Bias and Variance of an Estimator

- The **bias** of an estimator $\hat{\theta}$ is:
$$\text{Bias} = \mathsf{E}[\hat{\theta} - \theta] = \mathsf{E}[\hat{\theta}] - \theta$$
- The **variance** of an estimator $\hat{\theta}$ is:
$$\text{Var}(\hat{\theta}) = \mathsf{E}[(\hat{\theta} - \mathsf{E}[\hat{\theta}])^2]$$
- $\hat{\theta}_1$ is more **efficient** if $\text{Var}(\hat{\theta}_1) < \text{Var}(\hat{\theta}_2)$
- An estimator is **asymptotically unbiased** if $\lim_{n\to\infty} \mathsf{E}[\hat{\theta}] = \theta$

Least Squares Estimation

- Least squares estimation of a parameter θ is based on: $\min_{\theta} \mathsf{E}[(Y - \mathsf{E}[Y])^2]$
- The empirical version uses the criteria:
$$\min_{\theta} Q(\theta) = \min_{\theta} \sum_{i=1}^{n} (Y_i - \mathsf{E}_\theta[Y_i])^2$$
- The **normal equations**, derived by setting partial derivatives of Q equal to zero, may be used to obtain parameter estimates.

Maximum Likelihood Estimation

- The joint multivariate pdf of an independent sample $Y_1, \ldots, Y_n$ is given by:
$$f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = \prod_{i=1}^{n} f_{Y_i}(y_i) = L(\theta)$$
- $L(\theta)$ is the **likelihood function**.
- The **maximum likelihood estimator (MLE)**, $\hat{\theta}_{MLE}$, is defined as the point where $L(\theta)$ reaches its maximum.

Interval Estimation

- A $100(1-\alpha)$ % **confidence interval** for a parameter θ is a pair of statistics $\hat{\theta}_L$ and $\hat{\theta}_U$ such that:
$$P(\hat{\theta}_L < \theta < \hat{\theta}_U) = 1 - \alpha$$
- Let $Q = q(Y_1, \ldots, Y_n; \theta)$. If Q has a distribution that does not depend on θ, then Q is a **pivotal quantity**.

Hypothesis Testing

- A **hypothesis** is a statement about characteristics of a probability distribution.
- The **p-value** equals the probability that the test statistic is at least as extreme as the observed value.

$\alpha = P($**Type I error**$) = P(\text{Reject } H_0 | H_0 \text{ true})$

$\beta = P($**Type II error**$) = P(\text{Not reject } H_0 | H_0 \text{ F})$

$\text{Power } = 1 - \beta = P(\text{Reject } H_0 | H_0 \text{ false})$

The Simple Linear Regression Model

$$Y_i = \beta_0 + \beta_1 X_i + \varepsilon_i \qquad i = 1, 2, \ldots, n$$

- X_i is the known (fixed) predictor and Y_i the associated response for the ith observation.
- β_0 and β_1 are the regression coefficients.
- ε_i is a random variable such that:
  - $\mathsf{E}[\varepsilon_i] = 0$
  - $\text{Var}(\varepsilon_i) = \sigma^2$
  - $\varepsilon_i \perp \varepsilon_j$ for all $i \neq j$

Normal Equations

- $\sum_{i=1}^n Y_i = nb_0 + b_1 \sum_{i=1}^n X_i$
- $\sum_{i=1}^n X_i Y_i = b_0 \sum_{i=1}^n X_i + b_1 \sum_{i=1}^n X_i^2$

or

- $\sum_{i=1}^n (Y_i - b_0 - b_1 X_i) = 0$ (1)
- $\sum_{i=1}^n X_i (Y_i - b_0 - b_1 X_i) = 0$ (2)

Properties of Fitted Regression Line

- $\sum_{i=1}^n (Y_i - \hat{Y}_i) = \sum_{i=1}^n e_i = 0$
- $\sum_{i=1}^n Y_i = \sum_{i=1}^n \hat{Y}_i$
- $\sum_{i=1}^n X_i e_i = 0$
- $\sum_{i=1}^n \hat{Y}_i e_i = 0$
- $(\bar{X}, \bar{Y})$ is on the regression line.
- $\sum_{i=1}^n e_i^2$ is a minimum.

b_0 and b_1 as Linear Combinations of Y_i

$$b_1 = \sum k_i Y_i \qquad b_0 = \sum \left(\frac{1}{n} - \bar{X} k_i\right) Y_i$$

- $k_i = \dfrac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}$
- $\sum k_i = 0$
- $\sum k_i X_i = 0$
- $\sum k_i^2 = \dfrac{1}{\sum (X_i - \bar{X})^2}$