Expected Value and Variance

The **expected value** is given by:

$$\mu = \mathsf{E}[Y] = \begin{cases} \sum_Y y f_Y(y) & Y \text{ discrete} \\ \int_{-\infty}^{\infty} y f_Y(y) & Y \text{ continuous} \end{cases}$$

The population **variance** is:

$$\sigma^2 = \mathsf{Var}(Y) = \mathsf{E}\left[(Y-\mu)^2\right]$$

The population **standard deviation** is the positive square root of the variance.

Covariance and Correlation

- The **covariance** between X and Y is:

$$\mathsf{Cov}(X,Y) = \mathsf{E}\left[(X-\mu_X)(Y-\mu_Y)\right]$$

- The **correlation** between X and Y is:

$$\rho = \frac{\mathsf{Cov}(X,Y)}{\sigma_X \sigma_Y}$$

Linear Combinations of Random Variables

Suppose:

$$U_1 = a_0 + \sum_{i=1}^{n} a_i Y_i, \qquad U_2 = b_0 + \sum_{j=1}^{m} b_j X_j$$

- $\mathsf{E}[U_1] = a_0 + \sum_{i=1}^n a_i \mathsf{E}[Y_i]$
- $\mathsf{Var}(U_1) = \sum_{i=1}^n a_i^2 \mathsf{Var}(Y_i) + 2\sum_{i<j} \mathsf{Cov}(Y_i, Y_j)$
- $\mathsf{Cov}(U_1, U_2) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \mathsf{Cov}(Y_i, X_j)$

Properties of Expectation/Variance

- $\mathsf{E}[a] = a$ — a constant
- $\mathsf{E}[bY] = b\mathsf{E}[Y]$ — b constant
- $\mathsf{E}[a+bY] = a + b\mathsf{E}[Y]$
- $\mathsf{Var}(a) = 0$
- $\mathsf{Var}(a+bY) = \mathsf{Var}(bY) = b^2\mathsf{Var}(Y)$
- $\mathsf{Var}(Y) = \mathsf{E}[Y^2] - \mathsf{E}[Y]^2$

Properties of Covariance/Correlation

- $\mathsf{Cov}(X,Y) = \mathsf{E}[XY] - \mathsf{E}[X]\mathsf{E}[Y]$
- $\mathsf{Cov}(a_1 + a_2 Y, b_1 + b_2 X) = a_2 b_2 \mathsf{Cov}(X,Y)$
- $\mathsf{Cov}(Y,X) = \mathsf{Cov}(X,Y)$
- $\mathsf{Cov}(Y,Y) = \mathsf{Var}(Y)$

$\mathsf{Var}(Y_1+Y_2) = \mathsf{Var}(Y_1) + \mathsf{Var}(Y_2) + 2\mathsf{Cov}(Y_1,Y_2)$

$\mathsf{Var}(Y_1-Y_2) = \mathsf{Var}(Y_1) + \mathsf{Var}(Y_2) - 2\mathsf{Cov}(Y_1,Y_2)$

Normal Distribution

- $f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(y-\mu)^2\right]$
- If $Y_i \sim \mathsf{N}(\mu_i, \sigma_i^2)$ then:

$$U = a_0 + \sum_{i=1}^{n} a_i Y_i \sim \mathsf{N}\left(\mathsf{E}[U], \mathsf{Var}[U]\right)$$

- If $Y \sim \mathsf{N}(\mu, \sigma^2)$, then $Z = \frac{Y-\mu}{\sigma} \sim \mathsf{N}(0,1)$.

Central Limit Theorem

- Suppose $Y_1, \ldots, Y_n \sim i.i.d.\ \mathsf{N}(\mu, \sigma^2)$. Then:

$$\bar{Y} \sim \mathsf{N}(\mu, \sigma^2/n)$$

- If $Y_1, \ldots, Y_n$ i.i.d. from a population with mean μ and standard deviation σ, then:

$$\frac{\bar{Y}-\mu}{\sigma/\sqrt{n}} \to \mathsf{N}(0,1) \text{ in distribution}$$

t-Distribution and F-Distribution

- Suppose that $Z \sim \mathsf{N}(0,1)$, $W \sim \chi^2_\nu$, and $Z \perp W$. Then:

$$T = \frac{Z}{\sqrt{W/\nu}} \sim t_\nu$$

- Suppose that $W_1 \sim \chi^2_{\nu_1}$, $W_2 \sim \chi^2_{\nu_2}$, and $W_1 \perp W_2$. Then:

$$F = \frac{W_1/\nu_1}{W_2/\nu_2} \sim F_{\nu_1,\nu_2}$$

Chi-Square Distribution

- If $Z \sim \mathsf{N}(0,1)$ then $Z^2 \sim \chi^2_1$
- If $Z_i \sim i.i.d.\ \mathsf{N}(0,1)$, then $Y = \sum_{i=1}^n Z_i^2 \sim \chi^2_n$
- If $Y_i \sim \chi^2_{\nu_i}, Y_i \perp Y_j$, then:

$$W = \sum_{i=1}^{n} Y_i \sim \chi^2_{\nu_1+\cdots+\nu_n}$$

- For a random sample from a normal population:

$$\bar{Y} \perp S^2 \quad \text{and} \quad \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$