the γ_- version of (C-3) for the chosen value of τ that contains z_0 and with norm bounded for all t by $c_\varepsilon(\Delta + \mathrm{r}^{-1/2})$. Moreover, varying the data (τ, z_0) changes the corresponding solution in a smooth fashion; and the three-parameter family of solutions so defined is such that the derivative of $z(\cdot)$ with respect to changes of (τ, z_0) is bounded at each t by c_ε.

Step 5 This step uses the same notation as Step 4. Use z_+ to denote the point of υ_+ on the locus $(p^2+q^2)^{1/2} = \varepsilon_0$. The endpoint of υ_+ is on the $e^{-2(R-u)}(1-3\cos^2\theta) = \delta^2$ surface, this being the point $z_{\mathfrak{p}+}$. If $\mathrm{r} \ge c_\varepsilon$ then the segment of the integral curve of v in $\mathcal{H}_\mathfrak{p}$ that contains z_+ will intersect the surface $(p^2+q^2)^{1/2} = \frac{1}{2}\varepsilon_0$ where $u > 0$ at a point with distance at most $c_\varepsilon \mathrm{r}^{-1/2}$ from the point where υ_+ intersects this surface. It will also intersect the surface where $e^{-2(R-u)}(1-3\cos^2\theta) = \delta^2$. Introduce γ_+ to denote the segment of this integral curve of v that starts on the $(p^2+q^2)^{1/2} = \frac{1}{2}\varepsilon_0$ surface in the radius ε tubular neighborhood of γ and ends on the $e^{-2(R-u)}(1-3\cos^2\theta) = \delta^2$ surface. The radius c_ε^{-1} tubular neighborhood of γ_+ will intersect this surface in a disk, and it will intersect each surface in the radius ε tubular neighborhood of γ where $(p^2+q^2)^{1/2}$ is constant and between $\frac{3}{4}\varepsilon_0$ and $2\varepsilon_0$ as a disk in the $u > 0$ part of the surface in question.

Fix coordinates (t,z) for the radius c_ε^{-1} tubular neighborhood of γ from Part 4 of Section Aa with $|\nu|$ and $|\mu|$ bounded by c_0 and with the $t = 0$ point being the point z_+. The segment υ_+ appears in the coordinates (t,z) as a graph $t \mapsto (t, z(t))$ with $z(t)$ obeying the $\tau = 0$ version of a $\tau \in [0,1]$ family of equations that has the same form as that depicted in (C-3). This solution has $z(0) = 0$ and $|z(\cdot)| \le c\varepsilon \mathrm{r}^{-1/2}$. The solutions to the γ_+ and $\tau = 1$ version of (C-3) are integral curves of v.

The constant $c_{\mathfrak{p}\varepsilon}$ from Step 4 can be chosen so that there is a γ_+ analog of what is said in the final paragraph of Step 4. This is to say that the following is true: Choose any $\tau \in [0,1]$ and a point z_0 in the $p^2 + q^2 = \varepsilon_0$ surface with distance less than $c_{\mathfrak{p}\varepsilon}^{-1}$

from z_+. Use Δ to denote this distance. There is a unique solution to the γ_+ version of (C-3) for the chosen value of τ that contains z_0 and has norm bounded for all t by $c_\varepsilon(\Delta + \mathrm{r}^{-1/2})$. Varying the data (τ, z_0) changes the corresponding solution in a smooth fashion and the three-parameter family of solutions so defined is such that the derivative of $z(\cdot)$ with respect to changes of (τ, z_0) is bounded at each t by c_ε.

Step 6 Fix $\varepsilon \in (0, c_0^{-1})$ and suppose that υ intersects the radius ε tubular neighborhood of either $\hat{\gamma}_{\mathfrak{p}}^+$ or $\hat{\gamma}_{\mathfrak{p}}^-$. What follows considers the case when the curve in question is $\hat{\gamma}_{\mathfrak{p}}^+$. As the same analysis holds for the other case modulo some sign changes, the latter case is not discussed. Use γ now to denote $\hat{\gamma}_{\mathfrak{p}}^+$ and let $\theta_* \in \left(0, \frac{\pi}{2}\right)$ denote the angle with $\cos\theta_* = \frac{1}{\sqrt{3}}$, this the value of θ on γ. Coordinates for a neighborhood of γ are given by the $\mathbb{R}/(2\pi\mathbb{Z})$ function ϕ and $\mathbb{R}$–valued functions (x, y) that are defined by the rules $(u = bx, \theta = \theta_* + y)$, where $b = \frac{3}{2\sqrt{2}} e^R (x_0 + 4e^{-2R})^{1/2}$ is the constant that appears in (B-36) and in Steps 4 and 5. Introduce as in these same steps functions p and q given by $p = y + x$ and $q = y - x$.

If υ's intersection with the radius ε tubular neighborhood of γ lies entirely in α's zero locus, then the part of the curve υ where $(p^2+q^2)^{1/2} \le c_0^{-1}\varepsilon$ can be parametrized by an interval $I \subset \mathbb{R}$ as a map $t \mapsto (\phi = -t, p = p_\upsilon(t), q = q_\upsilon(t))$. No generality is lost in this case by taking I to contain the origin $0 \in \mathbb{R}$ and to take $t = 0$ to be the $u = 0$ point on υ. Thus $p_\upsilon(0) = q_\upsilon(0)$. If $(p^2+q^2)^{1/2} \le m^{-2} c_\upsilon \mathrm{r}^{-1/2}$ on $\upsilon \cap \mathcal{H}_{\mathfrak{p}}$, then it follows from what is said in Step 3 of the proof of Lemma B.11 that the part of the curve υ where $(p^2+q^2)^{1/2} \le c_0^{-1}\varepsilon$ can also be parametrized by an interval $I \subset \mathbb{R}$ containing 0; this parametrization has again the form $t \mapsto (\phi = -t, p = p_\upsilon(t), q = q_\upsilon(t))$. In this case the 0 point in I is taken as in Step 2 of the proof of Lemma B.11. It follows from (B-42) that $|p_\upsilon(0) - q_\upsilon(0)| \le c_0 m^{-4} c_\upsilon \mathrm{r}^{-1/2}$.

In all cases, the functions p_υ and q_υ obey an equation of the form that is depicted in (B-37). This equation is reproduced below:

$$\text{(C-4)} \quad \frac{d}{dt} p_\upsilon = \lambda p_\upsilon + \mathfrak{e}_p(p_\upsilon, q_\upsilon) + \mathfrak{r}_{p\upsilon} \quad \text{and} \quad \frac{d}{dt} q_\upsilon = -\lambda q_\upsilon + \mathfrak{e}_q(p_\upsilon, q_\upsilon) + \mathfrak{r}_{q\upsilon}.$$

By way of a reminder, $\lambda = 4\sqrt{6} e^{-R}(x_0 + 4e^{-2R})^{1/2}$ and the functions $\mathfrak{e}_p$ and $\mathfrak{e}_q$ are smooth and have absolute value bounded by $c_0(p^2+q^2)$. Meanwhile, $\mathfrak{r}_{p\upsilon}$ and $\mathfrak{r}_{q\upsilon}$ are smooth functions of t. Their absolute values are bounded at times $t \in I$ where $\upsilon(t) \subset \alpha^{-1}(0)$ by $c_0 \mathrm{r}^{-1/2}$; in particular, this occurs where $(p^2+q^2)^{1/2} \ge c_\upsilon \mathrm{r}^{-1/2}$. In general, their absolute values are bounded by $c_0 c_\upsilon \mathrm{r}^{-1/2}$. A smaller upper bound is given in the upcoming (C-5).

What follows says more about $\mathfrak{r}_{p\upsilon}$ and $\mathfrak{r}_{q\upsilon}$ at times $t \in I$ where their absolute value is greater than $c_0 \mathrm{r}^{-1/2}$. To this end, reintroduce the constant m that is used to define υ, and reintroduce $t_\diamond \in [0, 2\pi)$ from Step 2 of Lemma B.11. As done in this same Step 2, take the parametrization for I so that it is only necessary to consider times $t \in [-2\pi - t_\diamond, 2\pi + t_\diamond]$. The pair p_υ and q_υ on this interval are given in (B-42). Differentiate (B-42) and compare with (C-4) to see that $\mathfrak{r}_{p\upsilon}$ and $\mathfrak{r}_{q\upsilon}$ obey

(C-5)
- $\mathfrak{r}_{p\upsilon} \le c_0 m^{-6} c_\upsilon \mathrm{r}^{-1/2}$ for $t \in [0, 2\pi + t_\diamond]$ and $\mathfrak{r}_{p\upsilon} \le c_0 m^{-2} c_\upsilon \mathrm{r}^{-1/2}$ for $t \in [-2\pi - t_\diamond, 0]$;
- $\mathfrak{r}_{q\upsilon} \ge -c_0 m^{-6} c_\upsilon \mathrm{r}^{-1/2}$ for $t \in [-2\pi - t_\diamond, 0]$ and $\mathfrak{r}_{q\upsilon} \ge -c_0 m^{-2} c_\upsilon \mathrm{r}^{-1/2}$ for $t \in [0, 2\pi + t_\diamond]$;
- $\mathfrak{r}_{p\upsilon} \ge -c_0 m^{-6} c_\upsilon \mathrm{r}^{-1/2}$ and $\mathfrak{r}_{q\upsilon} \le c_0 m^{-6} c_\upsilon \mathrm{r}^{-1/2}$ for $t \in [-2\pi - t_\diamond, 2\pi + t_\diamond]$.

The constant m is left unspecified for now but ultimately chosen to be less than c_0. The choice of m determines in part a lower bound for Lemma C.1's constant κ.

To say something about the respective lengths of the $t > 0$ and $t < 0$ parts of I, introduce Δ to denote the value of $(p_\upsilon^2 + q_\upsilon^2)^{1/2}$ at $0 \in I$. Fix $\varepsilon_0 \in (0, \varepsilon)$ so that the coordinate functions p and q are defined where $(p^2 + q^2)^{1/2} < 2\varepsilon_0$. Let t_+ and t_- denote the respective values of t in I where $(p_\upsilon^2 + q_\upsilon^2)^{1/2} = \varepsilon_0$. As explained in Step 6,

$$|t_+ - \lambda^{-1} \ln(\varepsilon_0 \Delta^{-1})| \le c_0 \quad \text{and} \quad |t_- + \lambda^{-1} \ln(\varepsilon_0 \Delta^{-1})| \le c_0 \tag{C-6}$$

if $\varepsilon_0 \le c_0^{-1}$. These bounds imply in part that the lengths of the $t > 0$ and $t < 0$ parts of I differ by at most c_0.

Step 7 Introduce Y_0 and X_0 to denote the value of $y = \frac{1}{2}(p+q)$ and $x = \frac{1}{2}(p-q)$ at the $t = 0$ point on υ. Note that $\mathrm{X}_0 = 0$ if $p^2 + q^2 \ge m^{-2} c_\upsilon \mathrm{r}^{-1/2}$ on υ; and (B-39) and (B-41) imply that $|\mathrm{X}_0| \le c_0 m^{-6} c_\upsilon \mathrm{r}^{-1/2}$ otherwise. Meanwhile, $\mathrm{Y}_0 = 2^{-1/2}\Delta$ if $p^2 + q^2 \ge m^{-2} c_\upsilon \mathrm{r}^{-1/2}$ on υ and $|\mathrm{Y}_0 - 2^{-1/2} m^{-2} c_\upsilon \mathrm{r}^{-1/2}| \le c_0 m^{-4} c_\upsilon \mathrm{r}^{-1/2}$ otherwise.

Fix $\mathrm{Y} \in \left(\frac{1}{4}\mathrm{Y}_0, 4\mathrm{Y}_0\right)$. Given $\tau \in [0, 1]$, there is a unique map $t \mapsto (p_{\mathrm{Y},\tau}(t), q_{\mathrm{Y},\tau}(t))$ from a maximal interval $I_{\mathrm{Y},\tau} \subset \mathbb{R}$ to $\mathbb{R}^2$ that obeys the equation

$$\begin{aligned} \frac{d}{dt} p_{\mathrm{Y},\tau} &= \lambda p_{\mathrm{Y},\tau} + \mathfrak{e}_p(p_{\mathrm{Y},\tau}, q_{\mathrm{Y},\tau}) + (1-\tau)\mathfrak{r}_{p\upsilon}, \\ \frac{d}{dt} q_{\mathrm{Y},\tau} &= -\lambda q_{\mathrm{Y},\tau} + \mathfrak{e}_q(p_{\mathrm{Y},\tau}, q_{\mathrm{Y},\tau}) + (1-\tau)\mathfrak{r}_{q\upsilon} \end{aligned} \tag{C-7}$$

with $p_{\mathrm{Y},\tau}(0) = \mathrm{Y} + \mathrm{X}_0$ and $q_{\mathrm{Y},\tau}(0) = \mathrm{Y} - \mathrm{X}_0$ and with $(p_{\mathrm{Y},\tau}^2 + q_{\mathrm{Y},\tau}^2)^{1/2} \le \varepsilon_0$ for $t \in I_{\mathrm{Y},\tau}$ with equality only at each boundary point of $I_{\mathrm{Y},\tau}$. A proof of existence and uniqueness can be had using standard techniques from the theory of ordinary differential equations. Use $t_{\mathrm{Y},\tau+}$ and $t_{\mathrm{Y},\tau-}$ to denote the respective negative and positive endpoints of $I_{\mathrm{Y},\tau}$.

The first implication of (C-7) concerns the size of $q_{\mathrm{Y},\tau}$ relative to $p_{\mathrm{Y},\tau}$ where $t \geq 0$: As explained in a moment,

$$|q_{\mathrm{Y},\tau}| \leq c_0(\varepsilon_0 p_{\mathrm{Y},\tau} + m^{-2} c_v \mathrm{r}^{-1/2}) \tag{C-8}$$

for $t \in [0, t_{\mathrm{Y},\tau+}]$ when $\varepsilon_0 \leq c_0^{-1}$ and $m > c_0$. To prove this, fix $\varsigma > 0$ and set $w = |q_{\mathrm{Y},\tau}| - \varsigma |p_{\mathrm{Y},\tau}|$. It follows from (C-5) and (C-7) that

$$\frac{d}{dt} w \leq -\lambda w - 2\lambda\varsigma |p_{\mathrm{Y},\tau}| + c_0\varepsilon_0(|w| + |p_{\mathrm{Y},\tau}|) + c_0 m^{-2} c_v \mathrm{r}^{-1/2}. \tag{C-9}$$

Let c_* denote the version of c_0 that appears in this inequality. Take $\varsigma = 2c_*\lambda^{-1}\varepsilon_0$ to see that $\frac{d}{dt} w \leq -\lambda(w - \lambda^{-1} c_*\varepsilon_0) + c_0 m^{-2} c_v \mathrm{r}^{-1/2}$. (The negative multiple of $|p_{\mathrm{Y},\tau}|$ in (C-9) dominates the positive multiple for this choice of ς.) Moreover, supposing that ε_0 is chosen to be less than $\frac{1}{2}\lambda c_*^{-1}$ (which is greater than c_0^{-1}), this says that $\frac{d}{dt} w \leq -\frac{1}{2}\lambda w + c_0 m^{-2} c_v \mathrm{r}^{-1/2}$. Multiply both sides of this last inequality by e^{t/c_0} and integrate to obtain (C-8) (keep in mind that λ is a fixed positive number so it has a c_0 upper bound and a c_0^{-1} lower bound.)

With regards to $p_{\mathrm{Y},\tau}$, note first that (C-7) with (C-5) imply that $p_{\mathrm{Y},\tau}$ is an increasing function of t when t is positive. To say more about the size of $p_{\mathrm{Y},\tau}$ it proves useful to introduce the norm on $C^\infty([0,t];\mathbb{R})$ for $t \leq t_{\mathrm{Y},\tau+}$ given by $h \mapsto \|h\|_t = \sup_{s\in[0,t]} e^{-\lambda s}|h(s)|$. Use (C-8) with (C-5) and the right-hand equation in (C-7) to see that

$$|p_{\mathrm{Y},\tau} - e^{\lambda t}(\mathrm{Y} + \mathrm{X}_0)| \leq c_0(e^{\lambda t}\|p_{\mathrm{Y},\tau}\|_t^2 + m^{-6} c_v \mathrm{r}^{-1/2}). \tag{C-10}$$

Given that $\mathrm{Y} + \mathrm{X}_0 \geq c\varepsilon^{-1} m – 2 c_v \mathrm{r}^{-1/2}$, this last equation implies that

$$\begin{aligned}(1 - c_0\varepsilon_0)e^{\lambda t}(\mathrm{Y} + \mathrm{X}_0 - c_0 m^{-6} c_v \mathrm{r}^{-1/2}) \\ \leq p_{\mathrm{Y},\tau}(t) \\ \leq (1 + c_0\varepsilon)e^{\lambda t}(\mathrm{Y} + \mathrm{X}_0 + c_0 m^{-6} c_v \mathrm{r}^{-1/2})\end{aligned} \tag{C-11}$$

for $t \in [0, t_{Y,\tau+}]$ when $\varepsilon_0 \leq c_0^{-1}$ and $m \geq c_0$. Note that (C-11) with (C-8) implies that the function $1 - 3\cos^2\theta$ is positive along the trajectory $t \mapsto (p_{\mathrm{Y},\tau}(t), q_{\mathrm{Y},\tau}(t))$ where $(p_{\mathrm{Y},\tau}^2 + q_{\mathrm{Y},\tau}^2)^{1/2}$ is greater than $c_0 m^{-2} c_v \mathrm{r}^{-1/2}$.

The same sort of arguments can be used for $t \in [-2\pi - t_{D,\tau-}, 0]$ to see that

(C-12) • $|p_{\mathrm{Y},\tau}| \leq c_0(\varepsilon_0|q_{\mathrm{Y},\tau}| + m^{-2} c_v \mathrm{r}^{-1/2})$,

- $(1-c_0\varepsilon_0)e^{-\lambda t}(\mathrm{Y}-\mathrm{X}_0-c_0 m^{-6}c_v \mathrm{r}^{-1/2})$
$$\le q_{\mathrm{Y},\tau}(t) \le (1+c_0\varepsilon)e^{-\lambda t}(\mathrm{Y}-\mathrm{X}_0+c_0 m^{-6}c_v \mathrm{r}^{-1/2})$$

for $t \le [t_{Y,\tau-}, 0]$ if $\varepsilon_0 \le c_0^{-1}$ and $m \ge c_0$.

These bounds for $p_{\mathrm{Y},\tau}$ and $q_{\mathrm{Y},\tau}$ have the following implication with regards to the times $t_{\mathrm{Y},\tau+}$ and $t_{\mathrm{Y},\tau-}$. To say more, let $t_{\mathrm{Y},\tau*}$ denote either $t_{Y,\tau+}$ or $-t_{\mathrm{Y},\tau-}$. Then

$$|t_{\mathrm{Y},\tau*}-\lambda^{-1}\ln(\varepsilon_0\mathrm{Y}^{-1})| \le c_0\varepsilon_0 + c_\varepsilon \Delta m^{-6}c_v\mathrm{r}^{-1/2}. \tag{C-13}$$

Given that $\Delta m^{-6}c_v\mathrm{r}^{-1/2} \le c_0 m^{-4}$, the right-hand side of (C-12) is at most $c_0\varepsilon_0+c_\varepsilon m^{-4}$.

Step 8 Suppose next that (Y,τ) and (Y',τ') are as described in Step 7. Introduce P to denote $p_{\mathrm{Y},\tau}-p_{\mathrm{Y}',\tau'}$ and Q to denote $q_{\mathrm{Y},\tau}-q_{\mathrm{Y}',\tau'}$. Subtract their respective versions of (C-7) to derive equations for P and Q that can be written as

$$\begin{aligned} \frac{d}{dt}\mathrm{P} &= \lambda\mathrm{P} + \mathfrak{z}_{pp}\mathrm{P} + \mathfrak{z}_{pq}\mathrm{Q} + (\tau-\tau')\mathfrak{r}_{pv}, \\ \frac{d}{dt}\mathrm{Q} &= -\lambda\mathrm{Q} + \mathfrak{z}_{qp}\mathrm{P} + \mathfrak{z}_{qq}\mathrm{Q} + (\tau-\tau')\mathfrak{r}_{qv}, \end{aligned} \tag{C-14}$$

where each $\mathfrak{z}_{\bullet\bullet}$ is a function of t with norm bounded by

$$c_0(|p_{Y,\tau}|+|p_{Y',\tau'}|+|q_{Y,\tau}|+|q_{Y',\tau'}|).$$

These equations can be analyzed using the same tools used in Step 7 to draw the conclusions expressed by the following inequalities. The analysis for $t \ge 0$ finds

(C-15)
- $|\mathrm{Q}| \le c_0(\varepsilon_0|\mathrm{P}| + |\mathrm{Y}-\mathrm{Y}'| + m^{-2}c_v\mathrm{r}^{-1/2}|\tau-\tau'|)$.
- $|\mathrm{P}-e^{\lambda t}(\mathrm{Y}-\mathrm{Y}')|$
$$\le c_0 e^{\lambda t}\big((\varepsilon_0+\Delta|\ln(\varepsilon_0^{-1}\Delta)|)|\mathrm{Y}-\mathrm{Y}'| + m^{-6}c_v\mathrm{r}^{-1/2}|\tau-\tau'|\big).$$

Meanwhile, the analysis for $t \le 0$ leads to

(C-16)
- $|\mathrm{P}| \le c_0(\varepsilon_0\mathrm{Q}|_t + |\mathrm{Y}-\mathrm{Y}'| + m^{-2}c_v\mathrm{r}^{-1/2}|\tau-\tau'|)$,
- $|\mathrm{Q}-e^{-\lambda t}(\mathrm{Y}-\mathrm{Y}')|$
$$\le c_0 e^{-\lambda t}\big((\varepsilon_0+\Delta|\ln(\varepsilon_0^{-1}\Delta)|)|\mathrm{Y}-\mathrm{Y}'| + m^{-6}c_v\mathrm{r}^{-1/2}|\tau-\tau'|\big).$$

The bounds in (C-10)–(C-13) and (C-15)–(C-16) play central roles in what follows.

Step 9 The bounds in (C-10)–(C-13) and (C-15)–(C-16) can be used to say something about $t_{\mathrm{Y},\tau+}-t_{\mathrm{Y}',\tau'+}$ and $t_{\mathrm{Y},\tau-}-t_{\mathrm{Y}',\tau'-}$. To this end, suppose for the sake of argument that $t_{\mathrm{Y},\tau+} \ge t_{\mathrm{Y}',\tau'+}$. Write

$$p_{\mathrm{Y}',\tau'}^2 + q_{\mathrm{Y}',\tau'}^2 = p_{\mathrm{Y},\tau}^2 + q_{\mathrm{Y},\tau}^2 - 2(\mathrm{P}p_{\mathrm{Y},\tau}+\mathrm{Q}q_{\mathrm{Y},\tau}) + \mathrm{P}^2 + \mathrm{Q}^2, \tag{C-17}$$

and set $t = t_{\mathrm{Y},\tau}$. Use the fact that $(p_{\mathrm{Y},\tau}^2 + q_{\mathrm{Y},\tau}^2)^{1/2} = \varepsilon_0$ at $t = t_{\mathrm{Y},\tau+}$ with (C-8), (C-10) and (C-15) to see that $p_{\mathrm{Y}',\tau'}^2 + q_{\mathrm{Y}',\tau'}^2$ at $t = t_{\mathrm{Y},\tau+}$ is

$$p_{\mathrm{Y}',\tau'}^2 + q_{Y',\tau'}^2 = \varepsilon_0^2(1 - 2\mathrm{Y}^{-1}(\mathrm{Y} - \mathrm{Y}') + \mathfrak{e}), \tag{C-18}$$

where $|\mathfrak{e}| \le c_0(\varepsilon_0 \mathrm{Y}^{-1}|\mathrm{Y} - \mathrm{Y}'| + \mathrm{Y}^{-2}|\mathrm{Y} - \mathrm{Y}'|^2 + m^{-4}|\tau - \tau'|)$. This last inequality with (C-10) and (C-15) allows $t_{\mathrm{Y}',\tau'+}$ to be written as

$$t_{\mathrm{Y}',\tau'+} = t_{\mathrm{Y},\tau+} + \lambda^{-1}\mathrm{Y}^{-1}(\mathrm{Y} - \mathrm{Y}') + \mathfrak{e}, \tag{C-19}$$

where $\mathfrak{e}$ has the same absolute value bound as its namesake in (C-18). The same sort of arguments write $t_{\mathrm{Y}',\tau'-}$ as $t_{\mathrm{Y}',\tau'-} = t_{\mathrm{Y},\tau-} - \lambda^{-1}\mathrm{Y}^{-1}(\mathrm{Y} - \mathrm{Y}') + \mathfrak{e}$ with $\mathfrak{e}$ being different from its namesakes in (C-18) and (C-19) but obeying the same absolute value bound.

Step 10 The functions p and q are convenient to use on the radius ε tubular neighborhood of γ, but less so elsewhere on $\mathcal{H}_{\mathfrak{p}}$ and, in particular, less so near the boundary of $\mathcal{H}_{\mathfrak{p}}$. The function $\hbar = f(u)\cos\theta\sin^2\theta$ is far more convenient, this in part because the final arguments for Lemma C.1's proof are much the same as those used in the proof of Proposition II.2.7. At distance ε or less from γ, the function $\hbar$ can be readily written in terms of p and q, and doing so leads to the formula

$$\hbar = \frac{2}{3\sqrt{3}}(x_0 + 4e^{-2R}) + \frac{2}{\sqrt{3}}(x_0 + 4e^{-2R})pq + \mathfrak{h}, \tag{C-20}$$

where $\mathfrak{h}$ obeys $|\mathfrak{h}| \le c_0(p^2 + q^2)^{3/2}$ and $\left|\frac{\partial}{\partial p}\mathfrak{h}\right| + \left|\frac{\partial}{\partial q}\mathfrak{h}\right| \le c_0(p^2 + q^2)$.

Fix $\tau \in [0, 1]$ and fix Y as in Step 7 so as to define the interval $I_{\mathrm{Y},\tau}$ and the corresponding pair of functions $p_{\mathrm{Y},\tau}$ and $q_{\mathrm{Y},\tau}$ on $I_{\mathrm{Y},\tau}$. Of interest here is the function on $I_{\mathrm{Y},\tau}$ given by the rule $t \mapsto \hbar(p_{\mathrm{Y},\tau}(t), q_{\mathrm{Y},\tau}(t))$. This function is denoted in what follows by $\hbar_{\mathrm{Y},\tau}$. Of particular interest are the values $\hbar_{\mathrm{Y},\tau}$ at the $t = t_{\mathrm{Y},\tau+}$ and at $t = t_{\mathrm{Y},\tau-}$. In particular, (C-8) and (C-11) with (C-20) imply that its values at these times differ from $\frac{2}{3\sqrt{3}}(x_0 + 4e^{-2R})$ by at most $c_0\varepsilon_0^3$.

Consider now the functions $\hbar_{\mathrm{Y},\tau}$ and $\hbar_{\mathrm{Y}',\tau'}$ with $\tau, \tau' \in [0, 1]$ and with Y and Y' as in Step 8. Of interest is $\hbar_{\mathrm{Y},\tau}(t) - \hbar_{\mathrm{Y}',\tau'}(t')$ with t and t' being $t_{\mathrm{Y},\tau+}$ and $t_{\mathrm{Y}',\tau'+}$ or else t and t' being $t_{\mathrm{Y},\tau-}$ and $t_{\mathrm{Y}',\tau'-}$. Use t_* and t'_* to denote either of these pair of values for t. The absolute value $\hbar_{\mathrm{Y},\tau}(t_*) - \hbar_{\mathrm{Y}',\tau'}(t'_*)$ obeys the a priori bound

$$|\hbar_{\mathrm{Y},\tau}(t_*) - \hbar_{\mathrm{Y}',\tau'}(t'_*)| \le c_0\varepsilon_0^3(\mathrm{Y}^{-1}|\mathrm{Y} - \mathrm{Y}'| + m^{-4}|\tau - \tau'|); \tag{C-21}$$

this follows from (C-20) with (C-8), (C-11)–(C-13), (C-15)–(C-16) and Step 9's assertions.

Step 11 The arguments that follow in this step and Steps 12 and 13 focus almost entirely on the case when υ intersects but one $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$. The arguments in the general case are only outlined as they differ from those used for this simplest case in a straightforward fashion; and, in any event, they are much the same as those used for Proposition II.2.7. This step is a guide of sorts for Step 12.

Assume now that υ crosses only one handle from the set $\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p}}$. Let $\mathfrak{p} \in \Lambda$ denote the relevant pair. With an r–independent $\varepsilon > \varepsilon_0 > 0$ fixed in advance, it is sufficient given what is said in Step 3 to consider only the case where υ intersects the radius $\frac{1}{8}\varepsilon_0$ tubular neighborhood of either $\hat{\gamma}_{\mathfrak{p}}^{+}$ or $\hat{\gamma}_{\mathfrak{p}}^{-}$. The discussion that follows considers only the case of $\hat{\gamma}_{\mathfrak{p}}^{+}$ as the arguments for the other case are identical but for some sign changes and notation. This understood, the notation from Steps 4–9 will be used when referring to the radius ε tubular neighborhood of this curve. In particular, the curve $\hat{\gamma}_{\mathfrak{p}}^{+}$ is denoted below as γ. The constant ε_0 is chosen so that the locus in the radius ε tubular neighborhood where the coordinates p and q obey $(p^2+q^2)^{1/2} \le \varepsilon_0$ lies well inside this tubular neighborhood. The portion of υ in the radius ε tubular neighborhood of γ where $(p^2+q^2)^{1/2} \le \varepsilon_0$ is parametrized by the interval $I \subset \mathbb{R}$ as described in Step 6.

Fix $\tau \in [0,1]$. The next step constructs a 2–parameter family of continuous and piecewise smooth arcs in $M_\delta \cup \mathcal{H}_{\mathfrak{p}}$ that all start and end on the $f = \frac{3}{2}$ Heegaard surface Σ in M_δ. The starting and ending points are both very near υ's intersection with this surface. Any given member of this family starts near υ's intersection point with Σ follows υ to the $u = -R - \ln\delta$ sphere in $\mathcal{H}_{\mathfrak{p}}$. The arc stays close to υ through $\mathcal{H}_{\mathfrak{p}}$ so as to exit $\mathcal{H}_{\mathfrak{p}}$ through its $u = R + \ln\delta$ sphere in $\mathcal{H}_{\mathfrak{p}}$. It then follows υ in M_δ so as to end on the surface σ. Each arc from the family is the end-to-end concatenation of five smooth segments. The parameter space for the family of arcs is $[(1-\mathrm{R})\mathrm{Y}_0, (1+\mathrm{R})\mathrm{Y}_0] \times [-\mathrm{R}, \mathrm{R}]$ with $\mathrm{R} \le \frac{1}{4}$ to be determined ultimately by ε_0.

Step 12 Fix $\tau \in [0,1]$ and a pair $\mathrm{D} = (\mathrm{Y}, \sigma) \in [(1-\mathrm{R})\mathrm{Y}_0, (1+\mathrm{R})\mathrm{Y}_0] \times [-\mathrm{R}, \mathrm{R}]$. The corresponding continuous and piecewise smooth arc in $M_\delta \cup \mathcal{H}_{\mathfrak{p}}$ is denoted by $\upsilon_{\mathrm{D},\tau}$. As noted in the previous step, this arc is the end-to-end concatenation of five segments. It proves useful in this regard to describe the middle segment first, then the second and fourth segments and, at the end, the first and fifth segments. By way of notation, c_ε is used in what follows to denote a purely ε_0–dependent constant that is greater than 1. This constant can be assumed to increase between consecutive appearances.

The middle segment The middle segment crosses the $(p^2+q^2)^{1/2} \le \varepsilon_0$ portion of the radius ε tubular neighborhood of γ. This segment is parametrized as the map from the interval $I_{\mathrm{Y},\tau}$ given by the rule $t \mapsto (\phi = -t + \sigma, p = p_{\mathrm{Y},\tau}(t), q = q_{\mathrm{Y},\tau}(t))$ with $I_{\mathrm{Y},\tau}$ and the functions $p_{\mathrm{Y},\tau}$ and $q_{Y,\tau}$ as defined in Step 7. Use what is said in Steps 8 and 9 to see that the $t_{\mathrm{Y},\tau+}$ and $t_{\mathrm{Y}\tau-}$ endpoints of this segment on the $(p^2+q^2)^{1/2} = \varepsilon_0$ surface have distance at most $c_0\varepsilon_0(\mathrm{R} + m^{-4})$ from the $u > 0$ and $u < 0$ points where υ intersects this surface. Use $z_{\mathrm{D},\tau-}$ and $z_{\mathrm{D},\tau+}$ to denote these respective endpoints.

The second and fourth segments Let z_- and z_+ denote the respective $u < 0$ and $u > 0$ points where υ intersects the $(p^2 + q^2)^{1/2} = \varepsilon_0$ surface in the radius ε tubular neighborhood of γ. Introduce as in Steps 4 and 5 the segments υ_- and υ_+ of υ. By way of a reminder, υ_- starts on the $e^{-2(R+u)}(1-3\cos^2\theta) = \delta^2$ surface in $\mathcal{H}_{\mathrm{p}}$ and ends at z_-; and υ_+ starts at z_+ and ends on the $e^{-2(R-u)}(1-3\cos^2\theta) = \delta^2$ surface in $\mathcal{H}_{\mathrm{p}}$.

Steps 4 and 5 introduce the segments of integral curves of v, these being γ_- and γ_+. The former has $u < 0$, contains z_- and starts on the $e^{-2(R+u)}(1 - 3\cos^2\theta) = \delta^2$ surface in $\mathcal{H}_{\mathrm{p}}$ and the latter contains z_+ and ends on the $e^{-2(R-u)}(1-3\cos^2\theta) = \delta^2$ surface. Steps 4 and 5 describe parametrizations of the respective tubular neighborhoods of γ_- and γ_+ using coordinates (t, z) with $|z| \le c_\varepsilon^{-1}$ and with the $z = 0$ locus being γ_- or γ_+ as the case may be. The point $(0, 0)$ is z_- in the former case and z_+ in the latter. Reintroduce from the final paragraphs of Steps 4 and 5 the constant $c_{\mathrm{p}\varepsilon}$. Take R such that $\mathrm{R} < (c_0 c_{\mathrm{p}\varepsilon})^{-1}$ and take m such that $m > c_0 c_{\mathrm{p}\varepsilon}$. Granted these bounds, then the $t = t_{\mathrm{Y},\tau-}$ endpoint $z_{\mathrm{D},\tau-}$ of the middle segment has distance less than $c_{\mathrm{p}\varepsilon}^{-1}$ from z_- and the $t = t_{\mathrm{Y},\tau+}$ endpoint $z_{\mathrm{D},\tau+}$ of the middle segment has distance less than $c_{\mathrm{p}\varepsilon}^{-1}$ from z_+. Let $\Delta_{\mathrm{D},\tau-}$ and $\Delta_{\mathrm{D},\tau+}$ denote these distances.

Step 4 finds a solution to the (γ_-, τ) version of (C-3) that is defined for all $t \in \gamma_-$, contains $z_{\mathrm{Y},\tau-}$ and has pointwise norm bounded by $c_\varepsilon(\Delta_{\mathrm{D},\tau-} + \mathrm{r}^{-1/2})$ This solution defines a smoothly embedded arc in the $u < 0$ part of $\mathcal{H}_{\mathrm{p}}$ that starts on the $e^{-2(R+u)}(1 - 3\cos^2\theta) = \delta^2$ surface. Use $z_{\mathrm{D},\tau-}$ to denote the segment of this arc that starts on this surface and ends at $z_{\mathrm{D},\tau-}$. This arc is the second segment.

Step 5 finds a solution to the (γ_+, τ) version of (C-3) that is defined for each $t \in \gamma_+$, contains $z_{\mathrm{Y},\tau+}$ and has pointwise norm bounded by $c_\varepsilon(\Delta_{\mathrm{D},\tau+} + \mathrm{r}^{-1/2})$. This solution defines a smoothly embedded arc in the $u > 0$ part of $\mathcal{H}_{\mathrm{p}}$ that ends on the $e^{-2(R-u)}(1-3\cos^2\theta) = \delta^2$ surface. Use $z_{\mathrm{D},\tau+}$ in what follows to denote the segment of this arc that starts at $z_{\mathrm{D},\tau+}$ and ends on the $e^{-2(R-u)}(1 - 3\cos^2\theta) = \delta^2$ surface. This arc is the fourth segment.

The first and fifth segments The starting point of $z_{\mathrm{D},\tau-}$ on the surface in $\mathcal{H}_{\mathrm{p}}$ where $e^{-2(R+u)}(1 - 3\cos^2\theta) = \delta^2$ has distance at most $c_\varepsilon(\Delta_{\mathrm{D},\tau-} + \mathrm{r}^{-1/2})$ from

υ's intersection with this surface, the latter denoted in Step 1 by $z_{\mathrm{p}-}$. This being the case, it has distance at most $c_\varepsilon(\Delta_{\mathrm{D},\tau-} + \mathrm{r}^{-1/2})$ from the $(t = 2-\delta^2, z = 0)$ point in Step 1's coordinate cylinder. Meanwhile, the ending point of $z_{\mathrm{D},\tau+}$ on the $e^{-2(R-u)}(1-3\cos^2\theta) = \delta^2$ surface in $\mathcal{H}_{\mathrm{p}}$ has distance at most $c_\varepsilon(\Delta_{\mathrm{D},\tau+} + \mathrm{r}^{-1/2})$ from υ's intersection point with this same surface, and so this ending point of $z_{\mathrm{D},\tau+}$ has distance at most $c_\varepsilon(\Delta_{\mathrm{D},\tau-} + \mathrm{r}^{-1/2})$ from $(t = 1+\delta^2, z = 0)$ point in Step 1's coordinate cylinder.

With the preceding understood, suppose that $\mathrm{R} < (c_\varepsilon c_{\mathrm{p}\varepsilon})^{-1}$ and $m > c_\varepsilon c_{\mathrm{p}\varepsilon}$. Granted these bounds, then the starting point of $z_{\mathrm{D},\tau-}$ will be well inside the $t = 2-\delta^2$ boundary disk of Step 1's coordinate cylinder. Use $z_{\mathrm{D},\tau-}$ to denote the z–coordinate of this starting point of $z_{\mathrm{D},\tau-}$. Meanwhile, the ending point of $z_{\mathrm{D},\tau+}$ will be well inside the $t = 1+\delta^2$ boundary disk of Step 1's coordinate cylinder centered on γ_*. Use $z_{\mathrm{D},\tau+}$ to denote the z–coordinate of this ending point of $z_{\mathrm{D},\tau+}$.

According to Step 1, there is a solution $t \mapsto z(t)$ to (C-2) with $z(2-\delta^2) = z_{\mathrm{D},\tau-}$. Denote this solution by $z^{M}_{\mathrm{D},\tau-}$. The first segment is the $t \in \left[\frac{3}{2}, 2-\delta^2\right]$ part of the arc in M_δ given by the graph $t \mapsto (t, z^{M}_{\mathrm{D},\tau-}(t))$.

There is also a solution $t \mapsto z(t)$ to (C-2) with $z(1+\delta^2) = z_{\mathrm{D},\tau+}$. Denote the latter solution by $z^{M}_{\mathrm{D},\tau+}$. The fifth segment is the $t \in \left[1+\delta^2, \frac{3}{2}\right]$. part of the arc in M_δ given by the graph $t \mapsto (t, z^{M}_{\mathrm{D},\tau+})$.

Step 13 Introduce ϕ_+ and ϕ_- to denote the ϕ coordinates of the $(t = 1+\delta^2, z = 0)$ and $(t = 2-\delta^2, z = 0)$ points, respectively, on the two boundary disks of Step 1's coordinate cylinder. The ϕ coordinate of the ending point of $z_{\mathrm{D},\tau+}$ on the surface in $\mathcal{H}_{\mathrm{p}}$ where $e^{-2(R-u)}(1-3\cos^2\theta) = \delta^2$ can be written as $\phi_+ + \varphi_{\mathrm{D},\tau+}$ with $|\varphi_{\mathrm{D},\tau+}| \le c_0^{-1}$, and the ϕ coordinate of the starting point of $z_{\mathrm{D},\tau-}$ on the $e^{-2(R+u)}(1-3\cos^2\theta) = \delta^2$ surface can be written as $\phi_- + \varphi_{\mathrm{D},\tau-}$ with $|\varphi_{\mathrm{D},\tau-}| \le c_0^{-1}$. Write the respective values of the function $\hbar$ at these boundary points of $z_{\mathrm{D},\tau+}$ and $z_{\mathrm{D},\tau-}$ as $\hbar_{\mathrm{D},\tau+}$ and $\hbar_{\mathrm{D},\tau-}$.

It follows from what is said in Step 1 that the five-segment concatenated arc defined in Step 12 is a piecewise embedded loop in $M_\delta \cup \mathcal{H}_{\mathrm{p}}$ if

$$\text{(C-22)} \quad \varphi_{\mathrm{D},\tau+} = (-1)^{\hat{o}}(\hbar_{\mathrm{D},\tau-} + (1-\tau)u_1), \quad \hbar_{\mathrm{D},\tau+} = (-1)^{\hat{o}}(\varphi_{\mathrm{D},\tau-} + (1-\tau)u_2),$$

where the notation is such that $\hat{o} \in \{0,1\}$ is determined by the point $z_* \in \mathrm{C}_{p+} \cap \mathrm{C}_{p'}$, and where u_1 and u_2 are the respective real and imaginary parts of

$$\text{(C-23)} \qquad -2i\left(\int_{1+\delta^2}^{3/2} x_\upsilon - \int_{3/2}^{2-\delta^2} x_\upsilon\right)$$

with x_υ being the function in (C-2). As is explained next, there exists a smooth map, $\mathrm{D}(\cdot)\colon [0,1] \to ((1-\mathrm{R})\mathrm{Y}_0, (1+\mathrm{R})\mathrm{Y}_0) \times (-\mathrm{R},\mathrm{R})$ such that for each $\tau \in [0,1]$, the $\mathrm{D} = \mathrm{D}(\tau)$ version of (C-22) holds when the following conditions are met:

(C-24)
- $\varepsilon_0 \le c_0^{-1}$.
- $m \ge c_\varepsilon$ with $c_\varepsilon > 1$ being a purely ε–dependent constant.
- $\mathrm{R} \le c_\varepsilon m^{-1}$ with $c_\varepsilon > 1$ being a purely ε–dependent constant.
- $c_\upsilon \ge c_{\varepsilon,m,\mathrm{R}}$ with $c_{\varepsilon,m,\mathrm{R}} > 1$ being a constant that depends only on ε, m and R.
- $\mathrm{r} \ge \kappa_c$ with $\kappa_c > 1$ being a constant that depends only on c_υ.

To construct $\mathrm{D}(\cdot)$, take $\mathrm{D}(0)$ to be the pair $(\mathrm{Y} = \mathrm{Y}_0, \sigma = 0)$ which obeys (C-22) because the arc defined in Step 12 from $(\mathrm{Y} = \mathrm{Y}_0, \sigma = 0)$ is the smooth, embedded circle. The construction of $\mathrm{D}(\tau)$ for $\tau > 0$ requires a rewriting of (C-22). To set the stage for this, fix $\mathrm{D} = (\mathrm{Y},\sigma) \in [(1-\mathrm{R})\mathrm{Y}_0, (1+\mathrm{R})\mathrm{Y}_0] \times [-\mathrm{R},\mathrm{R}]$ and $\tau \in [0,1]$. Use (C-19) to write the difference between $\varphi_{\mathrm{D},\tau+}$ and its $(\mathrm{Y}_0, \sigma = 0)$, $\tau = 0$ analog as

$$\sigma - \lambda^{-1}\mathrm{Y}_0^{-1}(\mathrm{Y}-\mathrm{Y}_0) + \mathfrak{e}, \tag{C-25}$$

where $|\mathfrak{e}| \le c_0((\varepsilon_0 + \mathrm{R})\mathrm{Y}_0^{-1}|\mathrm{Y}-\mathrm{Y}_0| + m^{-4})$. This last formula and its $t_{\mathrm{Y},\tau-}$ analog allow (C-22) to be rewritten as

(C-26)
- $\sigma - \lambda^{-1}\mathrm{Y}_0^{-1}(\mathrm{Y}-\mathrm{Y}_0) - (-1)^{\hat{o}}(\mathfrak{h}_{\mathrm{Y},\tau}(t_{\mathrm{Y},\tau-}) - \mathfrak{h}_{\mathrm{Y}_0,0}(t_{\mathrm{Y}_0,0-})) + \mathfrak{e}_1 = 0,$
- $\sigma + \lambda^{-1}\mathrm{Y}_0^{-1}(\mathrm{Y}-\mathrm{Y}_0) - (-1)^{\hat{o}}(\mathfrak{h}_{\mathrm{Y},\tau}(t_{\mathrm{Y},\tau+}) - \mathfrak{h}_{\mathrm{Y}_0,0}(t_{\mathrm{Y}_0,0+})) + \mathfrak{e}_2 = 0,$

where $\mathfrak{e}_1$ and $\mathfrak{e}_2$ are functions of τ, Y and σ whose absolute values are bounded by $c_0((\varepsilon_0+\mathrm{R})\mathrm{Y}_0^{-1}|\mathrm{Y}-\mathrm{Y}_0|+m^{-4})$. The left-hand side of (C-26) defines a smooth map, $\mathcal{F}$, from $[0,1] \times [(1-\mathrm{R})\mathrm{Y}_0, (1+\mathrm{R})\mathrm{Y}_0] \times [-\mathrm{R},\mathrm{R}]$ to $\mathbb{R}^2$ with the property that $\mathcal{F} = 0$ if and only if (C-22) is obeyed. What follows is a crucial observation about this map: the differential of $\mathcal{F}$ along the domain's factor $[(1-\mathrm{R})\mathrm{Y}_0, (1+\mathrm{R})\mathrm{Y}_0] \times [-\mathrm{R},\mathrm{R}]$ is surjective if (C-24) holds, this being a consequence of (C-21) and what is said about $\mathfrak{e}$ in (C-25).

Suppose that $\tau_0 \in [0,1]$ is such that $\mathrm{D}(\cdot)$ has been defined on $[0,\tau_0)$. To extend $\mathrm{D}(\cdot)$ to a larger interval, use the fact that $[(1-\mathrm{R})\mathrm{Y}_0, (1+\mathrm{R})\mathrm{Y}_0] \times [-\mathrm{R},\mathrm{R}]$ is compact to see that there is a $\tau = \tau_0$ limit point $\mathrm{D}(\tau_0)$ for $\{\mathrm{D}(\tau)\}_{\tau\to\tau_0}$. It follows from (C-26) that this limit point is unique, that the extension of $\mathrm{D}(\cdot)$ to $[0,\tau_0]$ is continuous, and that $\mathrm{D}(\tau_0)$ obeys the $\tau = \tau_0$ version of (C-26). Write $\mathrm{D}(\tau_0)$ as (Y,σ). Use (C-19), (C-21) and the fact that (C-26) is obeyed to conclude that $|\sigma| + |\mathrm{Y}-\mathrm{Y}_0| \le c_0 m^{-4}$. This implies that $\mathrm{D}(\tau_0)$ lies in the interior of the parameter space $[(1-\mathrm{R})\mathrm{Y}_0, (1+\mathrm{R})\mathrm{Y}_0] \times [-\mathrm{R},\mathrm{R}]$ when

(C-24) holds. Since $\mathrm{D}(\tau_0)$ is not a boundary point of $[(1-\mathrm{R})\mathrm{Y}_0, (1+\mathrm{R})\mathrm{Y}_0] \times [-\mathrm{R}, \mathrm{R}]$, the fact that $\mathcal{F}$ has surjective differential along the $[(1-\mathrm{R})\mathrm{Y}_0, (1+\mathrm{R})\mathrm{Y}_0] \times [-\mathrm{R}, \mathrm{R}]$ factor of its domain implies via the inverse function theorem that $\mathrm{D}(\cdot)$ has a smooth extension to an open interval in $[0, 1]$ that contains $[0, \tau_0]$.

Step 14 Fix $\tau \in [0, 1]$. Let $\upsilon^{\alpha}_{\tau *}$ denote the continuous, piecewise smooth loop that is defined in Step 12 by the data set $\mathrm{D}(\tau)$ from the previous step. The loop $\upsilon^{\alpha}_{\tau *}$ is smooth on the interior of each of its five concatenating segments. The implicit function theorem construction implies that the assignment of $\tau \in [0, 1]$ to each of the five concatenating segments defines a smoothly varying arc in $M_\delta \cup \mathcal{H}_{\mathfrak{p}}$. Moreover, the assertions made by the four bullets in Lemma C.1 hold for each of these five $[0, 1]$–parametrized families of arcs. This follows directly from the implicit function theorem construction given what is said at the very end of Steps 4 and 5; and given (C-15)–(C-16), the bound for $\mathfrak{e}$ in (C-19) and the bound in (C-21).

The loop $\upsilon^{\alpha}_{\tau *}$ for $\tau \in (0, 1)$ is continuous, but its derivative may be discontinuous at four points, these being the loop's intersection points with the boundary spheres of the $|u| \le R + \ln \delta$ part of $\mathcal{H}_{\mathfrak{p}}$ and the $(p^2 + q^2)^{1/2} = \varepsilon_0$ surface in the radius ε tubular neighborhood of γ. Even so, the two concatenating segments near these points are smooth up to their endpoints on the relevant surface, and the corresponding tangent vectors differ by at most $c_0 \mathrm{r}^{-1/2}$ at these endpoints. This is because the tangent vector to each segment near these junctions differs from v by at most $c_0 \mathrm{r}^{-1/2}$. The preceding fact implies that the loop $\upsilon^{\alpha}_{\tau *}$ can be smoothed near the junctions of segments so that the result is a smoothly embedded loop that obeys the first three bullets of Lemma C.1. Moreover, it is a straightforward task to define this smoothing without changing the already smooth $\tau = 0$ and $\tau = 1$ versions so that the resulting $[0, 1]$–parameter family of smooth loops is smoothly parametrized and obeys the assertion of Lemma C.1's fourth bullet at each point. The details of this are straightforward and thus omitted.

Use $\{\upsilon^{\alpha}_{\tau}\}_{\tau \in [0,1]}$ to denote the resulting $[0, 1]$–parametrized family of smooth loops. This family obeys all of the requirements for Lemma C.1. □

Cc Increasing r

This part takes Proposition C.2's pair $(A_{\diamond 11}, \psi_{\diamond 11})$ as the starting point of a path in $\mathrm{Conn}(E) \times C^{\infty}(Y; \mathbb{S})$ whose end member is constructed from the same vortex solutions and loops in Y that are used to construct $(A_{\diamond 11}, \psi_{\diamond 11})$ but with the given choice of r replaced by a far larger choice. A result from [21] is brought to bear in the next section; it requires the larger value of r. This larger value of r is denoted by R. There is no upper bound to the value chosen but a lower bound $\mathrm{R} > c_0 c_v^4 \mathrm{r}$ is imposed.

The path is parametrized by $[0,1]$, and a given $\tau \in [0,1]$ member denoted by $(A_{\bullet\tau}, \psi_{\bullet\tau})$. The definition of $(A_{\bullet\tau}, \psi_{\bullet\tau})$ is identical to that of $(A_{\diamond 11}, \psi_{\diamond 11})$ given in Section Ca but for the replacement of r with $\mathrm{r}(\tau) = (1-\tau)\mathrm{r} + \tau \mathrm{R}$.

Keep in mind in what follows that the zero locus of the E summand of any $\tau \in [0,1]$ version of $\psi_{\bullet\tau}$ is identical to that of $\alpha_{\diamond 11}$ as is the degree of vanishing along any transverse disk centered on the zero locus. By way of a reminder, the zero locus of $\alpha_{\diamond 11}$ consists solely of closed integral curves of v in $M_\delta \cup \bigl(\bigcup_{\mathrm{p} \in \Lambda} \mathcal{H}_\mathrm{p}\bigr)$, these coming from two sets. The first set consisted of curves that intersect M_δ; this set was denoted by Θ^α. The second set is a subset of $\bigcup_{\mathrm{p}\in\Lambda} (\hat{\gamma}_\mathrm{p}^+ \cup \hat{\gamma}_\mathrm{p}^-)$.

Of interest is the spectral flow between the $\mathfrak{c} = (A_{\bullet 0}, \psi_{\bullet 0}) = (A_{\diamond 11}, \psi_{\diamond 11})$ version of $\mathfrak{L}_{\mathfrak{c},\mathrm{r}}$ and the $\mathfrak{c} = (\alpha_{\bullet 1}, \psi_{\bullet 1})$ version of $\mathfrak{L}_{\mathfrak{c},\mathrm{R}}$. Note in particular that the latter operator is defined using R rather than r. The proposition that follows asserts that there is an a priori upper bound for the norm of the spectral flow between these two operators that is independent of the original pair (A, ψ) and r and also R. This proposition uses $\mathfrak{c}(\tau)$ to denote the pair $(A_{\bullet\tau}, \psi_{\bullet\tau})$.

Proposition C.4 *There exists $\kappa \geq 100$ and, given $c_v \geq \kappa$, there exists $\kappa_{c_v} > \kappa$ with the following significance: Suppose that $\mathrm{r} \geq \kappa_{c_v} c_v^{10}$ and suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. The values of κ, c_v and r are suitable for defining $(A_{\diamond 11}, \psi_{\diamond 11})$ and any $\mathrm{R} \geq \kappa c_v^6 \mathrm{r}$ version of the family $\{(A_{\bullet\tau}, \psi_{\bullet\tau})\}_{\tau\in[0,1]}$. The norm of the spectral flow between the end members of the corresponding family $\{\mathfrak{L}_{\mathfrak{c}(\tau),\mathrm{r}(\tau)}\}_{\tau\in[0,1]}$ is bounded by κ.*

Proof It is assumed in what follows that κ, c_v and r are large enough to invoke the various results in the preceding subsections of Appendix C and those in Appendices A and B. The proof that follows has five parts.

Part 1 As explained directly, each member of the family $\{(A_{\bullet\tau}, \psi_{\bullet\tau})\}_{\tau\in[0,1]}$ obeys a version of Properties 1–5 in Section Ab. The proof that such is the case distinguishes between values of τ near 0 and larger values. To elaborate, note first that the various properties require the specification of constants c_0 and z. It is a straightforward matter to check that Properties 1, 2, 4 and 5 are obeyed using c_v in lieu of c_0 and $\mathrm{r}(\tau)$ in lieu of z. It is also a straightforward matter to verify the third and fourth bullets of

Property 3. Meanwhile, items (a) and (b) of the second bullet of Property 3 follow directly by virtue of the fact that the zero locus of the E summand of $\psi_{\bullet\tau}$ is a union of integral curves of v. The story with regards to the first bullet and items (c) and (d) of the second bullet of Property 3 is not so straightforward, the point being that $Y - Y_{\diamond z}$ is the union of tubular neighborhoods of the curves from the set $\bigcup_{p\in\Lambda}(\hat{\gamma}_p^+ \cup \hat{\gamma}_p^-)$ with radius proportional to $z^{-1/2}$. If a component of the zero locus of $\alpha_{\diamond 11}$ from Θ^α intersect the $z = r$ version of $Y - Y_{\diamond z}$, then the first bullet and items (c) and (d) of the second bullet of Property 3 will not hold when $z = r(\tau)$ for values of τ in certain subsets of $[0, 1]$.

To deal with this issue, fix $\tau \in [0, 1]$. The $c_0 = c_v$ and $z = r(\tau)$ version of Property 3 can fail if there exists a curve from Θ^α and a curve from $\bigcup_{p\in\Lambda}(\hat{\gamma}_p^+ \cup \hat{\gamma}_p^-)$ with the following property: Let v^α denote the curve from Θ^α and let γ denote the curve from $\bigcup_{p\in\Lambda}(\hat{\gamma}_p^+ \cup \hat{\gamma}_p^-)$. Then the minimum distance between the points in v^α and γ is no less than $(c_v^4 - 3c_v^3)r(\tau)^{-1/2}$ and no greater than $(c_v^4 + 3c_v^3)r(\tau)^{-1/2}$. This last observation has two immediate consequences, the first being that Property 3 can fail only in the case when $r(\tau) \le c_0 c_v^6 r$ and thus only if $\tau \le c_0 c_v^6 r/R$. This is so because the minimum distance between v^α and γ is in any event greater than $c_0^{-1} c_v r^{-1/2}$. To state the second consequence, introduce $c_1 = c_v - 2c_v^{-1}$. If the distance between v^α and γ is no less than $(c_v^4 - 3c_v^3)r(\tau)^{-1/2}$, then it is greater than $(c_1^4 + 3c_1^3)r(\tau)^{1/2}$ if $c_v \ge c_0$.

Given what was just said, it is a straightforward task to use the pointwise bounds given in Section Aa for the absolute values of α_0, a_0, y and ς to verify the following assertion: if $c_v \ge c_0$, then $(A_{\bullet\tau}, \psi_{\bullet\tau})$ obeys the $c_0 = c_v - c_0 c_v^{-1}$ and $z = r(\tau)$ version of Properties 1–5 if it does not obey the $c_0 = c_v$ version.

Part 2 A suitable bound for the absolute value of the spectral flow is obtained by studying the variation with τ of the spectrum of the family of operators $\{\mathfrak{L}_{c(\tau),r(\tau)}\}_{\tau\in[0,1]}$. This part of the subsection considers the values of τ when some closed integral curve of v from Θ^α and some curve from $\bigcup_{p\in\Lambda}(\hat{\gamma}_p^+ \cup \hat{\gamma}_p^-)$ have minimum distance at most $(c_0^4 + 3c_0^3)r(\tau)^{-1/2}$, where it is understood that $c_0 \in (c_v - c_0 c_v^{-1}, c_v]$ and that $(A_{\bullet\tau}, \psi_{\bullet\tau})$ obeys the c_0 and $z = r(\tau)$ version of Properties 1–5 in Section Ab. As noted in Part 1, this condition can hold only if $\tau \le c_0 c_v^6 r/R$ and so $r(\tau) \le c_0 c_v^6 r$. This understood, suppose in what follows that this minimum distance condition holds for $\tau \le c_0 c_v^6 r/R$ and that this minimum distance condition does not hold for $\tau \ge c_0 c_v^6 r/R$.

An almost verbatim repetition of the arguments used to prove Proposition C.2 finds an (A, ψ)– and r–independent bound for the absolute value of the spectral flow for the

$\tau \le c_0 c_v^6 \mathrm{r}/\mathrm{R}$ part of the family $\{\mathfrak{L}_{\mathrm{c}(\tau),\mathrm{r}(\tau)}\}_{\tau\in[0,1]}$. The bound for the absolute value of the spectral flow does, however, depend on the choice for c_v. The only salient changes to the arguments from the proof of Proposition C.2 involve Steps 1 and 2. Step 1 is replaced by Part 1 above. The change to Step 2 adds extra terms to the right-hand side of (C-1) to account for the fact that relevant version of $\frac{d}{d\tau}\mathfrak{L}_{\mathbb{V}\tau}$ is nonzero on the radius $c_0^{-1} c_v \mathrm{r}^{-1/2}$ tubular neighborhood of certain curves from the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$. In any event, the absolute value of the homomorphism $\frac{d}{d\tau}\mathfrak{L}_{\mathbb{V}\tau}$ is bounded by $\kappa_{c1}\mathrm{r}^{1/2}(\mathrm{R}/\mathrm{r})$. Steps 3 and 4 can be repeated with the only change being that the interval $[0,1]$ is replaced by $[0, c_0 c_v^6 \mathrm{r}/\mathrm{R}]$ and the latter is divided into some $m \le c_0 c_v^6 \kappa_c$ segments of length at most $\kappa_c^{-1}\mathrm{r}/\mathrm{R}$.

Part 3 Assume that $\tau_0 \in [0,1]$ is such that the following is true: Let υ^α denote a component of the zero locus of $\alpha_{\diamond 11}$ from Θ^α. Then υ^α has distance greater than $(c_0^4 + 3c_0^3)\mathrm{r}(\tau)^{1/2}$ from all curves from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$ with $c_0 \in (c_v - c_0 c_v^{-1}, c_v]$. Part 4 of the proof derives an (A,ψ)–, r– and R–independent upper bound for the absolute value of the spectral flow along the $[\tau_0, 1]$ part of the 1-parameter family of operators $\{\mathfrak{L}_{\mathbb{V}\tau}\}_{\tau\in[0,1]}$ under the assumption that $c_v \ge c_0$ and $\mathrm{r} \ge \kappa_c$ with $\kappa_c \ge 1$ denoting a purely c_v–dependent constant. This bound with the bound in Part 2 implies the assertions of Proposition C.4.

The arguments in Part 4 invoke the following auxiliary lemma:

Lemma C.5 *There exists $\kappa > 1$ with the following significance: Let $\upsilon \in Y$ denote a closed, integral curve of v that lies entirely in $M_\delta \cup \left(\bigcup_{\mathfrak{p}\in\Lambda}\mathcal{H}_{\mathfrak{p}}\right)$. Fix coordinates from Part 4 of Section Aa for a tubular neighborhood of υ. The corresponding version of the operator $\eta \mapsto \frac{i}{2}\frac{d}{dt}\eta + \nu\eta + \mu\bar{\eta}$ on $C^\infty(\gamma,\mathbb{C})$ has no eigenvalue between $-\kappa^{-1}$ and κ^{-1}.*

This lemma is proved in Part 5.

The arguments in Part 4 require a second auxiliary observation, this concerning the spectrum of operators that are associated to components of the zero locus of $\alpha_{\diamond 1}$ from the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$. These operators are versions of those depicted in (3-10) with the pair (ν,μ) in (3-8) being that from any $\gamma \in \bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$ version of (A-6) with both functions constant, and μ real and greater than $|\nu|$. With a positive integer, m, chosen, the relevant equivalence class from $\mathfrak{C}_m$ is that defined by the solution to (2-8) and (3-1) with $\alpha_0 = |\alpha_0|(z/|z|)^m$. This operator is denoted by $\mathcal{L}_m$. What follows is the second observation:

(C-27) Given $m_* \geq 1$ there exists $\kappa > 1$ such that if $m \leq m_*$, then the operator $\mathcal{L}_m$ has no eigenvalues with absolute value in the interval $(0, \kappa^{-1}]$.

Such κ exists because there are only m_* versions of (3-10) involved and each has discrete spectrum with no accumulation points. By way of a parenthetical remark, it is likely that these versions of (3-10) have trivial kernel and so lack eigenvalues in $[-\kappa^{-1}, \kappa^{-1}]$.

Part 4 This part assumes Lemma C.5 to complete the proof of Proposition C.4. This is done in the four steps that follow. These steps use c_c to denote a constant that is greater than 1 and depends only on c_v. Its value can increase between successive appearances. These steps also use κ_* to denote the smaller of the versions of κ that appear in Lemma C.5 and in (C-27).

Step 1 Lemmas A.8 and A.9 can be invoked if $c_v \geq c_0$ and $\mathrm{r} \geq c_c$ because the integer m that appears in Lemma A.9 is a priori bounded by c_0. This understood, what follows is a direct consequence of what is said by Lemmas A.6, A.8 and A.9, and then Lemmas B.5 and B.6: If $c_v \geq c_0$ and $\mathrm{r} \geq c_v$, then the number of linearly independent eigenvalues of any $\tau \in [0,1]$ version of $\mathfrak{L}_{\mathrm{c}(\tau),\mathrm{r}(\tau)}$ with eigenvalue between $-\frac{1}{100}\kappa_*^{-1}$ and $\frac{1}{100}\kappa_*^{-1}$ is bounded by c_0.

Step 2 As in the proof of Proposition C.2, no generality is lost by assuming that the parametrization of the family $\{\mathfrak{L}_{\mathrm{c}(\tau),\mathrm{r}(\tau)}\}_{\tau\in[0,1]}$ is real analytic so as to apply what is said in Part 1 of the proof of Proposition B.3. This understood, let $\{\lambda_{n\tau}\}_{n\in\mathbb{Z},\tau\in[0,1]}$ denote the corresponding family of eigenvalues. Let c_1 denote the dimension bound given in Step 1. Given what is said in Step 1, the absolute value of the spectral flow for the $[\tau_0, 1]$ part of the family $\{\mathfrak{L}_{\mathrm{c}(\tau),\mathrm{r}(\tau)}\}_{\tau\in[0,1]}$ is no greater than c_1 unless some $\tau \in [\tau_0, 1]$ version of $\mathfrak{L}_{\mathrm{c}(\tau),\mathrm{r}(\tau)}$ has an eigenvalue between $\frac{1}{100}\kappa_*^{-1}$ and $\frac{1}{50}\kappa_*^{-1}$. Suppose for the sake of argument that such is the case. Let $\mathfrak{f}$ denote the corresponding eigenfunction and λ its eigenvalue.

Let υ^α denote a given component of the zero locus of $\alpha_\diamond$ from Θ^α and let ζ denote the section of the $\gamma = \upsilon^\alpha$ version of the line bundle $\mathrm{Ker}_\vartheta|_\gamma \to \gamma$ that is described in Lemma A.8. Lemmas A.8 and C.5 are not mutually compatible if the L^2–norm of ζ is greater than $c_0 c_v^{-1} \|\mathfrak{f}\|_2$.

Step 3 Let $\gamma \in \bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_{\mathrm{p}}^+ \cup \hat{\gamma}_{\mathrm{p}}^-)$ denote a component of the zero locus of $\alpha_{\diamond 1}$ and let ζ denote the section of the bundle $\mathrm{Ker}_\vartheta|_\gamma \to \gamma$ that is described in Lemma A.9. Use m to denote the integer for γ's version of Lemma A.9. Note in particular that

$m \le c_0$. Introduce ζ_- to denote the L^2–orthogonal projection of ζ onto the span of the eigenvalues of $\mathcal{L}_m$ with eigenvalue 0 or less, and use ζ_+ to denote the L^2–orthogonal projection of ζ onto the span of the eigenvalues of $\mathcal{L}_m$ with eigenvalue greater than κ_*^{-1}. Note that $\zeta = \zeta_- + \zeta_+$. Lemma A.9 and (C-27) are not mutually compatible if the L^2–norm of either ζ_- or ζ_+ is greater than $c_0 c_\upsilon^{-1} \|\mathfrak{f}\|_2$.

Step 4 It follows from what is said in Steps 2 and 3 that $\|\Pi_\vartheta \mathfrak{f}\|_2 \le c_0 c_\upsilon \|f\|_2$. But if $c_\upsilon \ge c_0$, then this last conclusion is incompatible with what is said by Lemma A.6 if $\mathfrak{f}$ is not identically zero.

Part 5 This last part of the subsection contains the proof of Lemma C.5.

Proof Let L_υ denote the operator in question. Proposition II.2.7 asserts that γ is hyperbolic and such is the case if and only if L_υ has trivial kernel. This understood, the only issue is that of the size of the neighborhood of 0 that lacks eigenvalues. The six steps that follow in a moment prove that such a neighborhood contains an interval of the form $(-c_0^{-1}, c_0)$.

Keep in mind when reading the proof that L_υ is defined by the pair (ν, μ) and that the latter are defined by the choice of a unitary frame for $K^{-1}|_\upsilon$. This last fact has the following implication: Any two versions of (ν, μ) that arise from υ's version of (A-6) give isospectral versions of L_υ. This being the case, no generality is lost by choosing the coordinates so that $|\nu| + |\mu| \le c_0$.

Step 1 Fix $\varepsilon > 0$ so that the radius ε tubular neighborhood of any given curve in the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$ has coordinates (ϕ, x, y) with $x = b^{-1}u$ and $y = \theta_* + \theta$ with b denoting $\frac{2}{3\sqrt{3}} e^R (x_0 + 4e^{-2R})^{1/2}$ and with θ_* such that $\cos\theta_* = \pm\frac{1}{\sqrt{3}}$ as the case may be. These are the coordinates used in Step 4 and the subsequent steps of the proof of Lemma C.1. Set $p = y + x$ and $q = y - x$. Fix $\varepsilon_0 \in (0, c_0^{-1}\varepsilon)$ so that the surface $(p^2 + q^2)^{1/2} = \varepsilon_0$ lies well inside the radius ε tubular neighborhood.

Suppose that υ enters the $(p^2+q^2)^{1/2} < \frac{1}{2}\varepsilon_0$ part of the radius ε tubular neighborhood about a given $\gamma \in \bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$. The discussion that follows considers only the case when γ's version of $\cos\theta_*$ is equal to $\frac{1}{\sqrt{3}}$ as the discussion for the other case is identical but for some sign changes. Use λ to denote $4\sqrt{6}e^{-R}(x_0 + 4e^{-2R})^{1/2}$. The part of υ in the $p^2 + q^2 \le \varepsilon_0$ part of the radius ε tubular neighborhood of γ can be written in terms of the coordinates (ϕ, p, q) as the image of a map $t \mapsto (\phi = -t, p_\upsilon(t), q_\upsilon(t))$ with the domain being an interval $I_\upsilon \subset \mathbb{R}$ containing the origin and with the pair

(p_υ, q_υ) obeying the version of (C-4) with $\mathfrak{r}_{p\upsilon} = \mathfrak{r}_{q\upsilon} = 0$. They also obey analogs of (C-8), (C-11) and (C-12) with no terms proportional to $\mathrm{r}^{-1/2}$ and with $\mathrm{x}_0 = 0$.

As in the proof of Lemma C.1, no generality is lost by taking the $t = 0$ point so that $p_\upsilon = q_\upsilon$ at $t = 0$, this being the point where υ crosses the $u = 0$ sphere. With this choice understood, write I_υ as $[t_-, t_+]$. The pair t_- and t_+ obey (C-6) with Δ denoting the value of $(p_\upsilon^2 + q_\upsilon^2)^{1/2}$ at $t = 0$, this being the minimal value of $(p^2 + q^2)^{1/2}$ on γ.

Step 2 Fix $T > 1$ and let ζ denote an eigenvector of L_υ whose eigenvalue has absolute value no greater than $T^{-1}\lambda$. The eigenvalue equation for ζ on the part of υ in the radius ε tubular neighborhood of γ where $(p^2 + q^2)^{1/2} \le \varepsilon_0$ can be written as an equation for a pair of $\mathbb{R}$–valued functions $t \mapsto (\zeta_1(t), \zeta_2(t))$ on the interval I_υ. This equation has the form

$$\text{(C-28)}\qquad \frac{d}{dt}\zeta_1 = \lambda\zeta_1 + \mathfrak{e}_{11}\zeta_1 + \mathfrak{e}_{12}\zeta_2 \quad\text{and}\quad \frac{d}{dt}\zeta_2 = -\lambda\zeta_2 + \mathfrak{e}_{21}\zeta_1 + \mathfrak{e}_{22}\zeta_2,$$

where the $\mathfrak{e}_{ij}$ are smooth functions on I_υ that are bounded by $c_0(\varepsilon_0 + T^{-1}\lambda)$. The fact that $|\nu| + |\mu| \le c_0$ implies that $c_0^{-1}|\zeta|^2 \le |\zeta_1|^2 + |\zeta_2|^2 \le c_0|\zeta|^2$.

Step 3 The same argument that proves (C-8) can be used with (C-28) to prove that $|\zeta_2| \le c_0(\varepsilon_0 + T^{-1}\lambda)|\zeta_1|$ where $t \ge 0$, and it can be used to prove that $|\zeta_1| \le c_0(\varepsilon_0 + T^{-1}\lambda)|\zeta_2|$ where $t \le 0$. Granted these bounds, multiply the left-hand equation by ζ_1 and the right-hand by ζ_2. Integrate the resulting equalities to see that $|\zeta_1|^2 + |\zeta_2|^2$ at t_+ and t_- are at most $(\lambda + c_0(\varepsilon_0 + T^{-1}\lambda))\|\zeta\|_2$. It then follows from (C-28) that

$$\text{(C-29)}\qquad (|\zeta_1|^2 + |\zeta_2|^2)(t) \le c_0\|\zeta\|_2(e^{-\lambda(t_+ - t)/c_0} + e^{-\lambda(|t_-| + t)/c_0})$$

at each $t \in I_\upsilon$.

Fix $\mathrm{L} \ge 1$ and suppose that both t_+ and $|t_-|$ are greater than $c_0\lambda^{-1}2\mathrm{L}$. If such is the case, then (C-29) implies that

$$\text{(C-30)}\qquad (|\zeta_1|^2 + |\zeta_2|^2)(t) \le c_0\|\zeta\|_2 e^{-\mathrm{L}}$$

at times $t \in I$ with distance L or more from t_- and t_+.

Step 4 Use χ to construct a smooth, nonnegative function on γ that is equal to 1 except at points in I_υ with distance L or less from either t_- or t_+. This function should equal 0 at points on I_υ with distance greater than $\mathrm{L} + 1$ from both t_- and t_+, and its absolute value should be bounded by 4. Use $\chi_{\gamma,\mathrm{L}}$ to denote this function. What follows is a consequence of (C-30):

$$\text{(C-31)}\qquad \|\chi_{\gamma,\mathrm{L}}\zeta\|_2 \ge (1 - c_0e^{-\mathrm{L}})\|\zeta\|_2 \quad\text{and}\quad \|L_\upsilon(\chi_{\gamma,\mathrm{L}}\zeta)\|_2 \le T^{-1}\lambda\|\chi_{\gamma,\mathrm{L}}\zeta\|_2.$$

The function $\chi_{\gamma,\mathrm{L}}$ can be defined for each $\gamma \in \bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_\mathrm{p}^+ \cup \hat{\gamma}_\mathrm{p}^-)$ of the sort under consideration. Multiply ζ by all such functions and the result is a section of $K^{-1}|_\upsilon$ with compact support on the part of υ with distance greater than $c_0^{-1}\varepsilon_0 e^{-c_0\mathrm{L}}$ from all curves in the set $\bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_\mathrm{p}^+ \cup \hat{\gamma}_\mathrm{p}^-)$. This section is denoted by $\zeta_{\varepsilon,\mathrm{L}}$. What is said by (C-31) implies that $\|L_\upsilon \zeta_{\varepsilon,\mathrm{L}}\|_2 \le T^{-1}\lambda\|\zeta_{\varepsilon,\mathrm{L}}\|_2$.

Step 5 Use $\upsilon_{\varepsilon,\mathrm{L}}$ to denote the part of υ with distance $c_0^{-1}\varepsilon_0 e^{-c_0\mathrm{L}}$ or more from all curves in the set $\bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_\mathrm{p}^+ \cup \hat{\gamma}_\mathrm{p}^-)$. It follows from what is said in Step 2 that $\upsilon_{\varepsilon,\mathrm{L}}$ has length at most $c_0(\mathrm{L}+|\ln\varepsilon_0|)$. The fact that L_υ is a first-order operator, the fact that its coefficients are bounded by c_0, and the fact that $\upsilon_{\varepsilon,\mathrm{L}}$ has length at most $c_0(\mathrm{L}+|\ln\varepsilon_0|)$ has the following consequence: Let η denote a section of $K^{-1}|_\upsilon$ with compact support on $\upsilon_{\varepsilon,\mathrm{L}}$. Then $\|L_\upsilon\eta\|_2 \ge c_{\varepsilon,\mathrm{L}}^{-1}\|\eta\|_2$ with $c_{\varepsilon,\mathrm{L}}$ being a constant that is greater than 1 and depends only on ε_0 and L, but not on υ.

This last bound on $\|L_\upsilon\eta\|_2$ runs afoul of the inequality $\|L_\upsilon\zeta_{\varepsilon,\mathrm{L}}\|_2 \le T^{-1}\lambda\|\zeta_{\varepsilon,\mathrm{L}}\|_2$ unless T is less than $c_0 c_{\varepsilon,\mathrm{L}}\lambda$.

Step 6 Choose $\varepsilon_1 \ll c_0^{-1}\varepsilon_0 e^{-c_0\mathrm{L}}$. Suppose that υ is a closed, integral curve of v whose points have distance ε_1 or more from all curves in the set $\bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_\mathrm{p}^+ \cup \hat{\gamma}_\mathrm{p}^-)$. It follows as a consequence that υ's version of L_υ has no eigenvalues between $-c_{1\varepsilon}^{-1}$ and $c_{1\varepsilon}^{-1}$ with $c_{1\varepsilon} \ge 1$ depending only on ε_1. Such a constant exists because L_υ has trivial kernel, and because there is but a finite set of closed orbits of v in $M_\delta \cup \big(\bigcup_{\mathrm{p}\in\Lambda}\mathcal{H}_\mathrm{p}\big)$ that have distance ε_1 from $\bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_\mathrm{p}^+ \cup \hat{\gamma}_\mathrm{p}^-)$.

The bound in Step 5 and the bound in the preceding paragraph give a υ–independent, strictly positive lower bound to the absolute value of any eigenvalue of L_υ. □

Cd Decreasing r

A unique set of closed integral curves of v are defined by three properties, the first three following directly. By way of notation, the set in question is denoted here by Θ^0. The first property requires that all curves from Θ^0 lie in $M_\delta \cup \big(\bigcup_{\mathrm{p}\in\Lambda}\mathcal{H}_\mathrm{p}\big)$ and that none are from $\bigcup_{\mathrm{p}\in\Lambda}(\hat{\gamma}_\mathrm{p}^+ \cup \hat{\gamma}_\mathrm{p}^-)$. The second property requires that the union of the curves from Θ^0 intersects M_δ as G segments that give the same pairing of the index 1 and index 2 critical points of f as that given by the third bullet of Proposition 2.4 using the zero locus of α.

The statement of the third property requires introducing notation from Proposition II.2.7. This proposition characterizes a segment of an integral curve of v in a version of $\mathcal{H}_\mathrm{p}$ that starts on the $u<0$ boundary and ends on the $u>0$ boundary. Proposition II.2.7 characterizes such a segment by an integer, denoted by $\mathfrak{k}_{\mathfrak{p}}$. This $\mathfrak{k}_{\mathfrak{p}}$ is such that the total change in the ϕ angle along the segment in $\mathcal{H}_{\mathfrak{p}}$ can be written as $\sigma+2\pi\mathfrak{k}_{\mathfrak{p}}$ with $\sigma\in[0,2\pi)$. The first two properties imply that the union of the curves from Θ^0 intersect each $\mathfrak{p}\in\Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ as a single segment of the sort just described. This understood, the third property requires that each of the corresponding $\mathfrak{p}\in\Lambda$ versions of $\mathfrak{k}_{\mathfrak{p}}$ be 0.

Let Θ^1 denote the subset of pairs of the form (γ,m) where $\gamma\in\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^{+}\cup\hat{\gamma}_{\mathfrak{p}}^{-})$ is a component of the zero locus of $\alpha_{\diamond 11}$ and m is the integer that is used to define $(A_{\diamond 11},\psi_{\diamond 11})$ near γ via (A-44).

Part 1 of what follows uses the sets Θ^0 and Θ^1 and a real number $\mathrm{z}>c_0$ to specify a pair in $\mathrm{Conn}(E)\times C^{\infty}(Y;\mathbb{S})$. This pair is denoted in what follows by $\mathfrak{c}(\mathrm{z})$. Each such pair has its corresponding operator $\mathfrak{L}_{\mathfrak{c}(\mathrm{z}),\mathrm{z}}$. Part 2 of this subsection states and then proves two lemmas that supply an a priori upper bound for the absolute value of the spectral flow between any $\mathrm{z}=\mathrm{z}_0$ and $\mathrm{z}=\mathrm{z}_1$ version of $\mathfrak{L}_{\mathfrak{c}(\mathrm{z}),\mathrm{z}}$. Part 3 states and then proves a proposition that compares the absolute value of the spectral flow between any of the latter versions of $\mathfrak{L}_{\mathfrak{c}(\mathrm{z}),\mathrm{z}}$ and the version that is defined by taking R very large, $\mathrm{z}=\mathrm{R}$ and $\mathfrak{c}$ to be $(\alpha_{\bullet 1},\psi_{\bullet 1})$ as defined using the chosen value for R.

Part 1 This part of the subsection defines the pair $\mathfrak{c}(\mathrm{z})\in\mathrm{Conn}(E)\times C^{\infty}(Y;\mathbb{S})$ for a given $\mathrm{z}>c_0$. The definition of $\mathfrak{c}(\mathrm{z})$ on the radius $(c_v^4+3c_v^3)\mathrm{z}^{-1/2}$ tubular neighborhood of any given curve from Θ^1 is given by (A-44) with the integer m coming from the relevant pair in Θ^1.

The four steps that follow define $\mathfrak{c}(\mathrm{z})$ on the complement in Y of the union of the radius $c_v^4\mathrm{z}^{-1/2}$ tubular neighborhoods of the curves from Θ^1. By way of a look ahead, Section Aa's construction is used to define $\mathfrak{c}(\mathrm{z})$ on this part of Y.

Step 1 Let c_v denote the constant that is used to define $(A_{\diamond 11},\psi_{\diamond 11})$. Take $\mathrm{z}\ge c_0$ and introduce $Y_{*\Lambda}$ to denote the complement in Y of the union of the radius $c_v^4\mathrm{z}^{-1/2}$ tubular neighborhoods of the curves from the set $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^{+}\cup\hat{\gamma}_{\mathfrak{p}}^{-})$. Define $T_{*\Lambda}$ to be the subset of $Y-Y_{*\Lambda}$ that consists of the components that do not contain curves from Θ^1. The data consisting of c_v, $\rho_*=c_v^2\mathrm{z}^{-1/2}$, $T_{*\Lambda}$ and $\Theta=\Theta^0$ supply most but not all of what is needed in Section Aa to define a pair consisting of a Hermitian connection on $E|_{Y_{*\Lambda}\cup T_{*\Lambda}}$ and a section of $\mathbb{S}$ over $Y_{*\Lambda}\cup T_{*\Lambda}$.

The definitions in Section Aa requires the specification of coordinates from Part 4 of Section Aa for each curve in Θ^0. The latter are defined from a chosen isometric isomorphism over each such curve between K^{-1} and the product bundle. Make these choices.

Section Aa also requires isomorphisms between E and the product bundle over certain subsets of $Y_{*\Lambda} \cup T_{*\Lambda}$. These isomorphisms are defined in Step 4. Steps 2 and 3 supply necessary input for the definition in Step 4.

The pair $\mathfrak{c}(\mathrm{z})$ on $Y_{*\Lambda} \cup T_{*\Lambda}$ is the pair that is supplied by Section Aa using the data c_v, $\rho_* = c_v^2 \mathrm{z}^{-1/2}$, $T_{*\Lambda}$, $\Theta = \Theta^0$, the chosen isomorphisms over the curves in Θ^0 between K^{-1} and the product bundle, and the promised isomorphisms between E and the product bundle over the relevant subsets of $Y_{*\Lambda} \cup T_{*\Lambda}$.

Step 2 Section Aa introduces an open cover of $Y_{*\Lambda} \cup T_{*\Lambda}$ consisting of a set U_0 and a collection of sets $\{U_\gamma\}_{\gamma \in \Theta^0}$. The set U_0 is the complement of the union of the radius $c_v^2 \mathrm{z}^{-1/2}$ tubular neighborhoods of the curves in Θ^0. Meanwhile, each $\gamma \in \Theta^0$ version of U_γ is the radius $4c_v^2 \mathrm{z}^{-1/2}$ tubular neighborhood of γ. The construction of $\mathfrak{c}(\mathrm{z})$ requires an isomorphism between E and the product bundle over U_0 and an isomorphism between E and the product bundle over each set from the collection $\{U_\gamma\}_{\gamma \in \Theta^0}$.

Fix $\gamma \in \Theta^0$ to define the isomorphism between E and the product bundle over U_γ. To do this, note that the sets Θ^0 and Θ^α enjoy a $1{-}1$ correspondence with partnered elements being homotopic in $M_\delta \cup \big(\bigcup_{\mathfrak{p} \in \Lambda} \mathcal{H}_\mathfrak{p}\big)$. Moreover, the partners intersect M_δ as arcs that are isotopic via an isotopy that moves points a distance at most $c_0 \delta$, this being a consequence of Lemma II.2.5. Let υ^α denote γ's partner from Θ^α.

Choose a smoothly embedded, oriented surface in $[0,1] \times Y$ with the properties listed below:

(C-32)
- The surface intersects $[0, c_0^{-1}) \times Y$ as $[0, c_0^{-1}) \times \upsilon^\alpha$.
- The surface intersects $(c_0^{-1}, 1] \times Y$ as $(c_0^{-1}, 1] \times \gamma$.
- The surface intersects $[0,1] \times M_\delta$ as an embedded rectangle of width less than $c_0 \delta$ that intersects each constant f surface transversely as a single arc.
- The surface intersects the boundary of any radius δ coordinate ball in M_δ transversely as a single arc.
- The projection of the surface to Y intersects only the $\mathfrak{p} \in \Lambda$ versions of $\mathcal{H}_\mathfrak{p}$ that are crossed by υ^α and γ, and its projection in any such $\mathcal{H}_\mathfrak{p}$ is disjoint from $\hat{\gamma}_\mathfrak{p}^+$ and $\hat{\gamma}_\mathfrak{p}^-$.

Such a surface can be constructed by mimicking what is done in Step 3 of the proof of Lemma II.5.3 to construct the latter's surface Z_+. Use S_γ to denote the chosen surface.

Fix $\mathrm{R} > c_0 c_v^6 \mathrm{r}$ suitable for defining the path $\{(A_{\bullet\tau}, \psi_{\bullet\tau})\}_{\tau \in [0,1]}$ and in any event such that all points in S_γ have distance at least $(c_v^4 + 3c_v^3)\mathrm{R}^{-1/2}$ from each curve in the set $\bigcup_{\mathfrak{p} \in \Lambda} (\hat{\gamma}_\mathfrak{p}^+ \cup \hat{\gamma}_\mathfrak{p}^-)$. Let $U_S \subset [0,1] \times Y$ denote a tubular neighborhood of S_γ

that intersects $\{0\}\times Y$ as the radius $4c_{\upsilon}^2\mathrm{R}^{1/2}$ tubular neighborhood of υ^{α} and that it intersects $\{1\}\times Y$ as U_{γ}. Require in addition that points in U_S have distance $(c_{\upsilon}^4+3c_{\upsilon}^3)\mathrm{R}^{-1/2}$ from $\hat{\gamma}_{\mathrm{p}}^{+}$ and $\hat{\gamma}_{\mathrm{p}}^{-}$.

Step 3 Let $\pi\colon [0,1]\times Y\to Y$ denote the projection to the Y factor. As explained in the subsequent paragraphs, the section $\alpha_{\diamond 11}$ extends over $[0,1]\times Y$ as a section of π^*E with zero locus $\big(\bigcup_{\gamma\in\Theta^0} S\gamma\big)\cup\big(\bigcup_{(\gamma,m)\in\Theta^1}[0,1]\times\gamma\big)$ and with transversal zeros along each S_{γ}.

The explanation starts with the 1–cycle $\sum_{\upsilon^{\alpha}\in\Theta^{\alpha}}[\upsilon^{\alpha}]+\sum_{(\gamma,m)\in\Theta^1} m[\gamma]$, where $[\cdot]$ denotes the cycle defined by the fundamental class of the indicated loop. This sum is the weighted sum of the components of the zero locus of $\alpha_{\diamond 11}$ with the weight of a component being the degree of vanishing of $\alpha_{\diamond 11}$ on a small radius transverse disk centered on the given component. The class of this cycle in $H_1(Y;\mathbb{Z})$ is Poincaré dual to the first Chern class of E because $\alpha_{\diamond 11}$ is a section of E.

The first Chern class of E is also Poincaré dual to the class defined by the 1–cycle $\sum_{\gamma\in\Theta^0}[\gamma]+\sum_{(\gamma,m)\in\Theta^1} m[\gamma]$, and, as a consequence, the class of the relative 2–cycle $\sum_{\gamma\in\Theta^0}[S_{\gamma}]+\sum_{(\gamma,m)\in\Theta^1} m[[0,1]\times\gamma]$ on $[0,1]\times Y$ is Poincaré dual to the first Chern class of π^*E. This being the case, there is a section of π^*E whose zero locus defines this same relative 2–cycle. Moreover, there exists such a section with transverse zeros along each S_{γ} and the same local behavior as $\alpha_{\diamond 11}$ near the origin of any transverse disk in $\{0\}\times Y$ with center on a curve from Θ^1. Use $\hat{\alpha}$ to denote such a section and use $\hat{\alpha}|_0$ to denote its restriction to $\{0\}\times Y$. The latter can be written as $u\cdot\alpha_{\diamond 11}$ with u being a smooth map from the complement in Y of $\big(\bigcup_{\upsilon^{\alpha}\in\Theta^{\alpha}}\upsilon^{\alpha}\big)\cup\big(\bigcup_{(\gamma,m)\in\Theta^1}\gamma\big)$ to $\mathbb{C}-\{0\}$. The section $\alpha_{\diamond 11}$ has the desired extension if u extends as a map to $\mathbb{C}-\{0\}$ from the complement $[0,1]\times Y$ of $\big(\bigcup_{\gamma\in\Theta^0} S_{\gamma}\big)\cup\big(\bigcup_{(\gamma,m)\in\Theta^1}[0,1]\times\gamma\big)$.

Let Y^{α} denote the complement in Y of $\big(\bigcup_{\upsilon^{\alpha}\in\Theta^{\alpha}}\upsilon^{\alpha}\big)\cup\big(\bigcup_{(\gamma,m)\in\Theta^1}\gamma\big)$ and let X^{α} denote the complement in $[0,1]\times Y$ of $\big(\bigcup_{\gamma\in\Theta^0} S_{\gamma}\big)\cup\big(\bigcup_{(\gamma,m)\in\Theta^1}[0,1]\times\gamma\big)$. The map u will extend if the restriction homomorphism from $H^1(X^{\alpha};\mathbb{Z})$ to $H^1(Y^{\alpha};\mathbb{Z})$ is surjective; and this is the case if the inclusion homomorphism from $H_1(Y^{\alpha};\mathbb{Z})/\mathrm{tors}$ to $H_1(X^{\alpha};\mathbb{Z})/\mathrm{tors}$ is injective. To prove that this is so, note that its composition with the inclusion homomorphism $H_1(X^{\alpha};\mathbb{Z})$ to $H_1([0,1]\times Y;\mathbb{Z})$ is the same as the composition of the homomorphism from $H_1(Y^{\alpha};\mathbb{Z})$ to $H_1(Y;\mathbb{Z})$ with the isomorphism given by the pushforward of π. This understood, the claimed injectivity follows from the fact that the kernel of the inclusion homomorphism from $H_1(Y^{\alpha};\mathbb{Z})$ to $H_1(Y;\mathbb{Z})$ is generated by the linking circles of the transverse disks centered on the various curves from Θ^1.

Step 4 Let $\hat{\alpha}$ now denote an extension of $\alpha_{\diamond 11}$ to a section of π^*E with zero locus $\big(\bigcup_{\gamma\in\Theta^0} S_\gamma\big) \cup \big(\bigcup_{(\gamma,m)\in\Theta^1} [0,1]\times\gamma\big)$ that vanishes transversely along each S_γ and is equal to $\pi^*\alpha_{\diamond 11}$ near each curve from Θ^1. The restriction of this section to $\{1\}\times Y$ is denoted in what follows by $\hat{\alpha}|_1$. This is a section of E. The required isomorphism over U_0 between E and $U_0\times\mathbb{C}$ sends $\hat{\alpha}|_1$ to its absolute value, $\big|\hat{\alpha}|_1\big|$.

Fix a curve $\gamma\in\Theta^0$. The definition of the required isomorphism between $E|_{U_\gamma}$ and $U_\gamma\times\mathbb{C}$ uses the chosen isomorphism between $K^{-1}|_\gamma$ and $\gamma\times\mathbb{C}$ to define the coordinates (t,z) on U_γ from Part 4 of Section Aa. Granted these coordinates, the desired isomorphism over U_γ between E and the product bundle takes $\hat{\alpha}|_1$ to $\big|\hat{\alpha}|_1\big| z/|z|$.

Part 2 This part of the subsection supplies two lemmas that summarize some salient features of the pairs defined in Part 1.

Lemma C.6 *There exists $\kappa>1$ and, given $c_v>\kappa$, there exists $\kappa_{c_v}>\kappa$ with the following significance: Suppose that $\mathrm{r}\ge\kappa_{c_v}c_v^{10}$ and suppose that $(A,\psi=(\alpha,\beta))$ is a solution to the (r,μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. These values of c_v and r are suitable for defining $\mathfrak{c}(\mathrm{z})$ for $\mathrm{z}\ge\kappa_{c_v}$ given an isometric isomorphism between K^{-1} and the product bundle over each curve from Θ^0, and given also a surface of the sort described by (C-32) for each curve from Θ^0.*

- *The resulting $\mathfrak{c}(\mathrm{z})$ does not depend on the chosen set of isometric isomorphisms.*
- *The resulting $\mathfrak{c}(\mathrm{z})$ depends on the chosen surface and then the extension $\hat{\alpha}$ as follows:*
  - (a) *Respective versions of $\mathfrak{c}(\mathrm{z})$ that are defined by different sets of surfaces and extensions differ by the action of a map from Y to S^1.*
  - (b) *The homology class of this map defines a class in $H^1(Y;\mathbb{Z})$ that is Poincaré dual to a class from the $\bigoplus_{\mathfrak{p}\in\Lambda} H_2(\mathcal{H}_\mathfrak{p};\mathbb{Z})$ summand in (1-4).*

This lemma is proved in a moment.

The next lemma supplies an a priori bound for the absolute value of the spectral flow between versions of $\mathfrak{L}_{\mathfrak{c}(\mathrm{z}),\mathrm{z}}$ that are defined by distinct choices for z.

Lemma C.7 *There exists $\kappa>1$ and, given $c_v>\kappa$, there exists $\kappa_{c_v}>\kappa$ with the following significance: Suppose that $\mathrm{r}\ge\kappa_{c_v}c_v^{10}$ and suppose that $(A,\psi=(\alpha,\beta))$ is a solution to the (r,μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Use the data from α to define $\mathfrak{c}(\mathrm{z})$ for $\mathrm{z}\ge\kappa_{c_v}$. The absolute value of the spectral flow between the $\mathrm{z}=\kappa_{c_v}$ and any $\mathrm{z}=\mathrm{z}_1\ge\kappa_{c_v}$ version of $\mathfrak{L}_{\mathfrak{c}(\mathrm{z}),\mathrm{z}}$ is bounded by κ.*

The proof of this lemma is given directly. The proof assumes that the first bullet of Lemma C.6 is true.

Proof of Lemma C.7 Except for one added remark, the proof is identical to that used to prove Proposition C.4. The added remark concerns the use of Lemma A.8 in the proof. In particular, the bounds given for the term $\mathfrak{e}(f)$ in this lemma depend implicitly on bounds for the functions ν and μ. Meanwhile, the latter are defined by the coordinates from Part 4 of Section Aa and thus by the chosen isomorphism over the curve in question between K^{-1} and the product bundle. As Lemma C.6 asserts that $\mathfrak{c}(\mathrm{z})$ does not depend on the chosen isomorphism, choose one with $|\nu| + |\mu| \le c_0$. □

Proof of Lemma C.6 To prove the first bullet, assume that a choice of surfaces has been made for each curve from Θ^0. Fix $\gamma \in \Theta^0$ and choose an isometric isomorphism between $K^{-1}|_\gamma$ and $\gamma \times \mathbb{C}$ to define $\mathfrak{c}(\mathrm{z})$ on U_γ. The formulas for $\mathfrak{c}(\mathrm{z})$ are given in (A-8) and (A-9). Granted these formulas, the observations made in the first two paragraphs of Part 5 in Section Ba apply and prove that $\mathfrak{c}(\mathrm{z})$ does not change when the isomorphism changes.

To see about the second bullet, suppose that $\{S_\gamma\}_{\gamma\in\Theta^0}$ and $\{S'_\gamma\}_{\gamma\in\Theta^0}$ are two sets of surfaces of the sort described in (C-32). Let φ_0 and φ'_0 denote the corresponding isomorphism between E and the product bundle over U_0. Write φ'_0 as $u_0\varphi_0$ with u_0 being a map from U_0 to S^1. Fix a coordinates from Part 4 of Section Aa for each curve in Θ^0. Given $\gamma \in \Theta^0$, let φ_γ and φ'_γ denote the corresponding isomorphisms between E and the product bundle over U_γ. Write φ'_γ as $u_\gamma\varphi_\gamma$.

The respective primed and unprimed transition maps from $U_0 \cap U_\gamma$ to S^1 that identify the product structure for E over U_0 with that over U_γ are identical because the same coordinates for U_γ are used for the two cases. Use this fact with the formulas in Section Aa to conclude that $u_0 = u_\gamma$ on $U_0 \cup U_\gamma$. This being the case, the collection of maps consisting of u_0 and $\{u_\gamma\}_{\gamma\in\Theta^0}$ define a smooth map from Y to S^1 that relates the primed and unprimed versions of $\mathfrak{c}(\mathrm{z})$. Let $u\colon Y \to S^1$ denote this map.

Consider now the class defined by u in $H^1(Y;\mathbb{Z})$. This class is determined by the integral of $-\frac{i}{2\pi}u^{-1}du$ over a basis of cycles in Y that generate the free $\mathbb{Z}$–module $H_1(Y;\mathbb{Z})/\mathrm{tors}$. Part 4 of Section 1.2 describes the set $\{\gamma^{(z)}\}_{z\in¥}$ of $1+b_1(M)$ integral curves of v in $M_\delta \cup \mathcal{H}_0$ with the following property: the integral of $-\frac{i}{2\pi}u^{-1}du$ over these cycles detects the image in the summand $H_2(M;\mathbb{Z}) \oplus H_2(\mathcal{H}_0;\mathbb{Z})$ of the Poincaré dual in $H_2(Y;\mathbb{Z})$ of u's cohomology class. To prove that this image is zero, introduce $\widehat{\alpha}$ and $\widehat{\alpha}'$ to denote the corresponding $\{S_\gamma\}_{\gamma\in\Theta^0}$ and $\{S'_\gamma\}_{\gamma\in\Theta^0}$ extensions of $\alpha_{\diamond 11}$. Both $\widehat{\alpha}$ and $\widehat{\alpha}'$ are nonzero on the product of $[0,1]$ with the complement in $M_\delta \cup \mathcal{H}_0$ of the union of the radius $c_0\delta$ tubular neighborhoods of the component segments of $\bigcup_{\gamma\in\Theta^0}(\gamma\cap M_\delta)$. Let T denote this small radius tubular neighborhood of $\bigcup_{\gamma\in\Theta^0}(\gamma\cap M_\delta)$. Keep in mind that this set T is disjoint from the set $\bigcup_{z\in¥}([0,1]\times\gamma^{(z)})$. This fact can be used to exhibit a homotopy on a neighborhood of $\bigcup_{z\in¥}\gamma(z)$ between u and the constant map to $1\in S^1$: the desired homotopy is parametrized by $[0,1]$ with the $\tau\in[0,1]$ member of the homotopy being the restriction to $\{\tau\}\times((M_\delta\cup\mathcal{H}_0)-T)$ of $(\widehat{\alpha}/|\widehat{\alpha}|)(\widehat{\alpha}'/|\widehat{\alpha}'|)^{-1}$. □

Part 3 This part of the subsection concerns the spectral flow difference between very large R versions of $\mathfrak{L}_{\mathfrak{c},\mathrm{R}}$ as defined using $\mathfrak{c}=(A_{\bullet 1},\psi_{\bullet 1})$ and the corresponding $\mathrm{z}=\mathrm{R}$ version of the operator $\mathfrak{L}_{\mathfrak{c}(\mathrm{z}),\mathrm{z}}$. The proposition that follows says what is needed about this difference.

Proposition C.8 *There exists $\kappa\geq 100$, and, given $c_v\geq\kappa$, there exists $\kappa_{c_v}>\kappa$ with the following significance: Suppose that $\mathrm{r}\geq\kappa_{c_v}c_v^{10}$ and suppose that $(A,\psi=(\alpha,\beta))$ is a solution to the (r,μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. The values of κ, c_v and r are suitable for defining $(A_{\diamond 11},\psi_{\diamond 11})$ and any $\mathrm{R}\geq\kappa c_v^6\mathrm{r}$ version of $(A_{\bullet 1},\psi_{\bullet 1})$. Fix any sufficiently large R and use it to define the pair $(A_{\bullet 1},\psi_{\bullet 1})$. Define the $\mathrm{z}=\mathrm{R}$ version of $\mathfrak{c}(\mathrm{z})$ using any chosen set of isomorphisms between K^{-1} and the product bundle over the curves from Θ^0, and using any chosen set of surfaces of the sort described in (C-32) for the curves in Θ^0. The norm of the difference between the respective values of the spectral flow function $\mathfrak{f}_s$ at $(A_{\bullet 1},\psi_{\bullet 1})$ and at the $\mathrm{z}=\mathrm{R}$ version of $\mathfrak{c}(\mathrm{z})$ is bounded by κ.*

By way of a parenthetical remark, what is said in the second bullet of Lemma C.6 is consistent with what is said in Proposition C.8. This follows from three facts. Here is the first: The function $\mathfrak{f}_s$ is invariant under the action on $\mathrm{Conn}(E)\times C^\infty(Y;\mathbb{S})$ of the subgroup of maps from Y to S^1 whose corresponding class in $H^1(Y;\mathbb{Z})$ has zero cup product with the first Chern class of the line bundle $\det(\mathbb{S})$. The second fact concerns the cup product pairing between this first Chern class and a given class $\sigma\in H^1(Y;\mathbb{Z})$: this is the same as the pairing between the first Chern class of $\det(\mathbb{S})$ and the Poincaré dual of σ in $H_2(Y;\mathbb{Z})$. Here is the final fact: the first Chern class of $\det(\mathbb{S})$ annihilates the $\bigoplus_{\mathfrak{p}\in\Lambda}H^2(\mathcal{H}_{\mathfrak{p}};\mathbb{Z})$ summand of $H_2(Y;\mathbb{Z})$.

Proof of Proposition C.8 If R is sufficiently large, then the arguments from Section 2b of [21] with only notational changes can be imported to prove the proposition. □

Ce Proof of Proposition 2.6

This section gives the proof of Proposition 2.6. The argument has five steps.

Step 1 It is convenient to choose a finite set of surrogates for the pair (A_E, ψ_E). This set of surrogates is indexed by the set of all possible pairs of the form (Θ^0, Θ^1) that can arise in the previous subsection from large r and (r, μ) versions of (1-13). This indexing set is denoted by $\mathcal{Z}_{\mathrm{HF}} \times \mathcal{Z}^1$.

By way of a precise definition, the set $\mathcal{Z}_{\mathrm{HF}}$ has distinct elements of the following sort: Let Θ^0 denote a given element. This set Θ^0 consists of at most G distinct, closed integral curves of v. All curves in the set Θ^0 lie in $M_\delta \cup \big(\bigcup_{\mathfrak{p}\in\Lambda} \mathcal{H}_{\mathfrak{p}}\big)$ and none are from $\bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$. The union of the curves from Θ^0 intersects M_δ as G segments that give the same pairing of the index 1 and index 2 critical points of f. Finally, the union of the integral curves in v intersects each $\mathfrak{p} \in \Lambda$ version of $\mathcal{H}_{\mathfrak{p}}$ as a single segment that runs from the $u < 0$ boundary of $\mathcal{H}_{\mathfrak{p}}$ to the $u > 0$ boundary. The intersection is characterized by an integer, $\mathfrak{k}_{\mathfrak{p}}$, as in Proposition II.2.7, and the segment in question has $\mathfrak{k}_{\mathfrak{p}} = 0$.

As explained in [8; 9], the set $\mathcal{Z}_{\mathrm{HF}}$ determines a set of generators for the Heegaard Floer homology on M. In any event, $\mathcal{Z}_{\mathrm{HF}}$ has finitely many elements.

The set $\mathcal{Z}^1$ consists of elements of the following sort: Let Θ^1 denote a given element. This set Θ^1 consists of pairs of the form (γ, m) where $\gamma \in \bigcup_{\mathfrak{p}\in\Lambda}(\hat{\gamma}_{\mathfrak{p}}^+ \cup \hat{\gamma}_{\mathfrak{p}}^-)$ and where m is a positive integer. No two pairs share the same integral curve component. The integer m is bounded by $c_0 c_v^3$. The set Θ^1 is also finite.

Take each Θ^0 in $\mathcal{Z}_{\mathrm{HF}}$ and assign once and for all an isometric isomorphism between K^{-1} and the product bundle over each curve from Θ^0. Let κ_{c*} denote the larger of the versions of κ_{c_v} that appear in Lemmas C.6 and C.7. Fix $\mathrm{z}_0 = \kappa_{c*}^2$ and assign once and for all a product structure for E over the radius $4c_v^2 \mathrm{z}_0^{-1/2}$ tubular neighborhood of each curve from Θ^0. Fix once and for all a product structure for E over the complement in Y of the union of the radius $c_v^4 \mathrm{z}_0^{-1/2}$ tubular neighborhoods of the curves from Θ^0.

Take each pair $\hat{\mathrm{o}} = (\Theta^0, \Theta^1) \in \mathcal{Z}_{\mathrm{HF}} \times \mathcal{Z}^1$ and use the data c_v, $\mathrm{z} = \mathrm{z}_0$ with the product structures chosen in the preceding paragraph to construct the corresponding version of the pair $\mathfrak{c}(\mathrm{z} = \mathrm{z}_0)$ as instructed in Step 1 of Part 1 of Section Cd. Denote this pair by $\mathfrak{c}_{\hat{\mathrm{o}}}$. Write this pair as $(A_{\hat{\mathrm{o}}}, \psi_{\hat{\mathrm{o}}})$ and use $\alpha_{\hat{\mathrm{o}}}$ to denote the E summand component of $\psi_{\hat{\mathrm{o}}}$.

Since $\mathcal{Z}_{\mathrm{HF}} \times \mathcal{Z}^1$ is a finite set, there exists a purely c_v–dependent $\kappa_c > 1$ with the following property: Fix $\hat{\mathrm{o}} \in \mathcal{Z}_{\mathrm{HF}} \times \mathcal{Z}^1$. Then the connection $A_{\hat{\mathrm{o}}}$ can be written as

$A_E + \hat{a}_{\hat{o}}$ with $\hat{a}_{\hat{o}}$ being an $i\mathbb{R}$–valued 1–form with $|\hat{a}_{\hat{o}}| \le \kappa_c$. In addition, the norm of the difference between the respective values of $\mathfrak{f}_s$ at (A_E, ψ_E) and $\mathfrak{c}_{\hat{o}}$ is bounded by κ_c.

Step 2 Fix $c_v \ge c_0$ and $\mathrm{r} \ge \kappa_c c_v^{10}$ with κ_c being a purely c_v–dependent constant. Assume that c_v and κ_c are suitable for invoking the results in Appendices A and B and the previous subsections of this Appendix C. Suppose that $(A, \psi = (\alpha, \beta))$ is a solution to the (r, μ) version of (1-13) with μ a given element in Ω with $\mathcal{P}$–norm smaller than 1. Assume in addition that $|\mathrm{x}_{\mathbb{S}}(A)| \le c_0$. By assumption, the values of c_v and r are suitable for defining from (A, ψ) the pair $(A_{\diamond 11}, \psi_{\diamond 11})$ and any given $\mathrm{R} \ge \kappa c_v^6 \mathrm{r}$ version of $(A_{\bullet 1}, \psi_{\bullet 1})$. Fix any sufficiently large R and use it define both $(A_{\bullet 1}, \psi_{\bullet 1})$ and the $\mathrm{z} \in (\kappa_{c*}, \mathrm{R}]$ versions of $\mathfrak{c}(\mathrm{z})$. Use $\mathfrak{c}_{(A,\psi)}(\mathrm{z})$ to denote such a version. Write the pair $\mathfrak{c}_{(A,\psi)}(\mathrm{z})$ as $(A_{\mathrm{z}}, \psi_{\mathrm{z}})$ and write the E summand of ψ_{z} as α_{z}. Use $\hat{o} \in \mathcal{Z}_{\mathrm{HF}} \times \mathcal{Z}^1$ in what follows to denote the element (Θ^0, Θ^1) that is used to define $\mathfrak{c}_{(A,\psi)}(\mathrm{z})$. This element is determined by (A, ψ).

Step 3 Let T denote the union of the radius $c_0\delta$ tubular neighborhoods of the intersection between M_δ and the curves from Θ^0. This set T has distance at least c_0^{-1} from the curves in the set $\{\gamma(z)\}_{z \in ¥}$. Moreover, it contains the M_δ part of the zero locus of α and the M_δ part of the zero locus of $\alpha_{\hat{o}}$. With this understood, the section α on $(M_\delta \cup \mathcal{H}_0) - T$ can be written as $\alpha = |\alpha| u \alpha_{\hat{o}}$ with u being a map from $(M_\delta \cup \mathcal{H}_0) - T$ to S^1. Write $\nabla_A \alpha$ on $(M_\delta \cup \mathcal{H}_0) - T$ as $(d|\alpha| + (u^{-1} du + \hat{a}_A - \hat{a}_{\hat{o}})|\alpha|)\alpha_{\hat{o}}$.

Lemma 2.1 asserts that $1 - |\alpha| \le c_0 \mathrm{r}^{-1}$ and that $|\nabla_A \alpha| \le c_0$ on $(M_\delta \cup \mathcal{H}_0) - T$. Given that $A = A_E + \hat{a}_A$ and $A_{\hat{o}} = A_E + \hat{a}_{\hat{o}}$, it follows that $|u^{-1} du + \hat{a}_A - \hat{a}_{\hat{o}}| \le \kappa_c$ on $(M_\delta \cup \mathcal{H}_0) - T$ with $\kappa_c \ge 1$ being a purely c_v–dependent constant. The latter bound has the following consequence: the absolute value of the integral of $-\frac{i}{2\pi} u^{-1} du$ over any curve from the set $\{\gamma^{(z)}\}_{z \in ¥}$ is bounded by κ_c, with κ_c again denoting a purely c_v–dependent constant.

Fix $\mathrm{z} \in (\kappa_{c*}, \mathrm{R})$. The zero locus of α_{z} in M_δ also lies in T. This understood, write α_{z} on $(M_\delta \cup \mathcal{H}_0) - T$ as $\alpha_{\mathrm{z}} = u_{\mathrm{z}} |\alpha|^{-1} \alpha$ with u_{z} being a smooth map to S^1. It follows from what is said by Step 4 of Part 3 in Section Ca and by Step 4 of Part 1 in Section Cd that the integral of the 1–form $-\frac{i}{2\pi} u_{\mathrm{z}}^{-1} du_{\mathrm{z}}$ is zero over any curve from the set $\{\gamma^{(z)}\}_{z \in ¥}$. In fact, the extension $\hat{\alpha}$ used in Step 4 of Part 1 in Section 1.3 can be chosen so that $u_{\mathrm{z}} = 1$.

Step 4 The zero locus of α_{z} and that of $\alpha_{\hat{o}}$ are identical, it being the union of the curves from Θ^0 and Θ^1. The latter fact implies that α_{z} can be written on the complement of this zero locus as $\alpha_{\mathrm{z}} = |\alpha_{\mathrm{z}}|\,|\alpha_{\hat{0}}|^{-1}\hat{u}_{\mathrm{z}}\alpha_{\hat{0}}$ with $\hat{u}_{\mathrm{z}}$ being a smooth map to S^1 from the complement in Y of the union of the curves from Θ^0 and Θ^1. It follows from what was said in Step 3 that the integral of $-\frac{i}{2\pi}\hat{u}_{\mathrm{z}}^{-1}\,d\hat{u}_{\mathrm{z}}$ is zero over any curve from the set $\{\gamma^{(z)}\}_{z\in¥}$.

Take $\mathrm{z} = \mathrm{z}_0$ now, this being the value of z that is used to define $(A_{\hat{0}}, \psi_{\hat{0}})$. It follows from the first bullet of Lemma C.6 that the $\mathrm{z} = \mathrm{z}_0$ version of $\hat{u}_{\mathrm{z}}$ extends to define a smooth map from the whole of Y to S^1 and that the $\mathrm{z} = \mathrm{z}_0$ version of the pair $(A_{\mathrm{z}}, \psi_{\mathrm{z}})$ can be written as $(A_{\mathrm{z}0} = A_{\hat{0}} - \hat{u}_{\mathrm{z}0}^{-1}\,d\hat{u}_{\mathrm{z}0}, \psi_{\mathrm{z}0} = \hat{u}_{\mathrm{z}0}\psi_{\hat{0}})$ on the whole of Y.

Step 5 The integral of $-\frac{i}{2\pi}\hat{u}_{\mathrm{z}0}^{-1}\,d\hat{u}_{\mathrm{z}0}$ over the curves from $\{\gamma^{(z)}\}_{z\in¥}$ is zero, and this implies that the Poincaré dual in $H_2(Y;\mathbb{Z})$ of the class in $H^1(Y;\mathbb{Z})$ defined by $\hat{u}_{\mathrm{z}0}$ lies in the $\bigoplus_{\mathrm{p}\in\Lambda} H_2(\mathcal{H}_{\mathrm{p}};\mathbb{Z})$ summand of $H_2(Y;\mathbb{Z})$. As noted previously, this summand has zero pairing with the first Chern class of $\det(S)$. It follows as a consequence that the spectral flow function $\mathfrak{f}_s$ has the same value on $\mathfrak{c}_{(A,\psi)}(\mathrm{z}_0)$ as it has on $\mathfrak{c}_{\hat{0}}$. This being so, the absolute value of $\mathfrak{f}_s$ on $\mathfrak{c}_{(A,\psi)}(\mathrm{z}_0)$ is bounded by κ_c, with κ_c being a purely $c_{\mathfrak{v}}$–dependent constant.

Lemma C.7 asserts that the norm of the difference between the values of $\mathfrak{f}_s$ at $\mathfrak{c}_{(A,\psi)}(\mathrm{z}_0)$ and at $\mathfrak{c}_{(A,\psi)}(\mathrm{z} = \mathrm{R})$ is bounded by a purely $c_{\mathfrak{v}}$–dependent constant, and so the absolute value of $\mathfrak{f}_s$ at $\mathfrak{c}_{(A,\psi)}(\mathrm{z} = \mathrm{R})$ is also bounded by such a constant. Proposition C.8 asserts that the norm of the difference between values of $\mathfrak{f}_s$ at $\mathfrak{c}_{(A,\psi)}(\mathrm{z} = \mathrm{R})$ and $(A_{\bullet 1}, \psi_{\bullet 1})$ is also bounded by a purely $c_{\mathfrak{v}}$–dependent constant. Proposition C.4 asserts that such is also the case for the norm of the difference between the values of $\mathfrak{f}_s$ at $(A_{\bullet 1}, \psi_{\bullet 1})$ and $(A_{\diamond 11}, \psi_{\diamond 11})$. Proposition C.2 says the same thing for the norm of the difference between values of $\mathfrak{f}_s$ at $(A_{\diamond 11}, \psi_{\diamond 11})$ and at $(A_{\diamond}, \psi_{\diamond})$. Proposition B.13 says this about the norm of the difference between values of $\mathfrak{f}_s$ at $(A_{\diamond}, \psi_{\diamond})$ and (A_*, ψ_*), and Proposition B.3 says this about the norm of the difference between the values of $\mathfrak{f}_s$ at (A, ψ) and at (A_*, ψ_*).

Adding all of these absolute value bounds verifies that the absolute value of $\mathfrak{f}_s$ at (A, ψ) is bounded by a purely $c_{\mathfrak{v}}$–dependent constant. □

References

[1] **F Bourgeois**, **Y Eliashberg**, **H Hofer**, **K Wysocki**, **E Zehnder**, *Compactness results in symplectic field theory*, Geom. Topol. 7 (2003) 799–888 MR

[2] **M Hutchings**, *The embedded contact homology index revisited*, from "New perspectives and challenges in symplectic field theory" (M Abreu, F Lalonde, L Polterovich, editors), CRM Proc. Lecture Notes 49, Amer. Math. Soc., Providence, RI (2009) 263–297 MR

[3] **M Hutchings**, *Embedded contact homology and its applications*, from "Proceedings of the International Congress of Mathematicians" (R Bhatia, A Pal, G Rangarajan, V Srinivas, M Vanninathan, P Gastesi, editors), volume II, Hindustan Book Agency, New Delhi (2010) 1022–1041 MR

[4] **M Hutchings**, *Taubes's proof of the Weinstein conjecture in dimension three*, Bull. Amer. Math. Soc. 47 (2010) 73–125 MR

[5] **M Hutchings**, **M Sullivan**, *Rounding corners of polygons and the embedded contact homology of* T^3, Geom. Topol. 10 (2006) 169–266 MR

[6] **M Hutchings**, **C H Taubes**, *The Weinstein conjecture for stable Hamiltonian structures*, Geom. Topol. 13 (2009) 901–941 MR

[7] **P Kronheimer**, **T Mrowka**, *Monopoles and three-manifolds*, New Mathematical Monographs 10, Cambridge Univ. Press (2007) MR

[8] **Ç Kutluhan**, **Y-J Lee**, **C H Taubes**, $\mathrm{HF} = \mathrm{HM}$*, I: Heegaard Floer homology and Seiberg–Witten Floer homology*, Geom. Topol. 24 (2020) 2829–2854

[9] **Ç Kutluhan**, **Y-J Lee**, **C H Taubes**, $\mathrm{HF} = \mathrm{HM}$*, II: Reeb orbits and holomorphic curves for the ech/Heegaard–Floer correspondence*, Geom. Topol. 24 (2020) 2855–3012

[10] **Ç Kutluhan**, **Y-J Lee**, **C H Taubes**, $\mathrm{HF} = \mathrm{HM}$*, III: Holomorphic curves and the differential for the ech/Heegaard Floer correspondence*, Geom. Topol. 24 (2020) 3013–3218

[11] **Y-J Lee**, **C H Taubes**, *Periodic Floer homology and Seiberg–Witten–Floer cohomology*, J. Symplectic Geom. 10 (2012) 81–164 MR

[12] **P Ozsváth**, **Z Szabó**, *Holomorphic disks and topological invariants for closed three-manifolds*, Ann. of Math. 159 (2004) 1027–1158 MR

[13] **P Ozsváth**, **Z Szabó**, *An introduction to Heegaard Floer homology*, from "Floer homology, gauge theory, and low-dimensional topology" (D A Ellwood, P S Ozsváth, A I Stipsicz, Z Szabó, editors), Clay Math. Proc. 5, Amer. Math. Soc., Providence, RI (2006) 3–27 MR

[14] **C H Taubes**, $\mathrm{SW} \Rightarrow \mathrm{Gr}$*: from the Seiberg–Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. 9 (1996) 845–918 MR Reprinted in "Seiberg–Witten and Gromov invariants for symplectic 4–manifolds" (R Wentworth, editor), First International Press Lecture Series 2, International, Somerville, MA (2000) 1–97

[15] **C H Taubes**, $\mathrm{Gr} \Rightarrow \mathrm{SW}$*: from pseudo-holomorphic curves to Seiberg–Witten solutions*, J. Differential Geom. 51 (1999) 203–334 MR Reprinted in "Seiberg–Witten and Gromov invariants for symplectic 4–manifolds" (R Wentworth, editor), First International Press Lecture Series 2, International, Somerville, MA (2000) 163–273

[16] **C H Taubes**, *Asymptotic spectral flow for Dirac operators*, Comm. Anal. Geom. 15 (2007) 569–587 MR

[17] **C H Taubes**, *The Seiberg–Witten equations and the Weinstein conjecture*, Geom. Topol. 11 (2007) 2117–2202 MR

[18] **C H Taubes**, *The Seiberg–Witten equations and the Weinstein conjecture, II: More closed integral curves of the Reeb vector field*, Geom. Topol. 13 (2009) 1337–1417 MR

[19] **C H Taubes**, *Embedded contact homology and Seiberg–Witten Floer cohomology, I*, Geom. Topol. 14 (2010) 2497–2581 MR

[20] **C H Taubes**, *Embedded contact homology and Seiberg–Witten Floer cohomology, II*, Geom. Topol. 14 (2010) 2583–2720 MR

[21] **C H Taubes**, *Embedded contact homology and Seiberg–Witten Floer cohomology, III*, Geom. Topol. 14 (2010) 2721–2817 MR

[22] **C H Taubes**, *Embedded contact homology and Seiberg–Witten Floer cohomology, IV*, Geom. Topol. 14 (2010) 2819–2960 MR

[23] **C H Taubes**, *Embedded contact homology and Seiberg–Witten Floer cohomology, V*, Geom. Topol. 14 (2010) 2961–3000 MR

[24] **C H Taubes**, *Correction to a lemma in "Embedded contact homology and Seiberg–Witten Floer homology, IV"* (2018) arXiv

Department of Mathematics, University at Buffalo
Buffalo, NY, United States

Institute of Mathematical Sciences, The Chinese University of Hong Kong
Shatin, NT, Hong Kong

Department of Mathematics, Harvard University
Cambridge, MA, United States

kutluhan@buffalo.edu, yjlee@math.cuhk.edu.hk,
chtaubes@math.harvard.edu

Proposed: Tomasz Mrowka Received: 17 February 2012
Seconded: Tobias H Colding, Yasha Eliashberg Revised: 16 February 2019